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## **ON THE UNITARY INVARIANCE OF THE NUMERICAL RADIUS**

IVAN FILIPPENKO AND MARVIN DAVID MARCUS

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**A characterization is obtained of scalar multiples of unitary matrices in terms of the unitary invariance of a generalized numerical radius. The method of proof involves some rather delicate combinatorial considerations.**

**1. Introduction.** Let  $n$  and  $m$  be positive integers,  $1 \leq m \leq n$ , and denote by  $M_{n,m}(\mathbb{C})$  ( $M_n(\mathbb{C})$ ) the vector space of all  $n$ -by- $m$  ( $n$ -square) complex matrices. For a matrix  $A \in M_n(\mathbb{C})$ , define the  $m$ th decomposable numerical range of  $A$  to be the set

$$(1) \quad W_m^\wedge(A) = \{ \det(X^*AX) \mid X \in M_{n,m}(\mathbb{C}), \det(X^*X) = 1 \}$$

in the complex plane (the reason for this choice of terminology will become apparent in the next section). It is not difficult to verify that  $W_m^\wedge(A)$  is compact, so it makes sense to define the  $m$ th decomposable numerical radius of  $A$  by

$$(2) \quad r_m^\wedge(A) = \max_{z \in W_m^\wedge(A)} |z|.$$

When  $m = 1$ ,  $W_1^\wedge(A)$  is simply the classical numerical range

$$(3) \quad W(A) = \{ (Ax, x) \mid x \in \mathbb{C}^n, \|x\| = 1 \}$$

(here  $(\cdot, \cdot)$  denotes the standard inner product in the space  $\mathbb{C}^n$  of complex  $n$ -tuples), and  $r_1^\wedge(A)$  is the classical numerical radius

$$(4) \quad r(A) = \max_{z \in W(A)} |z|.$$

The numerical radius  $r(A)$  satisfies the interesting power inequality

$$(5) \quad r(A^k) \leq r(A)^k, \quad k = 1, 2, 3, \dots$$

[2, §176]. In general, the number  $r_m^\wedge(A)$  is an important function of the matrix  $A$ . For example, it is a bound for the moduli of all products of  $m$  eigenvalues of  $A$ . This is an immediate consequence of Proposition 1. Another easy consequence (Corollary 2) of Proposition 1 is that if  $A$

is a scalar multiple of a unitary matrix, then  $r_m^\wedge(A)$  remains invariant under pre- and postmultiplication of  $A$  by arbitrary unitary matrices. The purpose of the present paper is to prove that in fact this invariance property characterizes scalar multiples of unitary matrices (Theorem 1).

**2. Preliminary notions.** The  $m$ th Grassmann space over  $\mathbb{C}^n$ , denoted by  $\wedge^m \mathbb{C}^n$ , provides an appropriate setting for our investigation of the  $m$ th decomposable numerical radius. The standard inner product in  $\mathbb{C}^n$  induces an inner product in  $\wedge^m \mathbb{C}^n$ , given on decomposable symmetrized tensors

$$x^\wedge = x_1 \wedge \cdots \wedge x_m, y^\wedge = y_1 \wedge \cdots \wedge y_m \in \wedge^m \mathbb{C}^n$$

by

$$(x^\wedge, y^\wedge) = \det[(x_i, y_j)].$$

The *Grassmannian manifold*  $G_m(\mathbb{C}^n)$  is the set of all unit length decomposable symmetrized tensors in  $\wedge^m \mathbb{C}^n$ :

$$G_m(\mathbb{C}^n) = \left\{ x^\wedge \in \wedge^m \mathbb{C}^n \mid \|x^\wedge\| = 1 \right\}.$$

Let  $A \in M_n(\mathbb{C})$ , and let  $C_m(A)$  be the  $m$ th compound of  $A$ , so that for  $x_1, \dots, x_m \in \mathbb{C}^n$  we have

$$C_m(A)x_1 \wedge \cdots \wedge x_m = Ax_1 \wedge \cdots \wedge Ax_m.$$

If the columns of a matrix  $X \in M_{n,m}(\mathbb{C})$  are  $x_1, \dots, x_m$  in order, then

$$\det(X^*AX) = (C_m(A)x_1 \wedge \cdots \wedge x_m, x_1 \wedge \cdots \wedge x_m).$$

Furthermore,  $\det(X^*X) = 1$  if and only if  $x_1 \wedge \cdots \wedge x_m \in G_m(\mathbb{C}^n)$ . Thus from (1),

$$(6) \quad W_m^\wedge(A) = \{(C_m(A)x^\wedge, x^\wedge) \mid x^\wedge \in G_m(\mathbb{C}^n)\}.$$

Given  $x^\wedge = x_1 \wedge \cdots \wedge x_m \in G_m(\mathbb{C}^n)$ , it may in fact be assumed that the vectors  $x_1, \dots, x_m \in \mathbb{C}^n$  are orthonormal [4, p. 1]. Choose, then, a unitary matrix  $U \in M_n(\mathbb{C})$  such that

$$Ue_k = x_k, \quad k = 1, \dots, m,$$

where  $\{e_1, \dots, e_n\}$  is the standard orthonormal basis of  $\mathbb{C}^n$ , and compute that

$$\begin{aligned} (C_m(A)x^\wedge, x^\wedge) &= (C_m(A)C_m(U)e_1 \wedge \dots \wedge e_m, C_m(U)e_1 \wedge \dots \wedge e_m) \\ &= (C_m(U^*AU)e_1 \wedge \dots \wedge e_m, e_1 \wedge \dots \wedge e_m) \\ &= \det(U^*AU)[1, \dots, m \mid 1, \dots, m], \end{aligned}$$

where  $(U^*AU)[1, \dots, m \mid 1, \dots, m]$  indicates the submatrix of  $U^*AU$  lying in rows and columns  $1, \dots, m$ . In view of (6), this yields yet another formulation of the  $m$ th decomposable numerical range: denoting by  $U_n(\mathbb{C})$  the multiplicative group of  $n$ -square unitary matrices, we have

$$(7) \quad W_m^\wedge(A) = \{ \det(U^*AU)[1, \dots, m \mid 1, \dots, m] \mid U \in U_n(\mathbb{C}) \}.$$

From (6) we obtain

$$(8) \quad W_m^\wedge(A) \subset W(C_m(A))$$

and hence

$$(9) \quad r_m^\wedge(A) \leq r(C_m(A)).$$

Strict inequality may hold in (9); e.g., consider

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in M_4(\mathbb{C})$$

with  $m = 2$  [1].

We define  $P_m^\wedge(A)$ , the  $m$ th decomposable eigenpolygon of  $A$ , to be the convex polygon in the complex plane spanned by all products of  $m$  eigenvalues of  $A$ . Thus

$$(10) \quad P_m^\wedge(A) = \mathcal{H}\left(\left\{ \prod_{k=1}^m \lambda_{\omega(k)} \mid \omega \in Q_{m,n} \right\}\right),$$

where  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $A$ ,  $\mathcal{H}$  denotes convex hull, and  $Q_{m,n}$  is the set of all strictly increasing sequences of  $m$  integers chosen from  $\{1, \dots, n\}$ . When  $m = 1$ ,  $P_1^\wedge(A)$  is simply written  $P(A)$  and called the eigenpolygon of  $A$ . It should be observed that the sets  $W_m^\wedge(A)$  and  $P_m^\wedge(A)$  are both invariant under transformation of  $A$  by a unitary similarity, that is,

$$W_m^\wedge(U^*AU) = W_m^\wedge(A)$$

and

$$P_m^\wedge(U^*AU) = P_m^\wedge(A)$$

for any  $U \in U_n(\mathbb{C})$ .

**PROPOSITION 1.** *Let  $A \in M_n(\mathbb{C})$  have eigenvalues  $\lambda_1, \dots, \lambda_n$ , and let  $m \in \{1, \dots, n\}$ . Then*

$$(11) \quad \prod_{k=1}^m \lambda_{\omega(k)} \in W_m^\wedge(A), \omega \in Q_{m,n}.$$

Moreover, if  $A$  is normal then

$$(12) \quad W_m^\wedge(A) \subset P_m^\wedge(A).$$

*Proof.* Fix  $\omega \in Q_{m,n}$ . By the Schur triangularization theorem, there exists a matrix  $U \in U_n(\mathbb{C})$  such that  $U^*AU$  is an upper triangular matrix with first  $m$  main diagonal elements  $\lambda_{\omega(1)}, \dots, \lambda_{\omega(m)}$ . Then

$$\prod_{k=1}^m \lambda_{\omega(k)} = \det(U^*AU)[1, \dots, m \mid 1, \dots, m].$$

In view of (7), (11) is established.

Next, assume  $A \in M_n(\mathbb{C})$  is normal. Let  $\{u_1, \dots, u_n\}$  be an orthonormal basis of  $\mathbb{C}^n$  such that

$$Au_i = \lambda_i u_i, \quad i = 1, \dots, n.$$

Then

$$\{u_\omega^\wedge = u_{\omega(1)} \wedge \dots \wedge u_{\omega(m)} \in G_m(\mathbb{C}^n) \mid \omega \in Q_{m,n}\}$$

is an orthonormal basis of  $\wedge^m \mathbb{C}^n$  [3, p. 132]. Given  $x^\wedge \in G_m(\mathbb{C}^n)$ , we have

$$(13) \quad \begin{aligned} (C_m(A)x^\wedge, x^\wedge) &= \left( C_m(A) \sum_{\omega \in Q_{m,n}} (x^\wedge, u_\omega^\wedge) u_\omega^\wedge, \sum_{\omega \in Q_{m,n}} (x^\wedge, u_\omega^\wedge) u_\omega^\wedge \right) \\ &= \sum_{\omega \in Q_{m,n}} \left| (x^\wedge, u_\omega^\wedge) \right|^2 \prod_{k=1}^m \lambda_{\omega(k)}. \end{aligned}$$

Since

$$\sum_{\omega \in Q_{m,n}} \left| (x^\wedge, u_\omega^\wedge) \right|^2 = \|x^\wedge\|^2 = 1,$$

(13) expresses the element  $(C_m(A)x^\wedge, x^\wedge)$  of  $W_m^\wedge(A)$  as a convex combination of all products of  $m$  eigenvalues of  $A$ . This establishes (12).

**COROLLARY 1.** *Let  $A \in M_n(\mathbb{C})$  be normal and  $m \in \{1, \dots, n\}$ . Then  $r_m^\wedge(A)$  is the maximum modulus of a product of  $m$  eigenvalues of  $A$ .*

**COROLLARY 2.** *Let  $A = cZ \in M_n(\mathbb{C})$ , where  $Z \in U_n(\mathbb{C})$  and  $c \in \mathbb{C}$ , and let  $m \in \{1, \dots, n\}$ . Then*

$$r_m^\wedge(UAV) = r_m^\wedge(A)$$

for all  $U, V \in U_n(\mathbb{C})$ .

**3. Some lemmas.** In the following discussion let  $A \in M_n(\mathbb{C})$  be a fixed matrix,  $m \in \{1, \dots, n\}$  a fixed positive integer, and assume the rank of  $A$  is at least  $m$ . Denote the singular values of  $A$  by  $\alpha_1, \dots, \alpha_n$ , arranged so that

$$\alpha_1 \geq \dots \geq \alpha_n \geq 0,$$

and set

$$D = \text{diag}(\alpha_1, \dots, \alpha_n) \in M_n(\mathbb{C}).$$

It is well known that there exist matrices  $U_1, V_1 \in U_n(\mathbb{C})$  such that

$$A = U_1 D V_1.$$

Suppose momentarily that

$$(14) \quad r_m^\wedge(UAV) = r_m^\wedge(A)$$

for all  $U, V \in U_n(\mathbb{C})$ . Then clearly

$$(15) \quad r_m^\wedge(UDV) = r_m^\wedge(D)$$

for all  $U, V \in U_n(\mathbb{C})$ :

$$\begin{aligned} r_m^\wedge(UDV) &= r_m^\wedge(UU_1^*AV_1^*V) \\ &= r_m^\wedge(A) && \text{(by (14))} \\ &= r_m^\wedge(U_1^*AV_1^*) && \text{(by (14))} \\ &= r_m^\wedge(D). \end{aligned}$$

Fix  $U_0 \in U_n(\mathbf{C})$  and choose  $x_0^\wedge \in G_m(\mathbf{C}^n)$  so that

$$(16) \quad r_m^\wedge(U_0 D) = |(C_m(U_0 D)x_0^\wedge, x_0^\wedge)|.$$

Set

$$(17) \quad y_0^\wedge = C_m(U_0^*)x_0^\wedge \in G_m(\mathbf{C}^n).$$

Let  $\{e_1, \dots, e_n\}$  be the standard orthonormal basis of  $\mathbf{C}^n$ ; then

$$\{e_\omega^\wedge = e_{\omega(1)} \wedge \dots \wedge e_{\omega(m)} \in G_m(\mathbf{C}^n) \mid \omega \in Q_{m,n}\}$$

is the induced orthonormal basis of  $\wedge^m \mathbf{C}^n$ . Write

$$(18) \quad x_0^\wedge = \sum_{\omega \in Q_{m,n}} \chi_\omega e_\omega^\wedge, \chi_\omega \in \mathbf{C}, \omega \in Q_{m,n}$$

and

$$(19) \quad y_0^\wedge = \sum_{\omega \in Q_{m,n}} \eta_\omega e_\omega^\wedge, \eta_\omega \in \mathbf{C}, \omega \in Q_{m,n}.$$

LEMMA 1. *Assume*

$$r_m^\wedge(U_0 D) = r_m^\wedge(D).$$

*Then*

$$\alpha_1 \cdots \alpha_m = \alpha_{\omega(1)} \cdots \alpha_{\omega(m)}$$

*for every  $\omega \in Q_{m,n}$  for which  $\chi_\omega \neq 0$ . Moreover,*

$$|\chi_\omega| = |\eta_\omega|, \omega \in Q_{m,n}.$$

*Proof.* Notice that

$$\alpha_1 \cdots \alpha_m > 0$$

since  $A$  has rank at least  $m$ . We compute

$$\begin{aligned} \alpha_1 \cdots \alpha_m &= r_m^\wedge(D) && \text{(by Corollary 1)} \\ &= r_m^\wedge(U_0 D) && \text{(by hypothesis)} \\ &= |(C_m(U_0 D)x_0^\wedge, x_0^\wedge)| && \text{(by (16))} \end{aligned}$$

$$\begin{aligned}
 &= \left| \sum_{\omega \in Q_{m,n}} \alpha_\omega \chi_\omega \bar{\eta}_\omega \right| \quad (\alpha_\omega = \alpha_{\omega(1)} \cdots \alpha_{\omega(m)}) \\
 (20) \quad &\cong \sum_{\omega \in Q_{m,n}} \alpha_\omega |\chi_\omega| |\eta_\omega| \\
 &\cong \alpha_1 \cdots \alpha_m \sum_{\omega \in Q_{m,n}} |\chi_\omega| |\eta_\omega| \\
 &\cong \alpha_1 \cdots \alpha_m \left( \sum_{\omega \in Q_{m,n}} |\chi_\omega|^2 \right)^{\frac{1}{2}} \left( \sum_{\omega \in Q_{m,n}} |\eta_\omega|^2 \right)^{\frac{1}{2}} \\
 &= \alpha_1 \cdots \alpha_m \|x \hat{\circ}\| \|y \hat{\circ}\| \\
 &= \alpha_1 \cdots \alpha_m.
 \end{aligned}$$

The last inequality in (20) is the Cauchy–Schwarz inequality. Since equality holds throughout,  $\alpha_1 \cdots \alpha_m > 0$ , and  $x \hat{\circ}, y \hat{\circ} \neq 0$ , we conclude that

$$|\chi_\omega| = c |\eta_\omega|, \omega \in Q_{m,n}$$

for some  $c > 0$ . But then  $\|x \hat{\circ}\| = 1 = \|y \hat{\circ}\|$  implies  $c = 1$ . Thus

$$|\chi_\omega| = |\eta_\omega|, \omega \in Q_{m,n}.$$

It follows from equality in the second inequality in (20) that

$$\alpha_1 \cdots \alpha_m = \alpha_{\omega(1)} \cdots \alpha_{\omega(m)}$$

for every  $\omega \in Q_{m,n}$  for which  $\chi_\omega \neq 0$ .

Suppose now that  $\sigma$  is a permutation in  $S_n$ , the symmetric group of degree  $n$ , and  $U_\sigma^* \in U_n(\mathbf{C})$  is the permutation matrix corresponding to  $\sigma$ :

$$U_\sigma^* = P(\sigma) = [\delta_{\sigma(i)}].$$

In this situation, continuing with the above notation, we have

$$\begin{aligned}
 (21) \quad y \hat{\circ} &= C_m(P(\sigma))x \hat{\circ} \\
 &= \sum_{\omega \in Q_{m,n}} \chi_\omega C_m(P(\sigma))e_\omega \hat{\circ} \\
 &= \sum_{\omega \in Q_{m,n}} \chi_\omega e_{\sigma\omega(1)} \wedge \cdots \wedge e_{\sigma\omega(m)} \quad (\text{since } P(\sigma)e_i = e_{\sigma(i)}, i = 1, \cdots, n) \\
 &= \sum_{\omega \in Q_{m,n}} \epsilon_\omega \chi_\omega e_{\omega\sigma}.
 \end{aligned}$$



Here  $\omega_\sigma \in Q_{m,n}$  is the strictly increasing rearrangement of the sequence

$$(\sigma\omega(1), \dots, \sigma\omega(m)),$$

and  $\epsilon_\omega = \pm 1$  is the sign of the permutation

$$\begin{pmatrix} \sigma\omega(1) \cdots \sigma\omega(m) \\ \omega_\sigma(1) \cdots \omega_\sigma(m) \end{pmatrix}.$$

The mapping

$$\omega \mapsto \omega_\sigma, \omega \in Q_{m,n}$$

is clearly a bijection of  $Q_{m,n}$ . Hence from (19) and (21),

$$\begin{aligned} y_0^\wedge &= \sum_{\omega \in Q_{m,n}} \eta_\omega e_\omega^\wedge \\ &= \sum_{\omega \in Q_{m,n}} \eta_{\omega_\sigma} e_{\omega_\sigma}^\wedge \\ &= \sum_{\omega \in Q_{m,n}} \epsilon_\omega \chi_\omega e_{\omega_\sigma}^\wedge \end{aligned}$$

so that

$$(22) \quad \eta_{\omega_\sigma} = \epsilon_\omega \chi_\omega, \omega \in Q_{m,n}.$$

LEMMA 2. *Assume*

$$r_m^\wedge(P(\sigma)^T D) = r_m^\wedge(D).$$

Then

$$\alpha_1 \cdots \alpha_m = \alpha_{\omega_\sigma(1)} \cdots \alpha_{\omega_\sigma(m)}$$

for every  $\omega \in Q_{m,n}$  for which  $\chi_{\omega_\sigma} \neq 0$ . Moreover,

$$|\chi_{\omega_\sigma}| = |\chi_\omega|, \omega \in Q_{m,n}.$$

*Proof.* The first assertion is immediate from Lemma 1, as is the second:

$$\begin{aligned} |\chi_{\omega_\sigma}| &= |\eta_{\omega_\sigma}| \\ &= |\epsilon_\omega \chi_\omega| \quad (\text{by (22)}) \\ &= |\chi_\omega|, \omega \in Q_{m,n}. \end{aligned}$$

#### 4. The main result.

**THEOREM 1.** *Let  $A \in M_n(\mathbf{C})$  and let  $m$  be a positive integer,  $1 \leq m < n$ . Assume the rank of  $A$  is at least  $m$ . Then*

$$(23) \quad r_m^\wedge(UAV) = r_m^\wedge(A)$$

*for all  $U, V \in U_n(\mathbf{C})$  if and only if  $A$  is a scalar multiple of a unitary matrix.*

*Proof.* We have observed in Corollary 2 that the condition is sufficient.

To see that the condition is necessary, assume (23) holds for all  $U, V \in U_n(\mathbf{C})$ . Since there exist matrices  $U_1, V_1 \in U_n(\mathbf{C})$  such that

$$A = U_1 D V_1,$$

where

$$D = \text{diag}(\alpha_1, \dots, \alpha_n) \in M_n(\mathbf{C})$$

and

$$\alpha_1 \geq \dots \geq \alpha_n$$

are the singular values of  $A$ , it suffices to show that

$$\alpha_1 = \alpha_n.$$

Consider the full cycle

$$\varphi = (12 \dots n) \in S_n.$$

Choose  $x_0^\wedge \in G_m(\mathbf{C}^n)$  so that

$$r_m^\wedge(P(\varphi)^T D) = |(C_m(P(\varphi)^T D)x_0^\wedge, x_0^\wedge)|$$

and write

$$x_0^\wedge = \sum_{\omega \in Q_{m,n}} \chi_\omega e_\omega^\wedge, \chi_\omega \in \mathbf{C}, \omega \in Q_{m,n}.$$

Since

$$\sum_{\omega \in Q_{m,n}} |\chi_\omega|^2 = \|x_0^\wedge\|^2 = 1,$$

there exists  $\omega \in Q_{m,n}$  for which

$$(24) \quad \chi_\omega \neq 0.$$

Set

$$(25) \quad \gamma = \omega_{\varphi^{n-\omega(1)+1}} \in Q_{m,n}.$$

By (15) and Lemma 2 (with  $\sigma = \varphi^{n-\omega(1)+1}$ ),  $|\chi_\gamma| = |\chi_\omega|$  and hence by (24)

$$(26) \quad \chi_\gamma \neq 0.$$

Also observe that

$$\begin{aligned} \varphi^{n-\omega(1)+1}\omega(1) &= \varphi(\omega(1) + n - \omega(1)) \\ &= \varphi(n) \\ &= 1 \end{aligned}$$

implies  $\omega_{\varphi^{n-\omega(1)+1}}(1) = 1$ , i.e.,

$$\gamma(1) = 1.$$

The argument now splits into two cases.

*Case I.*  $\gamma(m) < n$ . Apply the permutation  $\varphi^{n-\gamma(m)}$  to

$$\gamma = (1, \gamma(2), \dots, \gamma(m))$$

to obtain

$$(27) \quad \begin{aligned} \varphi^{n-\gamma(m)}\gamma &= (1 + n - \gamma(m), \gamma(2) + n - \gamma(m), \dots, \gamma(m - 1) + n - \gamma(m), n) \\ &= \gamma_{\varphi^{n-\gamma(m)}} \end{aligned}$$

Since  $\gamma(m) < n$ , we have

$$\begin{aligned} 2 &\leq 1 + n - \gamma(m), \\ 3 &\leq \gamma(2) + n - \gamma(m), \\ &\vdots \\ m &\leq \gamma(m - 1) + n - \gamma(m). \end{aligned}$$

Therefore

$$(28) \quad \alpha_2 \alpha_3 \cdots \alpha_m \cong \alpha_{1+n-\gamma(m)} \alpha_{\gamma(2)+n-\gamma(m)} \cdots \alpha_{\gamma(m-1)+n-\gamma(m)}.$$

By (15) and Lemma 2 (with  $\sigma = \varphi^{n-\gamma(m)}$ ),  $|\chi_{\gamma_{\varphi^{n-\gamma(m)}}}| = |\chi_\gamma|$  and hence by (26)

$$\chi_{\gamma_{\varphi^{n-\gamma(m)}}} \neq 0.$$

Then Lemma 2 together with (27) implies

$$(29) \quad \alpha_1 \alpha_2 \alpha_3 \cdots \alpha_m = \alpha_{1+n-\gamma(m)} \alpha_{\gamma(2)+n-\gamma(m)} \cdots \alpha_{\gamma(m-1)+n-\gamma(m)} \alpha_n.$$

Since  $\alpha_1 \cdots \alpha_m > 0$  ( $A$  has rank at least  $m$ ), it follows from (28) and (29) that

$$\alpha_1 = \alpha_n.$$

Case II.  $\gamma(m) = n$ . In this case

$$\gamma = (1, \gamma(2), \cdots, \gamma(m-1), n).$$

Now  $m < n$  by hypothesis, so there exists a least positive integer  $k \in \{2, \cdots, m\}$  such that

$$k < \gamma(k).$$

Apply the permutation  $\varphi^{1-k}$  to

$$\gamma = (1, \cdots, k-1, \gamma(k), \cdots, \gamma(m-1), n)$$

to obtain

$$\begin{aligned} \varphi^{1-k} \gamma = & (n-k+2, n-k+3, \cdots, n-1, n, \gamma(k)-k+1, \cdots, \\ & \gamma(m-1)-k+1, n-k+1). \end{aligned}$$

Then

$$(30) \quad \gamma_{\varphi^{1-k}} = (\gamma(k)-k+1, \cdots, \gamma(m-1)-k+1, n-k+1, n-k+2, \\ n-k+3, \cdots, n-1, n).$$

Since  $k < \gamma(k)$ , we have

$$\begin{aligned}
 2 &\leq \gamma(k) - k + 1, \\
 3 &\leq \gamma(k + 1) - k + 1, \\
 &\vdots \\
 m - k + 1 &\leq \gamma(m - 1) - k + 1, \\
 m - k + 2 &\leq n - k + 1, \\
 m - k + 3 &\leq n - k + 2, \\
 &\vdots \\
 m &\leq n - 1.
 \end{aligned}$$

Therefore

$$(31) \quad \alpha_2 \alpha_3 \cdots \alpha_m \geq \alpha_{\gamma(k)-k+1} \alpha_{\gamma(k+1)-k+1} \cdots \alpha_{n-1}.$$

By (15) and Lemma 2 (with  $\sigma = \varphi^{1-k}$ ),  $|\chi_{\gamma\varphi^{1-k}}| = |\chi_\gamma|$  and hence by (26)

$$\chi_{\gamma\varphi^{1-k}} \neq 0.$$

Then Lemma 2 together with (30) implies

$$(32) \quad \alpha_1 \alpha_2 \alpha_3 \cdots \alpha_m = \alpha_{\gamma(k)-k+1} \alpha_{\gamma(k+1)-k+1} \cdots \alpha_{n-1} \alpha_n.$$

Once again, since  $\alpha_1 \cdots \alpha_m > 0$  it follows from (31) and (32) that

$$\alpha_1 = \alpha_n.$$

This completes the proof.

We remark that the restriction  $m \neq n$  in Theorem 1 is inevitable. Indeed, for any matrix  $A \in M_n(\mathbb{C})$ ,

$$\begin{aligned}
 r_n^\wedge(A) &= |\det(A)| \\
 &= |\det(UAV)| \\
 &= r_n^\wedge(UAV)
 \end{aligned}$$

for all  $U, V \in U_n(\mathbb{C})$ . The hypothesis that  $A$  have rank at least  $m$  is equally essential, since any matrix  $A \in M_n(\mathbb{C})$  of rank less than  $m$  satisfies

$$r_m^\wedge(A) = 0 = r_m^\wedge(UAV)$$

for all  $U, V \in U_n(\mathbb{C})$ .

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