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Results of Matsushima and Murakami, and Matsushima and Shimura, are applied to the study of certain subspaces of the cohomology of fiber varieties of Kuga–Satake type. These subspaces are defined in terms of the Kähler and fiber space bigradings of the cohomology of the fiber variety.

Let G be a real connected semi-simple Lie group with finite center, and K a maximal compact subgroup of G. The quotient space $X = G/K$ is then a symmetric space (of non-compact type). Assume that X is a symmetric domain. Consider a discrete subgroup Γ of G such that $\Gamma\backslash G$ is compact and Γ has no fixed point in X. Take a representation $\rho: G \to GL(F)$ of G in a real vector space F, such that ρ leaves invariant a nonsingular alternating form β on G. Suppose that $\rho|_\Gamma$ leaves invariant a lattice L in F, and that the form β takes integer values on $L \times L$. From the data $\{G, K, X, \Gamma, \rho, F, \beta, L\}$ Kuga has constructed an algebraic variety V, fibred by a family of abelian varieties, when ρ satisfies an additional condition.

In this paper the cohomology of the fiber variety V is studied. The cohomology of V is bigraded by the Kähler structure of V, and also by the fiber structure of V. In §1 it is shown that the Kähler and fiber bigradings:

$$\text{(k)} \qquad H^r(V)_{\mathbf{C}} = \sum_{p+q=r} H^{(p,q)}(V)$$

$$\text{(f)} \qquad H^r(V)_{\mathbf{C}} = \sum_{a+b=r} H^{\{a,b\}}(V)_{\mathbf{C}}$$

are compatible, and so lead to a finer decomposition of the cohomology of V:

$$H^r(V)_{\mathbf{C}} = \sum H^{a_1 a_2 b_1 b_2}.$$

In §3 this decomposition is related to the work of Matsushima and Murakami, ([5]) when the representation ρ satisfies a certain

[1] $S > 0$ means S is positive definite

condition. In §4 is studied the special case $G = SL(2,\mathbf{R})^N$, using results of Matsushima and Shimura [7]. The dimensions of the non-zero $H^{a_1a_2b_1b_2}$ are seen to be expressible in terms of the dimensions of spaces of Γ-automorphic forms. For $N = 1$, one obtains

$$H^{(p,p)}(V) = H^{\{0,2p\}}(V)_C + H^{\{2,2p-2\}}(V)_C.$$

This result used in the proof of the Hodge conjecture for the fiber variety V over an algebraic curve associated to an indefinite quaternion algebra over **Q**.

1. Preliminaries. First recall the construction of the fiber variety V. Let G, K, X, and Γ be as above. The quotient space $U = \Gamma\backslash X$ is then a compact complex manifold, and Kodaira has shown it is a Hodge manifold. Let B be an $N \times N$ integral skew-symmetric matrix with $\det B \neq 0$, and suppose there exists a representation $\rho: G \to Sp(B) = \{M \in GL(N\mathbf{R}): {}^tMBM = B\}$ such that $\rho|_\Gamma: \Gamma \to GL(N,\mathbf{Z})$. Denote the representation space $\mathbf{R}^N$ by F and the lattice $\mathbf{Z}^N$ by L. Form the semidirect product GF by defining $(g,u)(g',u') = (gg', \rho(g)u' + u)$. The group GF acts on $X \times F$ in a natural way, and ΓL is a discrete subgroup of GF which acts on $X \times F$ properly discontinuously and without fixed points. The quotient space $V = \Gamma L\backslash X \times F$ is then a compact manifold. The projection $V \to U$ makes V a fiber bundle over U.

Let $\mathfrak{G}$ and $\boldsymbol{k}$ denote the Lie algebras of G and K respectively, and let $\mathfrak{G} = \boldsymbol{k} + \boldsymbol{p}$ be the Cartan decomposition of $\mathfrak{G}$ with respect to $\boldsymbol{k}$. Then one can show there exists an $N \times N$ matrix S such that

1. ${}^tS = S, S > 0$[1]
2. $(B^{-1}S)^2 = -1_N$
3. $\rho(K) \subset 0(S) = \{M \in GL(N,\mathbf{R}): {}^tMSM = S\}$
4. ${}^td\rho(X)S = Sd\rho(X), X \in \boldsymbol{k}$

The pair $\{B, S\}$ is called a symplectic pair, and every such pair is conjugate to the pair

$$\left\{J = \begin{pmatrix} 0 & 1_{N/2} \\ -1_{N/2} & 0 \end{pmatrix}, 1_N\right\};$$

that is, there exists a real $N \times N$ matrix T such that ${}^tTBT = J$ and ${}^tTST = 1_N$. The mapping $M \to T^{-1}MT$ gives an isomorphism of $Sp(B)$ onto $Sp(J)$ and induces an isomorphism

$$X_0 = Sp(B)/0(S) \cap Sp(B) \cong Sp(J)/0(1_N) \cap Sp(J).$$

The latter space is identified with the Siegel space $\mathfrak{h}^{N/2} = \{Z \in M(N/2), \mathbf{C}): {}^tZ = Z \text{ and } \operatorname{Im} Z > 0\}$ by

$$Sp(J) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \to (Ai + B)(Ci + D)^{-1} \in \mathfrak{h}^{N/2}.$$

Of the two possible complex structures on X_0, we give the one making the mapping $X_0 \to \mathfrak{h}^{N/2}$ holomorphic.

Assume that the mapping $\tau: X \to X_0$ induced by $\rho: G \to Sp(B)$ is holomorphic. This assumption makes it possible to define a GF-invariant complex structure on $X \times F$. Let ρ': $G \xrightarrow{\rho} Sp(B) \xrightarrow{a} Sp(J)$ and let $\tau': X \to \mathfrak{h}^{N/2}$ be the induced mapping. Let $\xi(u)$ be the coordinate vector of $u \in F = \mathbf{R}^N$ with respect to the standard basis of $\mathbf{R}^N$, and let $\xi'(u) = T^{-1}\xi(u)$. Define

$$(1) \qquad Z(x,u) = \begin{pmatrix} Z^1(x,u) \\ \vdots \\ Z^{N/2}(x,u) \end{pmatrix} = (\tau'(x), 1_{N/2}) J\xi'(u).$$

If $w^1(x), \cdots, w^n(x)$ is a system of complex coordinates for X, then the system $w^1(x), \cdots, w^n(x)$, $Z^1(x,u), \cdots, Z^{N/2}(x,u)$ gives complex coordinates for $(x,u) \in X \times F$. With respect to this complex structure on $X \times F$, the action of GF is holomorphic. Give V the induced complex structure.

Let $x = gK$. Then $A(x) = {}^t\rho(g)^{-1}S\rho(g)^{-1}$ and $J(x) = B^{-1}A(x)$ are well-defined, and $J(x)$ induces the complex structure on the fiber over Γx in U. If ds_0^2 is a Hodge metric on U, then the metric $ds^2 = ds_0^2 + {}^td\xi(u)A(x)d\xi(u)$ makes V a Hodge manifold.

Let $\Omega^r(X)$ be the space of real valued r-forms on X. Let $\Omega^r(X, \Gamma, \rho) = \{\omega \in F \otimes \Omega^r(X): \omega \circ \gamma = \rho(\gamma)\omega\}$. The exterior differentiation d on $\Omega^*(X)$ gives a coboundary operator $1 \otimes d$ on $\Omega^*(X, \Gamma, \rho)$. Let $H^r(X, \Gamma, \rho)$ be the rth cohomology group of the complex $(\Omega^*(X, \Gamma, \rho), 1 \otimes d)$. Given a suitable inner product $(\,,)_F$ on F, the notion of harmonic element of $\Omega^r(X, \Gamma, \rho)$ can be defined. Let $\mathcal{H}^a(X, \Gamma, \rho) = \{\omega \in \Omega^r(X, \Gamma, \rho): \omega$ is harmonic$\}$. $\mathcal{H}^a(X, \Gamma, \rho)$ and $H^a(X, \Gamma, \rho)$ are naturally isomorphic.

Identify $\Lambda^b(F)^*$, the dual to the Grassmanian $\Lambda^b(F)$, with $\Lambda^b(F^*)$. Choose an ordered basis $\{\xi^{D_i}: i = 1, \cdots, \binom{N}{b}\}$ of $\Lambda^b(F)^*$, such that each ξ^{D_i} is of the form $\xi^D = \xi^{D_1} \wedge \cdots \wedge \xi^{d_b}$ where $D = (d_1, \cdots, d_b)$ and $\{\xi^1, \cdots, \xi^N\}$ is the usual coordinate system on $F = \mathbf{R}_N$. This notation will also be used for differential forms; e.g., $d\xi^D$ for $d\xi^{d_1} \wedge \cdots \wedge d\xi^{d_b}$. Let Λ^b be the natural representation of $GL(F)$ on $\Lambda^b(F)$, and let $\rho^{(b)} = (\Lambda^b \circ \rho)^*$ be the representation of G on $\Lambda^b(F)^*$, contragredient to $\Lambda^b \circ \rho$. The symmetric positive definite matrix S given above can be used to define a suitable inner product on $\Lambda^b(F)^*$, so that $\mathscr{H}^a(X, \Gamma, \rho^{(b)})$ is defined. (The condition of suitability is that, with respect to $(\,,\,)_F$, $\rho(X)$ is skew-symmetric if $X \in \boldsymbol{k}$ and symmetric if $X \in \boldsymbol{p}$.)

Let $\mathscr{H}^r(V)$ be the space of real harmonic forms on the Kähler manifold V, and let $H^r(V)$ be the rth real de Rham cohomology group of V. $H^r(V)$ and $\mathscr{H}^r(V)$ are naturally isomorphic; this isomorphism will be referred to as the Hodge isomorphism. One can define a mapping of $\mathscr{H}^a(X, \Gamma, \rho^{(b)})$ into $\mathscr{H}^{a+b}(V)$ by

$$\mathscr{H}^a(X, \Gamma, \rho^{(b)}) \ni \hat{\omega}(x) \to {}^t\hat{\omega}(x) \wedge \begin{bmatrix} d\xi^{D_1}(u) \\ \vdots \\ d\xi^{D\binom{N}{b}}(u) \end{bmatrix} = \omega(x, u) \in \mathscr{H}^{a+b}(V) \tag{2}$$

(Here, differential forms on V are considered to be ΓL-invariant forms on $X \times F$.) Let $\mathscr{H}^{\{a,b\}}(V)$ be the image of this mapping. Then the space $H^{\{a,b\}}(V)$ occurring in the decomposition (f) is the image of $\mathscr{H}^{\{a,b\}}(V)$ under the Hodge isomorphism. See [3] for details and proofs.

2. In this section the decomposition

$$\mathscr{H}^{\{a,b\}}(V)_{\mathbf{C}} = \sum_{\substack{a_1+a_2=a \\ b_1+b_2=b}} \mathscr{H}^{a_1 a_2 b_1 b_2}$$

is obtained. ($\mathscr{H}^{\{a,b\}}(V)_{\mathbf{C}}$ is the complexification $\mathscr{H}^{\{a,b\}}(V) \otimes \mathbf{C}$ of the real vector space $\mathscr{H}^{\{a,b\}}(V)$.) Rewrite equation (1) of §1 as

$$\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \begin{pmatrix} -1 & \tau' \\ -1 & \bar{\tau}' \end{pmatrix} \xi'.$$

By taking differentials obtain (1, 0)-forms $u^1(x, u), \cdots, u^{N/2}(x, u)$ defined by the equation

$$\begin{pmatrix} u \\ \bar{u} \end{pmatrix} = \begin{pmatrix} -1 & \tau' \\ -1 & \bar{\tau}' \end{pmatrix} d\xi' = \begin{pmatrix} dz & -d\tau' \cdot \xi'' \\ d\bar{z} & -d\bar{\tau}' \cdot \xi'' \end{pmatrix} \tag{3}$$

where

$$\xi'' = \begin{pmatrix} \xi'^{N/2+1} \\ \vdots \\ \xi'^{N} \end{pmatrix}.$$

For $D = (d_1, \cdots, d_b)$, put $|2D| = b$.

DEFINITION. Let $\omega \in \mathcal{H}^{\{a,b\}}(V)_{\mathbf{C}}$. ω is of type $a_1a_2b_1b_2$ if

$$\omega(x,u) = \Sigma \psi_{A_1A_2B_1B_2} dw^{A_1} \wedge d\bar{w}^{A_2} \wedge u^{B_1} \wedge \bar{u}^{B_2}$$

where $\psi_{A_1A_2B_1B_2} = 0$ unless $|A_i| = a_i$ and $|B_i| = b_i$, $i = 1,2$. The terminology "$a_1a_2b_1b_2$-component of $\omega \in \mathcal{H}^{\{a,b\}}(V)_{\mathbf{C}}$" will be used, whether or not that component is harmonic. Let

$$\mathcal{H}^{a_1a_2b_1b_2} = \{\omega \in \mathcal{H}^{\{a,b\}}(V)_{\mathbf{C}}: \omega \text{ is of type } a_1a_2b_1b_2\}.$$

Several definitions are needed. Let $\mathfrak{G}_{\mathbf{C}}$ be the complexification of the Lie algebra of G. The representation $\rho: G \to GL(F)$ induces a representation $\rho_{\mathbf{C}}: G \to GL(F) \subset GL(F_{\mathbf{C}})$ and a Lie algebra representation $\rho = d\rho_{\mathbf{C}}: \mathfrak{G}_{\mathbf{C}} \to GL(F_{\mathbf{C}})$. The isomorphism (2) of §1 given by $\hat{\omega} \to \omega$ extends to a $\mathbf{C}$-linear isomorphism of the complexifications

$$\mathcal{H}^a(X, \Gamma, \rho^{(b)})_{\mathbf{C}} \cong \mathcal{H}^{\{a,b\}}(V)_{\mathbf{C}}.$$

An invariant complex structure on $X_0 = Sp(B)/0(S) \cap Sp(B)$ is determined by the element $Z_0 = -1/2B^{-1}S$ belonging to the center of $\boldsymbol{k}_0$, the Lie algebra of $0(S) \cap Sp(B)$; and the invariant complex structure on X is determined by an element Z belonging to the center of $\boldsymbol{k}$.

Let $\mathfrak{G}_{\mathbf{C}} = \boldsymbol{k}_{\mathbf{C}} + \boldsymbol{p}_+ + \boldsymbol{p}_-$ be the decomposition of $\mathfrak{G}_{\mathbf{C}}$ into (respectively) the $0, +i$, and $-i$-eigenspaces of $\operatorname{ad} Z$. Then $\boldsymbol{k}_{\mathbf{C}}$ is orthogonal to $\boldsymbol{p}_{\mathbf{C}} = \boldsymbol{p}_+ + \boldsymbol{p}_-$ with respect to the Killing form K, and there exists a basis $Y_1, \cdots, Y_s$ of $\boldsymbol{k}_{\mathbf{C}}$ and a basis $X_1, \cdots, X_n$ of $\boldsymbol{p}_+$ such that

$$K(Y_a, Y_b) = -\delta_{ab} \quad a, b = 1, \cdots, s$$
$$K(X_i, \bar{X}_j) = \delta_{ij} \quad i, j = 1, \cdots, n.$$

Put $X_{\bar{j}} = \bar{X}_j \in \boldsymbol{p}_- = \bar{\boldsymbol{p}}_+$.

To $\omega \in \mathcal{H}^{\{a,b\}}(V)_{\mathbf{C}}$ one can associate a coordinate function $\omega^0(g)$ taking values in $\Lambda^b(F_{\mathbf{C}})^* \otimes \Lambda^a(\boldsymbol{p}_{\mathbf{C}})^*$. First identify $\Lambda^a(\boldsymbol{p}_{\mathbf{C}})^*$ with $\Lambda^a(\boldsymbol{p}_{\mathbf{C}})$ by means of the Killing form. Let $v: G \to X$ be the projection. For $I = (i_1, \cdots, i_{a_1})$, $J = (j_1, \cdots, j_{a_2})$, and $a_1 + a_2 = a$, define

$$\omega_{I\bar{J}}(g) = \rho^{(b)}(g^{-1}) v^* \hat{\omega}_g (X_{i_1}, \cdots, X_{i_{a_1}}, X_{\bar{j}_1}, \cdots, X_{\bar{j}_{a_2}}) \tag{4}$$

and let

$$\omega^o(g) = \Sigma \omega_{I\bar{J}}(g) X_{\bar{I}} \Lambda X_J.$$

Let C be the Casimir operator of G,

(5) $$C = \sum_{k=1}^{n} X_k X_{\bar{k}} + X_{\bar{k}} X_k - \sum_{a=1}^{s} Y_a^2,$$

and let $\rho^{(b)}(C)$ be the Casimir matrix of the representation $\rho^{(b)}$. Then one can show

PROPOSITION 1. $\omega \in \mathcal{H}^{\{a,b\}}(V)_{\mathbf{C}} \Leftrightarrow$
(i) $\omega^o(\gamma g) = \omega^o(g) \forall \gamma \in \Gamma, g \in G$
(ii) $\omega^o(gk^{-1}) = \rho^{(b)}(k) \otimes \Lambda^a o Ad(k) \forall g \in G, k \in K$
(iii) $C\omega^o = \rho^{(b)}(C) \otimes 1 \omega^o$
(for the proof, cf. [3]). *(Ad is the adjoint representation of K on* $\mathfrak{p}_{\mathbf{C}}$.) Let $V^{b_1 b_2}$ be the $(b_2 - b_1)i/2$-eigenvalue of $d\Lambda^b\begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}$ in $\mathbf{C}^{\binom{N}{b}}$. Recall there exists $T \in GL(N, \mathbf{R})$ such that ${}^tTBT = J$ and ${}^tTST = 1$, and that $\rho' = T^{-1}\rho T$. Let $L = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}$, and let $L_0 = TL^{-1}$. Then

$$Z_0 = -\frac{1}{2}B^{-1}S = L_0\begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}L_0^{-1}$$

so that $F^{b_1 b_2} = \Lambda^b(L_0)^*(V^{b_1 b_2})$ is the $(b_2 - b_1)i/2$-eigenspace of $d\Lambda^b(Z_0)^*$. Now take the $a_1 a_2 b_1 b_2$-component η of $\omega \in \mathcal{H}^{\{a,b\}}(V)_{\mathbf{C}}$ and make the F-valued form $\hat{\eta}$ and F-valued functions $\eta_{I\bar{J}}$ as in (4). Then

PROPOSITION 2. *$\eta_{I\bar{J}}$ takes values in $F^{b_1 b_2}$ and*

$$\eta_{I\bar{J}} = 0 \text{ unless } |I| = a_1 \text{ and } |J| = a_2.$$

Proof. Since $\hat{\eta}$ is of type (a_1, a_2) and the action of $\mathrm{ad}(Z)$ on ρ determines, via the projection $G \to X$, the complex structure on X, one sees that $\eta_{I\bar{J}} = 0$ unless $|I| = a_1$ and $|J| = a_2$. Since

$$\begin{pmatrix} u \\ \bar{u} \end{pmatrix} = \begin{pmatrix} -1 & \tau' \\ -1 & \bar{\tau}' \end{pmatrix} T^{-1} d\xi \quad \text{and} \quad \eta = {}^t\hat{\eta} \wedge \begin{pmatrix} \vdots \\ d\xi^{D_i} \\ \vdots \end{pmatrix},$$

one sees that the form

$$\zeta = \Lambda^b\left(\begin{pmatrix} -1 & \tau' \\ -1 & \bar{\tau}' \end{pmatrix} T^{-1}\right)^* \hat{\eta}$$

takes values in $V^{b_1 b_2}$. Now

$$\eta_{I\bar{J}}(g) = \Lambda^b\left(T\rho'(g^{-1})\begin{pmatrix} -1 & \tau' \\ -1 & \bar{\tau}' \end{pmatrix}^{-1}\right)^* v^* \zeta_g(X_{i_1}, \cdots, X_{\bar{j}a_2}).$$

Let $\rho'(g)=\begin{pmatrix} A & B \\ C & D\end{pmatrix}$, $\tau'=\tau'(gk)$ and $j={}^t(Ci+D)^{-1}$. Then using the relations

$$\operatorname{Im}\tau' = j^{-1}t_{\bar{j}},$$
$$\rho'(g)\begin{pmatrix} i & -i \\ 1 & 1\end{pmatrix}=\begin{pmatrix} \tau' & \bar{\tau}' \\ 1 & 1\end{pmatrix}\begin{pmatrix} t_j^{-1} & 0 \\ 0 & t_{\bar{j}^{-1}}\end{pmatrix},$$

one finds

$$T\rho'(g)^{-1}\begin{pmatrix} -1 & \tau' \\ -1 & \bar{\tau}'\end{pmatrix}^{-1} = L_0\begin{pmatrix} \sqrt{-2}j^{-1} & 0 \\ 0 & \sqrt{-2}\bar{j}^{-1}\end{pmatrix}.$$

Since $V^{b_1b_2}$ is $\Lambda^b\begin{pmatrix} \sqrt{-2}j^{-1} & 0 \\ 0 & \sqrt{-2\bar{j}^{-1}}\end{pmatrix}$-invariant, $\eta_{I\bar{J}}\in\Lambda^b(L_0)^*(V^{b_1b_2})=F^{b_1b_2}$, proving the proposition.

Since the representation ρ induces a holomorphic mapping $\tau\colon X\to X_0$, we have $\rho\,|_{p_+}\colon p_+\to p_{0+}$ and $\rho\,|_{p_-}\colon p_-\to p_{0-}$, where p_{0+} (resp p_{0-}) is the $+i$ (resp. $-i$)-eigenspace of $\operatorname{ad} Z_0$. Hence

$$\rho^{(b)}(X_k)\colon F^{b_1b_2}\to F^{b_1-1,b_2+1}$$

and

$$\rho^{(b)}(X_{\bar{k}})\colon F^{b_1b_2}\to F^{b_1+1,b_1-1}.$$

From (5), one sees that $F^{b_1b_2}$ is invariant under $\rho^{(b)}(C)$. Furthermore, for any $k\in K$, $\rho^{(b)}(k)\otimes\Lambda^a\circ Ad(k)$ sends $F^{b_1b_2}\otimes\Lambda^{a_1}(p_-)\otimes\Lambda^{a_2}(p_+)$ to itself. This implies that conditions (i) thru (iii) of Proposition 1 are satisfied by the coordinate function

$$\eta^o=\Sigma\eta_{I\bar{J}}(g)X_{\bar{I}}\wedge X_J;$$

hence the $a_1a_2b_1b_2$-component of a harmonic form is harmonic. In summary:

PROPOSITION 3. *Let* $\omega\in\mathcal{H}^{\{a,b\}}(V)_{\mathbf{C}}$. *Then* $\omega\in\mathcal{H}^{a_1a_2b_1b_2}\Leftrightarrow\omega^o$ *takes values in* $F^{b_1b_2}\otimes\Lambda^{a_1}(p_-)\otimes\Lambda^{a_2}(p_+)$.

The following theorem is now immediate.

THEOREM 1.

(1) $$\mathcal{H}^{\{a,b\}}(V)_{\mathbf{C}}=\sum_{\substack{a_1+a_2=a\\ b_1+b_2=b}}\mathcal{H}^{a_1a_2b_1b_2}$$

$$\mathscr{H}^{(p,q)}(V) = \sum_{\substack{a_1+b_1=p \\ a_2+b_2=p}} \mathscr{H}^{a_1a_2b_1b_2}. \tag{2}$$

3. In this section assume that the representation ρ satisfies the condition

$$\rho(Z) = Z_0. \tag{6}$$

This assumption allows the application of some results of Matsushima and Murakami ([5]) to the study of the spaces $\mathscr{H}^{a_1a_2b_1b_2}$.

Let Σ be the root system of $\mathfrak{G}_C$ with respect to a Cartan subalgebra contained in $\boldsymbol{k}_C$. Choose an ordering of the roots such that the roots belonging to $\boldsymbol{p}_+$ are positive. Let Σ^+ (resp., Σ^-) denote the set of positive (resp. negative) roots. Let ψ denote the set of positive noncompact roots and let θ denote the set of positive compact roots. Let W be the Weyl group of $\mathfrak{G}_C$, and let W_1 be the Weyl group of $\boldsymbol{k}_C$, considered as a subgroup of W. Let R_1 be the involution in W_1 satisfying $R_1\theta = -\theta$. For $T \in W$ put $\phi_T = T(\Sigma^-) \cap \Sigma^+$, and put $\langle\phi_T\rangle = \Sigma_{\alpha\in\phi_T}\alpha$. Let $\delta = \frac{1}{2}\Sigma_{\alpha\in\Sigma^+}\alpha$. Then $\langle\phi_T\rangle = \delta - T\delta$. Let $W' = \{T \in W\colon \phi_T \subset \psi\}$. Let $n(T) = \mathrm{Card}(\phi(T))$. There is a direct sum decomposition

$$\Lambda^{a_1}(\boldsymbol{p}_-) \otimes \Lambda^{a_2}(\boldsymbol{p}_+) = \sum_{\substack{n(S)=a_1 \\ n(T)=a_2}} P^{-\langle\phi_S\rangle} \otimes Q_{\langle\phi_T\rangle}. \tag{7}$$

of $\Lambda^{a_1}(\boldsymbol{p}_-) \otimes \Lambda^{a_2}(\boldsymbol{p}_+)$ into a sum of $\boldsymbol{k}_C$-invariant submodules (with respect to the representation $\mathrm{ad}_-^{a_1} \otimes \mathrm{ad}_+^{a_2}$); the submodule $P^{-\langle\phi_S\rangle}$ (resp. $Q_{\langle\phi_T\rangle}$) is the (unique) irreducible $\boldsymbol{k}_C$-submodule of $\Lambda^{a_1}(\boldsymbol{p}_-)$ (resp. $\Lambda^{a_2}(\boldsymbol{p}_+)$) having highest weight $-\langle\phi_S\rangle$ (resp. lowest weight $\langle\phi_T\rangle$).

Let F now be a complex vector space and $\rho^\Lambda\colon G \to GL(F)$ an irreducible representation of G with highest weight Λ. Then the spaces $H^a(X, \Gamma, \rho^\Lambda)$, and, when a suitable hermitian inner product is given on F, the spaces $\mathscr{H}^a(X, \Gamma, \rho^\Lambda)$ of harmonic F-valued forms on V can be defined. To $\omega \in \mathscr{H}^a(X, \Gamma, \rho^\Lambda)$ one can associate a coordinate vector ω^o as in §1. For each $g \in G$, $\omega^o(g)$ determines a "harmonic representative" of a Lie algebra cohomology class. (See [5] for explanation and details.)

Now consider F to be a $\boldsymbol{k}_C$-module by restricting the differential of the representation ρ^Λ to the subalgebra $\boldsymbol{k}_C$ of $\mathfrak{G}_C$. Let F^μ be irreducible $\boldsymbol{k}_C$-submodule of F with highest weight μ. Let $\mathfrak{h}(\Lambda, \mu, S, T) = \{C \in F^\mu \otimes P^{-\langle\phi_S\rangle} \otimes Q_{\langle\phi_T\rangle}\colon C \text{ is "harmonic"}\}$. Let Λ' denote the lowest weight of the representation ρ^Λ. In [5], the following lemma is proved:

LEMMA 1. $\mathfrak{h}(\Lambda, \mu, S, T) = 0$ *unless* $\mu = S\Lambda = R_1^{-1}T\Lambda'$.

Identify $\mathscr{H}^a(X,\Gamma,\rho^{(b)})_C$ and $\mathscr{H}^a(X,\Gamma,\rho_C^{(b)})$, and fix a direct sum decomposition

(8) $$\Lambda^b(F_C)^* = \Sigma F_\iota(\Lambda)$$

of $\Lambda^b(F_C)^*$ into a direct sum of irreducible $\mathfrak{G}_C$-submodules such that $\rho^{(b)}|_{F_\iota(\Lambda)} \sim \rho^\Lambda$, ρ^Λ being irreducible, with highest weight Λ. By the assumption (6), $\rho^{(b)}(Z) = d\Lambda^b(Z_0)^*$, and so

(9) $$F^{b_1b_2} = \Sigma F_\iota^{b_1b_2}(\Lambda)$$

where $F_\iota^{b_1b_2}(\Lambda) = F_\iota(\Lambda) \cap F^{b_1b_2}$. One can consider $F_\iota(\Lambda)$ as a representation space of K_C. Then one obtains a direct sum decomposition

$$F_\iota(\Lambda) = \Sigma F_{\iota j}(\Lambda, \mu)$$

of $F_\iota(\Lambda)$ into irreducible $\boldsymbol{k}_C$-submodules $F_{\iota j}(\Lambda, \mu)$, μ being the highest weight of the representation on $F_{\iota j}(\Lambda,\mu)$. Now define, for $S, T \in W'$ and fixed Λ and μ as above

(10) $$\mathscr{H}^{\{a,b\}}(\Lambda,\mu,S,T,) = \{\omega \in \mathscr{H}^{\{a,b\}}(V)_C\colon \omega^o \text{ takes values in } \Sigma F_{\iota j}(\Lambda,\mu)\otimes P^{-\langle\phi_S\rangle}\otimes Q_{\langle\phi_T\rangle}\}.$$

Since $F_{\iota j}(\Lambda,\mu)\otimes P^{-\langle\phi_S\rangle}\otimes Q_{\langle\phi_T\rangle}$ is $\rho^{(b)}(C)^-$ and $\rho^{(b)}(k)\otimes\Lambda^a\circ Ad(k)$-invariant,

(11) $$\mathscr{H}^{\{a,b\}}(V)_C = \sum_{n(S)+n(T)=a} \mathscr{H}^{\{a,b\}}(\Lambda,\mu,S,T).$$

The subspace

(12) $$V_\iota(\Lambda) = \{\omega \in \mathscr{H}^{\{a,b\}}(V)_C\colon \omega^o \text{ takes values in } F_\iota(\Lambda)\otimes\Lambda^a(\boldsymbol{p}_C)^*\}$$

is isomorphic to $\mathscr{H}^a(X,\Gamma,\rho^\Lambda)_C$. If $\omega \in V_\iota(\Lambda) \cap |\mathscr{H}^{\{a,b\}}(\Lambda,\mu,S,T)$, then, letting $F = F_\iota(\Lambda)$, $\omega^o(g) \in \mathfrak{h}(\Lambda,\mu,S,T)$, and so Lemma 1 implies that this intersection is $\{0\}$ unless $\mu = S\Lambda = R_1^{-1}T\Lambda'$, where Λ' is the lowest weight of ρ^Λ. Let $d(\Lambda,\mu,S,T) = \dim V_\iota(\Lambda)\cap\mathscr{H}^{\{a,b\}}(\Lambda,\mu,S,T)$. (This dimension is independent of i.)

PROPOSITION 4. *Let $a(\Lambda, b)$ denote the multiplicity of the irreducible representation ρ^Λ in the representation $\rho_C^{(b)}$, and let $S, T \in W'$. Then*

(1) $\mathscr{H}^{\{a,b\}}(\Lambda,\mu,S,T) = 0$ *unless* $n(S)+n(T) = a$ *and* $\mu = S\Lambda = R_1^{-1}T\Lambda'$

(2) $\dim \mathcal{H}^{\{a,b\}}(\Lambda,\mu,S,T) = a(\Lambda,b)d(\Lambda,\mu,S,T)$

(3) *If* $\mathcal{H}^{\{a,b\}}(\Lambda,\mu,S,T) \neq 0$, *then* $\mathcal{H}^{\{a,b\}}(\Lambda,\mu,S,T) \subset H^{a_1a_2b_1b_2}$ *where* $a_1 = n(S)$, $a_2 = n(T)$ *and* b_1 *and* b_2 *are determined by the conditions*: $b_1 + b_2 = b$ *and* $(b_2 - b_1)i/2 = \mu(Z)$.

(4) $\mathcal{H}^{a_1a_2b_1b_2} = \Sigma \mathcal{H}^{\{a,b\}}(\Lambda,\mu,S,T)$, *the summation being taken over* $S, T \in W'$ *such that* $n(S) = a_1$, $n(T) = a_2$ *and* μ *satisfying the conditions of* (1) *and* (3).

Proof. (1) and (2) follow form (10), (12) and the remarks preceding the proposition, and (4) follows from (3), (11), and Theorem 1. Now, (since Z belongs to the center of $k_{\mathbf{C}}$) $F_{ij}(\Lambda,\mu)$ belongs to the $\mu(Z)$-eigenspace of $\rho^{(b)}(Z)$. By assumption, $\rho^{(b)}(Z) = d\Lambda^b(Z_0)^*$ so $F_{ij}(\Lambda,\mu) \subset F^{b_1b_2}$, where $b_1 + b_2 = b$ and $(b_2 - b_1)i/2 = \mu(Z)$. Use (7) and Proposition 3 to complete the proof of 3.

4. Now consider the special case where V is a fiber variety associated to $G = SL(2,\mathbf{R})^N$ (= the direct product of N copies of $SL(2,\mathbf{R})$), $K = SO(2,\mathbf{R})^N$, $\rho: G \to Sp(B)$ a representation equivalent to h copies of the identity representation, and Γ an irreducible subgroup of G satisfying the conditions given before. The quotient $X = G/K$ may be identified with N copies of the upper half plane.

Denote by Z^k the complex coordinate on the k-th factor, and let $z = (z^1,\cdots,z^N)$. The irreducible representations of G are of the form $\rho_{(m)} = \rho_{m_1} \otimes \cdots \otimes \rho_{m_N}$ where ρ_{m_k} is the symmetric tensor representation of $SL(2,\mathbf{R})$ of degree $m_k + 1$. Denote the representation space of $\rho_{(m)}$ by $F_{(m)}$, and the highest weight of $\rho_{(m)}$ by $\Lambda_{(m)}$. In **[6]**, the subspaces $\mathfrak{h}(k_1,\cdots,k_p;l_1,\cdots,l_q;\rho_{(m)})$ of $\mathcal{H}^{p+q}(X,\Gamma,\rho_{(m)})$ are studied. $\mathfrak{h}(k_1,\cdots,k_p;l_1,\cdots,l_q;\rho_{(m)})$ contains all the harmonic $F_{(m)}$-valued forms

$$\omega_z = f(z)dz^{k_1} \wedge \cdots \wedge dz^{k_p} \wedge d\bar{z}^{l_1} \wedge \cdots d\bar{z}^{l_q}$$

which satisfy the condition: $(\omega \circ \gamma)_z = \rho_{(m)}(\gamma)\omega_z$ for all $\gamma \in \Gamma$. It is proved in **[6]** that $\dim \mathfrak{h}(k_1,\cdots,k_p;l_1,\cdots,l_q;\rho_{(m)}) = 0$ unless $(m) = (m_1,\cdots,m_N) = (0,\cdots,0), p = q$ and $\{k_1,\cdots,k_p\} = \{l_1,\cdots,l_q\}$; or unless $\{k_1,\cdots,k_p, l_1,\cdots,l_q\} = \{1,\cdots,N\}$. In the latter cases, $\dim \mathfrak{h}(k_1,\cdots,k_p;l_1,\cdots,l_q;\rho_{(m)}) = \dim \gamma_{(m)+2}(\Gamma)$, where $\gamma_{(m)+2}(\Gamma)$ is the space of holomorphic Γ-automorphic forms $f(z)$ transforming by

$$f(\gamma z) = \pi_{k=1}^{N}(c_k z^k + d_k)^{m_k+2} f(z),$$

if $\gamma = (\gamma_1,\cdots,\gamma_N) \in \Gamma$ and $\gamma_k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$, $k = 1,\cdots,N$.

The coordinate vector ω^o of $\omega \in \mathfrak{h}(k_1,\cdots,k_p;l_1,\cdots,l_q;\rho_{(m)})$ takes values in $F_{(m)} \otimes p_-^{k_1} \wedge \cdots \wedge p_-^{k_p} \otimes p_+^{l_1} \wedge \cdots \wedge p_+^{l_q}$; note that $p_-^{k_1} \wedge \cdots \wedge p_-^{k_p} = p^{-\langle \phi_S \rangle}$ for the $S \in W'$ acting nontrivially on the $k_1, \cdots, k_p$th factors and trivially on the other factors. Similarly, $p_+^{l_1} \wedge \cdots \wedge p_+^{l_q}$ is a $Q_{\langle \phi_T \rangle}$, $T \in W'$. A G-module isomorphism $F_{(m)} \to F_\iota(\Lambda_{(m)}) \subset \Lambda^b(F_{\mathbf{C}})^*$ (notations as in §3) induces isomorphisms $\mathcal{H}^a(X, \Gamma, \rho_{(m)}) \cong V_\iota(\Lambda_{(m)})$ taking $\mathfrak{h}(k_1, \cdots, k_p; l_1, \cdots, l_{a-p}; \rho_{(m)})$ onto $V_\iota(\Lambda_{(m)}) \cap \mathcal{H}^{\{a,b\}}(\Lambda_{(m)}, \mu, S, T)$.

In this case, μ, S, and T are related by: $\mu = S\Lambda_{(m)} = -T\Lambda_{(m)}$. (When $\mathfrak{h}(k_1, \cdots, k_p; l_1, \cdots, l_q; \rho_{(m)}) \neq 0$, the condition $S\Lambda_{(m)} = R_1^{-1} T \Lambda'_{(m)}$ is automatically satisfied.) Since $\rho \sim id \otimes \cdots \otimes id$ and since ρ induces a holomorphic mapping on X, the condition (6) of §3 is satisfied. In view of Proposition 4 of §3 and the preceding discussion, one has

PROPOSITION 5. *$\mathcal{H}^{a_1 a_2 b_1 b_2} = 0$ unless $a_1 + a_2 = N$ or $a_1 = a_2$ and $b_1 = b_2$. Furthermore, from Theorem 1 and Proposition 5 we obtain*

PROPOSITION 6. $\mathcal{H}^{(p,p)}(V) = \Sigma_{a+b=p} \mathcal{H}^{\{2a,2b\}}(V)_{\mathbf{C}} + \Sigma \mathcal{H}^{a,N-a,p-a,p-N+a}$ *where the second summation is taken over integers a with $\min\{a, N-a, p-a, p-N+a\} \geqq 0$. In particular, if $p \leqq N/2$,*

$$\mathcal{H}^{(p,p)}(V) = \sum_{a+b=p} \mathcal{H}^{\{2a,2b\}}(V)_{\mathbf{C}}.$$

In Case $N = 1$, one can prove more. The irreducible representations occuring in $\rho^{(b)}$ are the ρ_m with $m \equiv b \bmod(2)$ and $0 \leqq m \leqq b$. Also, $W = W' \cong \{\pm 1\}$. By Proposition 4, $\mathcal{H}^{0,1,p,p-1}$ is a sum of subspaces $\mathcal{H}^{\{1,2p-1\}}(\Lambda_m, \mu, S, T)$ and each such nonzero subspace must satisfy the conditions $n(S) = 0$, $n(T) = 1$, $\mu = s\Lambda_m = -T\Lambda_m$, and $\mu(Z) = [(p-1) - p]i/2 = -i/2$. But $S = 1$, $T = -1$ and so $\mu(Z) = \Lambda_m(Z) = mi/2$, with $m \geqq 0$. Thus $\mathcal{H}^{0,1,p,p-1} = 0$. Similarly, $\mathcal{H}^{1,0,p-1,p} = 0$. This gives

THEOREM 2. *In Case $N = 1$, $\mathcal{H}^{(p,p)}(V) = \mathcal{H}^{\{0,2p\}}(V)_{\mathbf{C}} + \mathcal{H}^{\{2,2(p-1)\}}(V)_{\mathbf{C}}$.*

Application 1. Since the subspaces $\mathcal{H}^{\{a,b\}}(V)$ are defined over $\mathbf{Q}$, (see **[1]**, **[8]**) $\mathcal{H}^{(p,p)}(V)$ is defined over $\mathbf{Q}$ when $p \leqq N/2$.

Application 2. The Hodge conjecture for V a fiber variety over an algebraic curve attached to an indefinite quaternion algebra over $\mathbf{Q}$. — see **[2]**.

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