

Pacific Journal of Mathematics

REGULAR SEQUENCES AND LIFTING PROPERTY

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Let A be a commutative noetherian ring, E a finite A -module and let M be an arbitrary A -module. Let $\varphi: E \rightarrow M$ be a homomorphism of A -modules.

In this note we prove in an elementary way that an M -sequence $\underline{x} = (x_1, \dots, x_n)$ being taken to lie in the (Jacobson-) radical $\text{rad}(A)$ of A , is also an E -sequence if $\underline{x}E$ is the contraction $\varphi^{-1}(\underline{x}M)$ of $\underline{x}M$ in E .

As a corollary of this lifting property we obtain very easily the so-called delocalization-lemma for regular sequences (also [2], Cor. 1 for local rings A and [4] Chap. I, §4). Then we exemplify that the condition $\varphi^{-1}(\underline{x}M) = \underline{x}E$ is not necessary for the statement of our theorem (see Example 3); otherwise it is easily seen that generally the theorem (especially Corollary 2) becomes false without any additional condition (see Examples 1 and 2).

Recall that a sequence x_1, \dots, x_n of elements of A is said to be (M -regular or) an M -sequence if, for each $0 \leq i \leq n-1$, a_{i+1} is a non-zero-divisor on $M/(x_1, \dots, x_i)M$ and $M \neq (x_1, \dots, x_n)M$.

2. First we consider the case $n = 1$.

LEMMA. *The notations being as above. Let x be a M -regular element in the radical $\text{rad}(A)$ of A and suppose that*

$$(1) \quad \ker \varphi \subseteq xE^1.$$

Then x is an E -regular element too and φ is injective.

Proof. We put $F = \ker \varphi$. Clearly x is E/F -regular, hence $xE \cap F = xF$, hence $F = xF$ by (1). Therefore we get $F = 0$ by Nakayama's lemma, hence φ is injective and x is E -regular.

THEOREM. *Let E be a finite A -module, M an arbitrary A -module and $\varphi: E \rightarrow M$ a module-homomorphism. Let $\underline{x} = (x_1, \dots, x_n)$ be an M -sequence in $\text{rad}(A)$ and suppose that*

¹ We denote by xE or $\underline{x}E$ the product $(x)E$ or $(\underline{x})E$ respectively, where (x) or (\underline{x}) is the ideal generated by x or x_1, \dots, x_n respectively.

$$(2) \quad \varphi^{-1}(\underline{x}M) = \underline{x}E^1.$$

Then \underline{x} is an E -sequence and φ is injective.

Proof. (By induction on n): Note that $\ker \varphi \subseteq \varphi^{-1}(\underline{x}M)$, hence the case $n = 1$ results from the lemma.

For $n > 1$ we put $\underline{x}' = (x_1, \dots, x_{n-1})$, $E' = E/\underline{x}'E$, $M' = M/\underline{x}'M$ and $\varphi' = \varphi \otimes 1_{A/\underline{x}'A} : E' \rightarrow M'$. Then we have

$$\ker \varphi' = \varphi^{-1}(\underline{x}'M)/\underline{x}'E \cap \varphi^{-1}(\underline{x}'M) = \varphi^{-1}(\underline{x}'M)/\underline{x}'E \subseteq \varphi^{-1}(\underline{x}M)/\underline{x}'E,$$

hence we get by (2):

$$\ker \varphi' \subseteq \underline{x}E/\underline{x}'E = x_n \cdot E'.$$

Since x_n is M' -regular we are in the situation of the lemma with x_n and $\varphi' : E' \rightarrow M'$ instead of x and φ . Therefore φ' is injective, i.e.

$$(3) \quad \varphi^{-1}(\underline{x}'M) = \underline{x}'E,$$

and x_n is an E' -regular element. The sequence \underline{x}' is M -regular by assumption, hence by (3) and by induction on n , \underline{x}' is E -regular and φ is injective. This concludes the proof.

COROLLARY 1.1 (*Delocalization Lemma, 1. form*): Let A be a noetherian ring, E a finite A -module and $x_1, \dots, x_n \in \text{rad}(A)$. Let $U \subset A$ be the set of nonzerodivisors modulo \underline{x} for E . Suppose that \underline{x} is an E_U -sequence. Then \underline{x} is an E -sequence.

Proof. Let $\varphi : E \rightarrow E_U$ be the natural homomorphism. No element of U is zerodivisor for $E/\underline{x}E$, hence $\varphi^{-1}(\underline{x}E_U) = \underline{x}E$, proving the corollary.

It results from the following Corollary 1.2 that the conditions for \underline{x} in the two Corollaries 1.1 and 1.2 are equivalent.

COROLLARY 1.2 (*Delocalization Lemma, 2. form*): Let E be a finitely generated module over a noetherian ring A and $x_1, \dots, x_n \in \text{rad}(A)$. If \underline{x} is an E_η -sequence for all $\eta \in \text{Ass}(E/\underline{x}E)$, then \underline{x} is an E -sequence.

Proof. Let $M = \bigoplus E_\eta$ for $\eta \in \text{Ass}(E/\underline{x}E)$, and φ the homomorphism $E \rightarrow M$ defined by $u \rightarrow \sum_\eta \varphi_\eta(u)$, where φ_η denotes the natural map $E \rightarrow E_\eta$. [Note that $\text{Ass}(E/\underline{x}E)$ is a finite set.]

Since \underline{x} is an E_η -sequence for all $\eta \in \text{Ass}(E/\underline{x}E)$, it must be an M -sequence too. We want to apply our theorem to finish the proof. For that we show that $\varphi^{-1}(\underline{x}M) = \underline{x}E$:

Since E is finitely generated, the submodule $\underline{x}E$ has an irredundant primary decomposition $\underline{x}E = Q_1 \cap \dots \cap Q_r$ corresponding to the ideals $\eta_i \in \text{Ass}(E/\underline{x}E)$. Localizing $\underline{x}E$ by any ideal $\eta \in \text{Ass}(E/\underline{x}E)$ we obtain [5]:

$$\varphi_\eta^{-1}((Q_i)_\eta) = \begin{cases} Q_i & \text{if } \eta_i \subseteq \eta \\ E & \text{if } \eta_i \not\subseteq \eta, \end{cases}$$

hence $\bigcap_\eta \varphi_\eta^{-1}(\underline{x}E_\eta) = \underline{x}E$.

On the other hand we have $\underline{x}M = \bigoplus_\eta \underline{x}E_\eta$, hence $\varphi^{-1}(\underline{x}M) = \bigcap_\eta \varphi_\eta^{-1}(\underline{x}E_\eta)$. This concludes the proof of the corollary.

3. Now let $f: A \rightarrow B$ be a ring-homomorphism. If \mathfrak{a} and \mathfrak{b} are ideals in A and B respectively we define as usual \mathfrak{a}^e to be the extension $f(\mathfrak{a})B$ of \mathfrak{a} and \mathfrak{b}^c to be the contraction $f^{-1}(\mathfrak{b})$ of \mathfrak{b} .

COROLLARY 2. *Let $f: A \rightarrow B$ be a homomorphism of noetherian rings. Let \mathfrak{a} be an ideal generated by elements $x_1, \dots, x_n \in \text{rad}(A)$. Suppose that $f(x_1), \dots, f(x_n)$ form a B -sequence and suppose that $\mathfrak{a}^e = \mathfrak{a}$. Then x_1, \dots, x_n is an A -sequence.*

Proof. Regard B as an A -module relatively to f . We consider the module-homomorphism $\varphi: A \rightarrow B$ given by $\varphi(a) = f(a)$ for all $a \in A$. Then, by assumption all conditions of the theorem are fulfilled, proving the corollary.

REMARKS. (i) The proof of Corollary 2 shows that the delocalization Lemma 1.1 or 1.2 respectively can be formulated for rings too.

(ii) If $f: A \rightarrow B$ is faithfully flat then any sequence $x_1, \dots, x_n \in \text{rad}(A)$ is A -regular $\Leftrightarrow f(x_1), \dots, f(x_n)$ is B -regular. This well-known statement (s. [4] or [3]) is an easy consequence of Corollary 2: f is faithfully flat says that f is flat and the induced map ${}^a f: \text{Spec } B \rightarrow \text{Spec } A$ is surjective. But if f is flat then the last condition is equivalent to $\mathfrak{a}^e = \mathfrak{a}$ (s. [1], p. 45), where \mathfrak{a} is generated by x_1, \dots, x_n . Hence Corollary 2 works for \Leftarrow ; the other direction is trivial.

(iii) Let B be a surjectively-free A -algebra [i.e. $A = \sum_\psi \psi(B)$, where ψ runs over $\text{Hom}_A(B, A)$]. Then for any ideal \mathfrak{a} of A one has

$$\mathfrak{a}^e = \mathfrak{a}B \cap A = \mathfrak{a},$$

and the induced map $\text{Spec } B \rightarrow \text{Spec } A$ is surjective; see [5], (5, E), p. 37.

4. We are indebted to L. Badescu for pointing to the following

EXAMPLE 1. Consider the ring

$$R = \{f \in k[x, y] \mid f(1, 0) = f(-1, 0)\} \subset k[x, y],$$

where k denotes say the field of complex numbers. Then R is the finitely generated subring $k[x - x^3, x^2, xy, y]$ of $k[x, y]$; clearly $k[x, y]$ is integrally dependent on R and with the same quotient field. Therefore $Y := \text{Spec } R$ is not normal. Write X for the normal affine variety $\text{Spec}(k[x, y]) \cong k^2$. Let the inclusion of R in $k[x, y]$ define the proper morphism

$$\pi: X \rightarrow Y.$$

Then if x_1, x_2 are the points $(1, 0)$ and $(-1, 0) \in k^2$, we have $\pi(x_1) = \pi(x_2) =: y_0$, and

$$\text{res } \pi: X - \pi^{-1}\{y_0\} \rightarrow Y - \{y_0\}$$

is an isomorphism. In particular, Y is normal at all points except y_0 [this is also clear by the connectedness theorem, because $\pi^{-1}\{y_0\}$ is not connected]. To be more in detail, take

$$v_1 = 1 - x^2, v_2 = xy, v_3 = y, v_4 = x - x^3.$$

Then

$R =$

$$k[v_1, v_2, v_3, v_4]/(v_4v_3 - v_2v_1; v_2^2 - v_3^2 + v_1v_3^2; v_4^2 + v_3^3 - v_1^2; v_1v_2 - v_2v_4 - v_1^2v_3).$$

So Y can be regarded as an affine surface in k^4 , which is nonsingular in codimension 1, but not normal in the origin (corresponds to the point $y_0 \in \text{Spec } R = Y$). Therefore $0_{Y, y_0}$ is not a Cohen Macaulay-ring by the criterion of normality, [3], 5.8.6.

We fix the notations:

$$A = 0_{Y, y_0} = R_{\mathfrak{y}} \quad \text{with } \mathfrak{y} = (x - 1, y) \cap R = (x + 1, y) \cap R;$$

$$B = 0_{X, x_1} = k[x, y]_{\mathfrak{y}} \quad \text{with } \mathfrak{y} = (x - 1; y);$$

$f: A \rightarrow B$ the corresponding local homomorphism; $a_1 = (x - 1)(x + 1)$, $a_2 = y$ in A and $f(a_1) = (x - 1)(x + 1)$, $f(a_2) = y$ regarded as being in B . Then B is a regular local ring, and

$f(a_1), f(a_2)$ generate its maximal ideal \mathfrak{m}_B . Since $\text{depth } A \cong \dim A = 2$ the sequence a_1, a_2 will not be A -regular. And indeed we have $\mathfrak{a}^e = \mathfrak{m}_B$ and $\mathfrak{a}^{ec} \mathfrak{m}_A \neq \mathfrak{a}$.

EXAMPLE 2. Let (A, \mathfrak{m}) be a one-dimensional local noetherian ring which is not a Cohen Macaulay-ring. Let $x \in \mathfrak{m}$ be a parameter of A and η a minimal prime over-ideal of zero in A . Take $B = A/\eta$. Then by assumptions $f(x)$ is B -regular, but x is not A -regular. And we have $\mathfrak{a}^{ec} \neq \mathfrak{a}$ (otherwise η would be zero).

EXAMPLE 3. Let (A, \mathfrak{m}) be a local Cohen Macaulay-ring of dimension 1 which is not regular. Then the maximal ideal \mathfrak{m} can be generated in this way:

$$\mathfrak{m} = xA + m_1A + \cdots + m_rA,$$

where x denotes an A -regular element.

Let \mathfrak{a} be the ideal generated by x and $B = A[m_i/x] \subset A_x$. Since x is A -regular the natural homomorphism $f: A \rightarrow B$ is injective, and clearly x is B -regular.

But now we have $\mathfrak{a}^e = xB = \mathfrak{m}B$, hence $\mathfrak{a}^{ec} = \mathfrak{m}B \cap A = \mathfrak{m}$, hence $\mathfrak{a}^{ec} \neq \mathfrak{a}$ because A is not regular.

This example shows that condition (2) is not necessary for the statement of the theorem.

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Received April 20, 1977.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

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