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FOURIER TRANSFORMS AND THEIR LIPSCHITZ CLASSES

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We define a class of functions A_α for each $\alpha > 0$. We show that the Fourier transform of every function of A_α exists and is Lipschitz of order α . We construct examples to show that the converse is not true in general. However, we show that for a certain class of function k (e.g., $k \in L_2$) if its Fourier transform \hat{k} is Lipschitz of order α then $k \in A_\beta$ for all $\beta < \alpha$.

Boas ([1] and [2]) studied this problem in the case where the function k is nonnegative and gave a complete solution in this case. In connection with this question several authors (e.g. Hirschman [5]; Liang Shin Hahn [4], Drobot, Naparstek and Sampson [3]) have proved mapping properties of convolution operators with kernel k , by studying the behavior of \hat{k} . To be more precise, they have proved mapping theorems when $\hat{k} \in \text{Lip}(\alpha)$ with additional conditions on k . In our applications we prove a similar result (see §3, Theorem 4).

Notations and Definitions.

L_{loc} shall denote the set of all Lebesgue measurable functions integrable over all finite intervals. In this paper, the functions $f, g, k, \dots \in L_{\text{loc}}$.

For $0 < \alpha \leq 1$, a function f is Lipschitz of order α ($f \in \text{Lip}(\alpha)$) if there is a positive constant A such that

$$\sup_{x \in \mathbb{R}} |f(x+h) - f(x)| \leq A|h|^\alpha.$$

For $\alpha > 1$, we say that a function f is Lipschitz of order α if

- (i) $f^{(m)} \in L_\infty$ for all $m < [\alpha]$ and
- (ii) $f^{([\alpha])} \in \text{Lip}(\alpha - [\alpha])$.

When we use the symbol

$$\int_a^b g(t, x) dt \quad \text{for} \quad -\infty \leq a < b \leq \infty.$$

We are assuming that $g(t, x) \in L_{\text{loc}}$ as a function of t for each x and moreover the integral exists in the following sense:

$$(0) \quad \int_a^b g(t, x) dt = \lim_{\substack{\alpha \rightarrow a \\ \beta \rightarrow b}} \int_\alpha^\beta g(t, x) dt.$$

We write $h(x_1, x_2, \dots, x_n, y) = O(y^a)$ to mean that there exists a positive number C independent of x_1, x_2, \dots, x_n, y so that

$$\sup_{\substack{x_i \in \mathbb{R} \\ 1 \leq i \leq n}} |h(x_1, x_2, \dots, x_n, y)| \leq C |y|^a.$$

In particular, we say $h(x_1, x_2, \dots, x_n) = O(1)$ to mean that there exists a C independent of x_1, x_2, \dots, x_n so that

$$\sup_{\substack{x_i \in \mathbb{R} \\ 1 \leq i \leq n}} |h(x_1, x_2, \dots, x_n)| \leq C.$$

We number each section independently.

1. Sufficient conditions.

LEMMA 1. *Let a and b be numbers so that $0 < a < b$. Then for each $y > 0$*

$$(1) \quad \int_0^y t^b f(t, x) dt = O(y^a) \Leftrightarrow \int_y^\infty f(t, x) dt = O(y^{a-b})$$

$$(2) \quad \int_{-y}^0 |t|^b f(t, x) dt = O(y^a) \Leftrightarrow \int_{-\infty}^{-y} f(t, x) dt = O(y^{a-b}).$$

Proof. A similar lemma can be found in Boas's paper [1]. Thus we will be brief. We will prove (1); the argument for (2) is similar.

\Rightarrow : Let $F(x, y) = \int_0^y t^b f(t, x) dt$, then we get

$$\int_y^\infty f(t, x) dt = t^{-b} F(x, t) \Big|_y^\infty + b \int_y^\infty t^{-b-1} F(x, t) dt.$$

Since $F(x, y) = O(y^a)$ and also $a < b$ then we are through.

\Leftarrow : Let $F(x, y) = \int_y^\infty f(t, x) dt$, then we get

$$\int_0^y t^b f(t, x) dt = -t^b F(x, t) \Big|_0^y + b \int_0^y t^{b-1} F(x, t) dt.$$

Since $F(x, y) = O(y^{a-b})$, then we are through.

LEMMA 2. *Let $h > 0$.*

$$(3) \quad \text{If } \int_{1/h}^{\infty} f(t, x) dt = O(h^\alpha), \text{ then}$$

$$(4) \quad \int_0^{1/h} f(t, x) \sin th dt = O(h^\alpha) \quad \text{for } 0 < \alpha < 1$$

$$(5) \quad \int_0^{1/h} f(t, x) (1 - \cos th) dt = O(h^\alpha) \quad \text{for } 0 < \alpha < 2.$$

We get a similar result for $\int_{-\infty}^{-1/h} f(t, x) dt$.

Proof. From Lemma 1 we get,

$$(6) \quad \int_{1/h}^{\infty} f(t, x) = O(h^\alpha) \Leftrightarrow \int_0^{1/h} tf(t, x) dt = O(h^{\alpha-1}) \quad \text{for } 0 < \alpha < 1.$$

$$(7) \quad \Leftrightarrow \int_0^{1/h} t^2 f(t, x) dt = O(h^{\alpha-2}) \quad \text{for } 0 < \alpha < 2.$$

To see this, it suffices to take $y = 1/h$, $a = b - \alpha$ where $b = 1$ for (6) and $b = 2$ for (7).

The function $\varphi(t) = (th)^{-1} \sin th$ is decreasing and nonnegative for $t \in (0, 1/h)$. By the second-mean-value theorem for integrals we get,

$$\int_0^{1/h} tf(t, x) \varphi(t) dt = \int_0^\xi tf(t, x) dt \quad \text{for some } \xi \in (0, 1/h).$$

Hence by hypothesis and (6) we conclude

$$\int_0^{1/h} tf(t, x) \varphi(t) dt = O(\xi^{1-\alpha}) = O(h^{\alpha-1}).$$

Consequently, we have

$$\int_0^{1/h} f(t, x) \sin th dt = O(h^\alpha).$$

The proof for (5) is similar with $\varphi(t) = (th)^{-2}(1 - \cos th)$.

DEFINITION 3. Let α be a positive number. We say that $k \in A_\alpha$ if $k \in L_{\text{loc}}$ and satisfies the following two conditions:

$$(8) \quad \int_{1/h}^{\infty} f(t)e^{-itx} dt = O(h^\alpha) \quad \text{and}$$

$$\int_{-\infty}^{-1/h} k(t)e^{-itx} dt = O(h^\alpha).$$

LEMMA 4. If $k \in A_\alpha$ then the Fourier transform $\hat{k}(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} k(t)e^{-itx} dt$ exists for all x and $\hat{k} \in L_\infty$.

Proof. Since $k \in L_{loc}$ by (8) we get that \hat{k} exists for each x . It also follows that $\hat{k} \in L_\infty$.

LEMMA 5. Let $0 < \alpha < 1$. If $k \in A_\alpha$ then $\hat{k} \in \text{Lip } \alpha \cap L_\infty$.

Proof. By Lemma 4, \hat{k} exists for each x and $\hat{k} \in L_\infty$.

Now we show that $\hat{k} \in \text{Lip } \alpha$. Let $h > 0$,

$$\begin{aligned} \hat{k}(x+h) - \hat{k}(x) &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} k(t)e^{-itx} (e^{-ith} - 1) dt \\ &= (2\pi)^{-1/2} \left\{ \int_{-\infty}^{-1/h} + \int_{-1/h}^{1/h} + \int_{1/h}^{\infty} \right\} \\ &\quad \int_{-1/h}^0 + \int_0^{1/h} = O(h^\alpha) \quad \text{by Lemma 2.} \end{aligned}$$

By hypothesis we get,

$$\int_{-\infty}^{-1/h} + \int_{1/h}^{\infty} = O(h^\alpha).$$

Therefore, $\hat{k}(x+h) - \hat{k}(x) = O(h^\alpha)$.

We are going to show that the above lemma can be extended to $\alpha > 1$ and $\alpha \notin \mathbb{N}^+$ (\mathbb{N}^+ : set of positive integers). For $\alpha \in \mathbb{N}^+$ we will give another sufficient condition. We are able also to give a sufficient condition on k so that \hat{k} is differentiable.

LEMMA 6. If k satisfies

$$(9) \quad \left| \int_0^{\infty} tk(t)e^{-itx} dt \right| < \infty \quad \text{for each } x, \text{ then}$$

$$\lim_{h \rightarrow 0^+} \int_0^{1/h} k(t)e^{-itx} \frac{e^{-ith} - 1}{h} dt = -i \int_0^{\infty} tk(t)e^{-itx} dt \quad \text{for each } x.$$

For the negative side we get a similar result.

Proof. It suffices to show

$$(10) \quad \lim_{h \rightarrow 0^+} \int_0^{1/h} tk(t)e^{-itx} \frac{\sin th}{th} dt = \int_0^\infty tk(t)e^{-itx} dt$$

and

$$(11) \quad \lim_{h \rightarrow 0^+} \int_0^{1/h} tk(t)e^{-itx} \frac{1 - \cos th}{th} dt = 0.$$

Let $\epsilon > 0$ be given and let x be such that (9) is true. Here, we keep x fixed throughout the entire argument.

From (9), we conclude that there exists an $N > 0$ and $0 < h_0 < 1/N$ such that for all h satisfying $0 < h < h_0$

$$(12) \quad \left| \int_N^{1/h} tk(t)e^{-itx} dt \right| < \epsilon.$$

Since the function $(1 - \cos th)(th)^{-1}$ is monotonic and nonnegative in $(N, 1/h)$ there exists a $\xi \in (N, 1/h)$ so that

$$(13) \quad \left| \int_N^{1/h} tk(t)e^{-itx} \frac{\cos th - 1}{th} dt \right| \leq \left| \int_\xi^{1/h} tk(t)e^{-itx} dt \right|.$$

It follows from (12) and (13) that

$$(14) \quad \left| \int_N^{1/h} tk(t)e^{-itx} \frac{\cos th - 1}{th} dt \right| < \epsilon.$$

On the other hand, since $tk(t) \in L_{\text{loc}}$, by the Lebesgue dominated convergence theorem (for N fixed),

$$\lim_{h \rightarrow 0^+} \int_0^N tk(t)e^{-itx} \frac{\cos th - 1}{th} dt = 0.$$

Thus we get (11).

Now to show (10).

From (9), there exists N_0 such that for all N

$$(15) \quad N > N_0 \Rightarrow \left| \int_0^\infty tk(t)e^{-itx} dt - \int_0^N tk(t)e^{-itx} dt \right| < \epsilon.$$

Since $(\sin th)(th)^{-1}$ is nonnegative and monotonic in $(N, 1/h)$, then by a similar argument, there exists a fixed $N > N_0$, h_0 and $h_1 < (1/N)$ such that

$$(16) \quad \left| \int_N^{1/h} tk(t)e^{-tx} \frac{\sin th}{th} dt \right| < \epsilon \quad \text{for all } 0 < h < h_0$$

$$(17) \quad \left| \int_0^N tk(t)e^{-tx} \frac{\sin th}{th} dt - \int_0^N tk(t)e^{-tx} dt \right| < \epsilon$$

for all h satisfying $0 < h < h_1$.

From (15), (16) and (17) we conclude that

$$\left| \int_0^\infty tk(t)e^{-tx} dt - \int_0^{1/h} tk(t)e^{-tx} \frac{\sin th}{th} dt \right| < 3\epsilon$$

for all h satisfying $0 < h < \min(h_0, h_1)$.

Therefore we get (10) and we are through.

THEOREM 7. *Let $\alpha > 1$, $m < \alpha$ and $m \in \mathbf{N}^+$. If $k \in A_\alpha$ then \hat{k} is m times differentiable at each x and $\hat{k}^{(m)} \in L_\infty$. In fact*

$$\hat{k}^{(m)}(x) = (2\pi)^{-1/2}(-i)^m \int_{-\infty}^\infty t^m k(t)e^{-tx} dt.$$

Proof. By Lemma 1,

$$(18) \quad \int_{1/h}^\infty k(t)e^{-tx} dt = O(h^\alpha) \Rightarrow \int_0^{1/h} t^{\alpha+1} k(t)e^{-tx} dt = O(h^{-1}).$$

Hence by Lemma 1 again, for all $m < \alpha$

$$(19) \quad \int_{1/h}^\infty t^m k(t)e^{-tx} dt = O(h^{\alpha-m}).$$

Thus $t^m k(t) \in A_{\alpha-m}$.

It follows from Lemma 4 that

$$(20) \quad f_m(x) = (2\pi)^{-1/2}(-i)^m \int_{-\infty}^\infty t^m k(t)e^{-tx} dt$$

exists for each x and $f_m \in L_\infty$.

To prove the theorem, we first show that $\hat{k}'(x)$ exists for each x . Since $k \in A_\alpha$, by Lemma 4 \hat{k} exists and we have,

$$\frac{\hat{k}(x+h) - \hat{k}(x)}{h} = (2\pi)^{-1/2} h^{-1} \left\{ \int_{-\infty}^{-1/h} k(t) e^{-ix} (e^{-ih} - 1) dt + \int_{-1/h}^0 + \int_0^{1/h} + \int_{1/h}^{\infty} \right\}.$$

Since $k \in A_\alpha$ ($\alpha > 1$) by (8) we have $\lim_{h \rightarrow 0} h^{-1} \left(\int_{1/h}^{\infty} + \int_{-1/h}^{-\infty} \right) = 0$. By (19) and Lemma 6 with $m = 1$, it follows that $\hat{k}'(x) = f_1(x)$.

The theorem is then true for $m = 1$. Now we suppose that the theorem is true up to $m - 1$, i.e. $\hat{k}^{(m-1)}(x) = f_{m-1}(x)$.

$$\frac{\hat{k}^{(m-1)}(x+h) - \hat{k}^{(m-1)}(x)}{h} = (-1)^{m-1} \frac{\hat{g}(x+h) - \hat{g}(x)}{h}$$

where $g(t) = t^{m-1} k(t)$.

Since $\alpha - m + 1 > 1$, the above argument starting with (19) can be applied to $g(t)$ and we get,

$$\lim_{h \rightarrow 0^+} \frac{\hat{g}(x+h) - \hat{g}(x)}{h} = -i(2\pi)^{-1/2} \int_{-\infty}^{\infty} t g(t) e^{-ix} dt.$$

Thus $\hat{k}^{(m)}(x) = f_m(x)$.

THEOREM 8. Let $\alpha > 0$ and $\alpha \notin \mathbf{N}^+$.

If $k \in A_\alpha$ then $\hat{k} \in \text{Lip } \alpha \cap L_\infty$.

Proof. For $0 < \alpha < 1$, this is Lemma 5.

Now look at the case where $\alpha > 1$. By Lemma 4, \hat{k} exists for each x and $\hat{k} \in L_\infty$. Due to Theorem 7 we can conclude that for all $m \leq [\alpha]$, $\hat{k}^{(m)}$ exists and $\hat{k}^{(m)} \in L_\infty$. Moreover

$$(21) \quad \hat{k}^{([\alpha])}(x) = (-i)^{[\alpha]} (2\pi)^{-1/2} \int_{-\infty}^{\infty} t^{[\alpha]} k(t) e^{-ix} dt.$$

From (19) we get

$$\int_{1/h}^{\infty} t^{[\alpha]} k(t) e^{-ix} dt = O(h^{\alpha - [\alpha]}).$$

Hence $t^{[\alpha]}k(t) \in A_{\alpha-[\alpha]}$. It follows from Lemma 5 that

$$\widehat{t^{[\alpha]}k(t)} \in \text{Lip}(\alpha - [\alpha]).$$

Hence by (21) we get our result.

THEOREM 9. *Let $\alpha \in \mathbb{N}^+$. If $k \in L_{\text{loc}}$ and satisfies*

$$(22) \quad \int_0^\infty t^\alpha k(\pm t)e^{\mp itx} dt = O(1)$$

and

$$(23) \quad \int_{1/h}^\infty k(\pm t)e^{\mp itx} dt = o(h^\alpha)$$

then $\hat{k} \in \text{Lip } \alpha \cap L_\infty$.

Proof. First assume $\alpha = 1$. By (23), $k \in A_1$. By Lemma 4, \hat{k} exists for each x and $\hat{k} \in L_\infty$. In (23) we are using the little “ o ” notation. We note

$$(24) \quad \frac{\hat{k}(x+h) - \hat{k}(x)}{h} = (2\pi)^{-1/2} h^{-1} \left\{ \int_{-\infty}^{-1/h} + \int_{-1/h}^{1/h} + \int_{1/h}^\infty \right\}.$$

Hence by Lemma 6 and (23) we get

$$\hat{k}'(x) = -i(2\pi)^{-1/2} \int_{-\infty}^\infty tk(t)e^{-itx} dt.$$

From (22) we conclude that \hat{k} is absolutely continuous. Hence $\hat{k} \in \text{Lip}(1)$.

For the case $\alpha > 1$, we use induction. The argument is similar to that given in Theorem 7 and will be omitted here.

2. Necessary conditions. We know that for each α ($0 \leq \alpha < 1$) there exists a function g such that $\hat{g} \in \text{Lip}(\alpha)$ but $g \notin A_\alpha$. We give this example in §4. However, we have succeeded in showing that $\hat{k} \in \text{Lip}(\alpha)$ implies $k \in A_\beta$ for all $\beta < \alpha$ with some other conditions placed on k . One of the conditions that k must satisfy is the following:

$$(1) \quad \int_{v/2}^v k(w)e^{-uw} dw = i(2\pi)^{-1/2} \int_{-\infty}^\infty \frac{\hat{k}(2x+t) - \hat{k}(x+t)}{x} e^{ux} dx.$$

Condition (1) is merely Parseval's identity, however we have only been able to show (1) holds for a certain class of functions. This result appears in Lemma 3. In the case where $k \in L_p$ ($1 \leq p \leq 2$) and \hat{k} is continuous, we can show (1) holds (the argument is similar to that given in Lemma 3).

Another condition that appears is,

DEFINITION 1. Let $\epsilon \geq 0$. We say that $f \in V_\epsilon$ if there exists some constant A such that

$$(2) \quad \int_{|x| \geq A} \frac{|f(2x+t) - f(x+t)|}{|x|^{1-\epsilon}} dx = O(1).$$

It is obvious that if $f \in L_p$ then $f \in V_\epsilon$ for $\epsilon < 1/p$. Furthermore, all constant functions belong to V_ϵ for all $\epsilon \geq 0$.

THEOREM 2. If some $0 < \epsilon < 1$, $\hat{k} \in \text{Lip } \alpha \cap V_\epsilon$ ($0 < \alpha \leq 1$) and k satisfies (1) then $k \in A_\beta$ for all $0 < \beta < \alpha$.

COROLLARY. Let $k \in L_p$ for some $1 < p \leq 2$. If $\hat{k} \in \text{Lip } (\alpha)$, then $k \in A_\beta$ for all $0 < \beta < \alpha$.

REMARK. In the above corollary \hat{k} is defined as usual in the L_q sense ($1/p + 1/q = 1$). We note that for $0 < n \leq v \leq 2^n$ $\int_n^v k(w)e^{-uw} dw = \int_n^{2^{2n}} + \dots + \int_{c^v}^v$ where $1/2 \leq c \leq 1$. Next we note that a formula similar to (1) holds for the term

$$\int_{c^v}^v k(w)e^{-uw} dw, \quad 1/2 \leq c \leq 1.$$

And now the Corollary follows.

Proof of Theorem 2. Let

$$\varphi(x, t) = \frac{\hat{k}(2x+t) - \hat{k}(x+t)}{x}.$$

From (1) it follows for $v > 1$ (note $k \in L_{\text{loc}}$)

$$(3) \quad \begin{aligned} & 2 \int_{v/2}^v k(w)e^{-uw} dw \\ &= \left(\int_{-\infty}^{-2\pi/v} + \int_{-2\pi/v}^{\pi/v} + \int_{\pi/v}^{\infty} \right) (\varphi(x, t) - \varphi(x + \pi/v, t)) e^{ux} dx. \end{aligned}$$

For the middle term on the right hand side of (3) we note that there is some constant C such that,

$$|\varphi(x, t)| \leq C |x|^{\alpha-1} \quad \text{and} \quad |\varphi(x + (\pi/v))| \leq C |x + (\pi/v)|^{\alpha-1}.$$

It follows that

$$(4) \quad \int_{-2\pi/v}^{\pi/v} = O(v^{-\alpha}).$$

For the remaining terms we write,

$$(5) \quad \varphi(x, t) - \varphi(x + \pi/v, t) = \psi(x, v, t) + \pi/v \left(\frac{\hat{k}(2x + t) - \hat{k}(x + t)}{x(x + \pi/v)} \right)$$

where

$$\psi(x, v, t) = \frac{\hat{k}(2x + t) - \hat{k}(x + t) - \hat{k}(2x + (2\pi/v) + t) + \hat{k}(x + (\pi/v) + t)}{x + (\pi/v)}.$$

To complete our argument we need to show that

$$\left(\int_{-\infty}^{-2\pi/v} + \int_{\pi/v}^{\infty} \right) \psi(x, v, t) e^{ix} dx = O(v^{-\beta}) \forall \beta < \alpha.$$

The second term on the right hand side of (5) can be handled in a straightforward manner. We will give the argument for $\int_{-\infty}^{-2\pi/v} \psi e^{ix} dx$; the proof for $\int_{\pi/v}^{\infty} \psi e^{ix} dx$ is similar. First let $\mu = \alpha/\epsilon$ and $s = -v^\mu$. We get

$$(6) \quad \left| \int_{-\infty}^{-2\pi/v} \psi(x, v, t) e^{ix} dx \right| \leq \left(\int_{-\infty}^s + \int_s^{-2\pi/v} \right) |\psi(x, v, t)| dx.$$

Since $\hat{k} \in \text{Lip } \alpha$ we have

$$|\psi(x, v, t)| \leq Cv^{-\alpha} |x + (\pi/v)|^{-1}$$

for some constant C independent of $x, v,$ and t . It follows that

$$(7) \quad \int_s^{-2\pi/v} |\psi(x, v, t)| dx \leq Cv^{-\alpha} \int_s^{-2\pi/v} \frac{dx}{|x + (\pi/v)|} = O(v^{-\beta}) \forall \beta < \alpha.$$

For the other term we have,

$$\begin{aligned} \int_{-\infty}^s |\psi(x, v, t)| dx &\leq \int_{-\infty}^s \frac{|\hat{k}(2x + t) - \hat{k}(x + t)|}{|x + (\pi/v)|^{1-\epsilon} |x + (\pi/v)|^\epsilon} dx \\ &\quad + \int_{-\infty}^s \frac{|\hat{k}(2x + (2\pi/v) + t) - \hat{k}(x + (\pi/v) + t)|}{|x + (\pi/v)|^{1-\epsilon} |x + (\pi/v)|^\epsilon} dx. \end{aligned}$$

Since $\hat{k} \in V_\epsilon$ by the second mean value theorem for integrals we conclude that

$$(8) \quad \int_{-\infty}^s |\psi(x, v, t)| dx = O(v^{-\alpha}).$$

Hence the proof is complete.

LEMMA 3. *If k is a real valued function such that:*

$$(9) \quad \hat{k} \text{ is continuous at } t,$$

$$(10) \quad \int_a^b k(w)e^{-iwv} dw = O(1)^1 \text{ and}$$

$$(11) \quad \int_{|u| \geq A} \left| \frac{\hat{k}(u)}{u} \right| du < \infty \text{ for some } A > 0.$$

Then

$$\int_{v/2}^v k(w)e^{-iwv} dw = i(2\pi)^{-1/2} \lim_{\epsilon \rightarrow 0^+} \int_{|x| \geq \epsilon} \frac{\hat{k}(2x+t) - \hat{k}(x+t)}{x} e^{ixv} dx.$$

Proof. From (10) it follows that,

$$(12) \quad \int_a^b k(w)dw = O(1) \text{ and } \hat{k} \in L_\infty.$$

We will assume $v > 0$, the proof for $v < 0$ is similar. Let $P_\delta(u) = \delta/(\delta^2 + u^2)$ which is the well-known Poisson kernel. We begin by showing that

$$(13) \quad \int_{v/2}^v k(u)e^{-iuu} du = \lim_{\delta \rightarrow 0^+} 1/\pi \int_{v/2}^v e^{-iuw} \int_{-\infty}^{\infty} k(u)P_\delta(w-u) dudw.$$

Using (12) we note that

$$\lim_{\delta \rightarrow 0^+} \int_{v/2}^v e^{iuw} \left(\int_{-\infty}^{v/4} + \int_{2v}^{\infty} \right) k(u)P_\delta(w-u) dudw = 0.$$

Hence since $k \in L_{loc}$ we get

¹ Refer to page 2 with $g(u, a, b) = O(1)$.

$$\begin{aligned} \lim_{\delta \rightarrow 0^+} \int_{v/2}^v e^{-i\pi w} \int_{v/4}^{2v} k(u) P_\delta(w-u) du dw \\ = \lim_{\delta \rightarrow 0^+} 1/\pi \int_{v/4}^{2v} k(u) \int_{v/2}^v e^{-i\pi w} P_\delta(w-u) dw du. \end{aligned}$$

Now (13) follows immediately.

By (10) and the second mean value theorem for integrals we get

$$(14) \quad \int_{-\infty}^{\infty} e^{-\delta|u|} \hat{k}(u) e^{i\pi u} du = 2(2\pi)^{-1/2} \int_{-\infty}^{\infty} k(u) P_\delta(w-u) du.$$

From (13) and (14) and using the fact that $\hat{k} \in L_\infty$ we get

$$\begin{aligned} (15) \quad \int_{v/2}^v k(w) e^{i\pi w} dw &= \lim_{\delta \rightarrow 0^+} (2\pi)^{-1/2} \int_{v/2}^v e^{-i\pi w} \int_{-\infty}^{\infty} e^{-\delta|u|} \hat{k}(u) e^{i\pi u} du dw \\ &= \lim_{\delta \rightarrow 0^+} \left(\int_{|u-t| \leq \epsilon} + \int_{|u-t| \geq \epsilon} \right) e^{-\delta|u|} \hat{k}(u) \rho(u-t, v) du \end{aligned}$$

where $\rho(u, v) = (2\pi)^{-1/2} (iu)^{-1} (e^{i\pi v} - e^{i\pi v/2})$.

We note that there exists some constant C independent of δ and ϵ such that

$$(16) \quad \left| \int_{|u-t| \geq \epsilon} e^{-\delta|u|} \hat{k}(u) \rho(u-t, v) du \right| \leq C\epsilon.$$

By (11) we can conclude that,

$$\lim_{\delta \rightarrow 0^+} \int_{|u-t| \geq \epsilon} e^{-\delta|u|} \hat{k}(u) \rho(u-t, v) du = \int_{|u-t| \geq \epsilon} \hat{k}(u) \rho(u-t, v) du.$$

After substitution we have,

$$\begin{aligned} (17) \quad & i \int_{|u-t| \geq \epsilon} \hat{k}(u) \rho(u-t, v) du \\ &= (2\pi)^{-1/2} \left\{ \int_{|x| \geq \epsilon} \frac{\hat{k}(x+t) - \hat{k}(2x+t)}{x} e^{i\pi x} dx - \int_{\epsilon/2 \leq |x| \leq \epsilon} \frac{\hat{k}(2x+t)}{x} e^{i\pi x} dx \right\}. \end{aligned}$$

We have,

$$\begin{aligned} & \left| \int_{\epsilon/2 \leq |x| \leq \epsilon} \frac{\hat{k}(2x+t)}{x} e^{i\pi x} dx \right| \\ & \leq \int_{\epsilon/2 \leq |x| \leq \epsilon} \frac{|\hat{k}(2x+t) - \hat{k}(t)|}{|x|} dx + \left| \hat{k}(t) \int_{\epsilon/2 \leq |x| \leq \epsilon} \frac{e^{i\pi x}}{x} dx \right|. \end{aligned}$$

Since $\hat{k} \in L_\infty$ and \hat{k} is continuous at t ,

$$(18) \quad \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon/2 \leq |x| \leq \epsilon} \frac{\hat{k}(2x+t)}{x} e^{ix} dx = 0.$$

The conclusion follows from (15), (16), (17) and (18).

3. Applications.

1. If $k(t) = e^{i|t|^a} / (|t|^b + 1)$ where $a(a-1) \neq 0$ and $b + a/2 - 1 > 0$, then $\hat{k} \in \text{Lip}(b + a/2 - 1)$. This follows immediately from Theorem 8 of §1, and van der Corput's Lemma (see [6]).

2. We adopt the following definitions:

DEFINITION 1. We say that $k \in L_p^c$ if for all $f \in L_\infty^c$ (set of L_∞ functions with compact support)

$$\int |k * f|^p \leq C \int |f|^p$$

where C is independent of f .

DEFINITION 2. We say that $k \in L_2^*$ if for all $f \in L_0^c$, there is a constant C such that

$$\left| \left\{ x : \left| \int_0^\infty k(\pm t) f(x \pm t) dt \right| > y \right\} \right| \leq \frac{C}{y^2} \|f\|_2^2, \quad \text{for all } y > 0.$$

Here, C is independent of f and y .

LEMMA 3. (Jurkat and Sampson). If $k \in L_2^*$ and $\int_s^{2s} |k(t)| dt = O(1)$ for all s , then $\int_a^b k(t) e^{-itx} dt = O(1)$.

Proof. Let f be the characteristic function of $[0, 2b]$ with $b > 0$. For all $u \in [0, b]$ we have, for fixed x ,

$$(1) \quad \int_0^\infty f(u+t) k(t) e^{-itx} dt = \left(\int_0^b + \int_b^{2b-u} \right) k(t) e^{-itx} dt.$$

But

$$\left| \int_b^{2b-u} k(t) e^{-itx} dt \right| \leq \sup_{s \in \mathbb{R}} \left| \int_s^{2s} |k(t)| dt \right| = M.$$

Therefore if $\left| \int_0^b k(t)e^{-ixt} dt \right| \leq 2M$ the proof is over. Now suppose that $\left| \int_0^b k(t)e^{-ixt} dt \right| > 2M$. In this case, from (1) it follows that

$$(2) \quad \left| \left\{ u : \left| \int_0^\infty k(t)e^{-ixt} f(u+t) dt \right| > \frac{1}{2} \left| \int_0^b k(t)e^{-ixt} dt \right| \right\} \right| \geq b.$$

Since $k \in L_2^*$, there exists some constant C independent of x and b such that

$$(3) \quad \left| \left\{ u : \left| \int_0^\infty k(t)e^{-ix(t+u)} f(u+t) dt \right| > \frac{1}{2} \left| \int_0^b k(t)e^{-ixt} dt \right| \right\} \right| \leq \frac{C}{\left| \int_0^b k(t)e^{-ixt} dt \right|^2} \int_0^\infty |f(t)|^2 dt.$$

From (2) and (3) it follows that $\left| \int_0^b k(t)e^{-ixt} dt \right| \leq \sqrt{C}$ where C is independent of b and x . A similar argument works for $b < 0$ and hence we get our result.

THEOREM 4. *If k is real-valued and satisfies the following conditions:*

$$(4) \quad \int_s^{2s} |k(t)| dt = O(1)$$

$$(5) \quad \hat{k} \in \text{Lip } \alpha \quad \text{for some } 0 < \alpha < 1$$

$$(6) \quad |x|^\lambda \hat{k}(x) = O(1) \quad \text{for some } \lambda > 0, \quad \text{and}$$

$$(7) \quad k \in L_2^*.$$

Then $k \in L_p^p$ for $1 < p < \infty$.

Proof. By Lemma 3, (4) and (7) imply that $\int_a^b k(t)e^{-ixt} dt = O(1)$. By Lemma 3 in §2 we can conclude that (1) in §2 holds. Furthermore (6) implies that $\hat{k} \in V_\epsilon$ for some $\epsilon > 0$. Hence due to Theorem 2 in §2, $k \in A_\beta \forall \beta < \alpha$. The conclusion follows from [3, Theorem 1.17].

4. Examples.

LEMMA 1. Let l , m , a and b be given numbers. Set $M = \max(|l|, |m|)$, $L = \min(|l|, |m|)$ and $V = \max(|a|, |b|)$. Then,

$$(1) \quad \left| \int_a^b \frac{\sin lu \cos mu}{u} du \right| = O(1)$$

$$(2) \quad \left| \int_{-b}^b \frac{\sin lue^{-mu}}{u} du \right| = O(1)$$

(3) if M and V are sufficiently large,

$$\left| \int_a^b \frac{\sin lu}{u} e^{-mu} du \right| = O(\log V + L).$$

Proof. We get (1) since $\left| \int_c^d \sin u/udu \right| \leq A$ where A is a positive constant independent of c and d ; also, (2) follows immediately from (1). For (3) it suffices to dominate

$$(4) \quad \left| \int_a^b \frac{\sin lu}{u} \sin mudu \right|.$$

Since the expression (4) is even and symmetric in l and m we can assume w.l.o.g. that $0 < l \leq m$. Furthermore, we can assume that $0 \leq a < b$ since the integrand is an odd function in u . Thus

$$(5) \quad \left| \int_a^b \frac{\sin lu}{u} \sin mudu \right| \leq l \quad \text{if } 0 \leq a < b < 1,$$

$$(6) \quad \left| \int_a^b \right| \leq \left| \int_a^1 \right| + \left| \int_1^b \right| \leq l + \log b \quad \text{if } 0 \leq a \leq 1 < b,$$

and

$$(7) \quad \left| \int_a^b \right| \leq \log b \quad \text{if } 1 < a < b.$$

Hence we get our result.

THEOREM 2. For each $0 < \alpha < 1$ there exists a function $g \in L_p$ ($1 \leq p < \infty$) such that $\hat{g} \in \text{Lip } \alpha$ but $g \notin A_\alpha$.

Proof. It suffices to show that there exists a $g \in L_p$ such that $\hat{g} \in \text{Lip}(\alpha)$ and for some sequences $\{h_n\} \rightarrow 0$, $\{x_n\}$, and $\{B_n\} \rightarrow \infty$ then

$$(8) \quad \left| h_n^{-\alpha} \int_{1/h_n}^{2/h_n} g(t) e^{-itx} dt \right| > B_n \quad \text{for} \quad |x - x_n| \leq 1/a_n.$$

Consider

$$g(t) = \sum_{m=1}^{\infty} \frac{\sin(m(t - c_m))}{m^{\gamma-1} a_m^\alpha (t - c_m)} \chi_{[a_m, b_m]}^{(t)}$$

where γ is a fixed positive integer ≥ 3 and $\gamma \geq (1 - \alpha)^{-1}$. Also,

$$(9) \quad a_m = 2^{m\gamma}, \quad b_m = 2a_m \quad \text{and} \quad c_m = 3/2a_m$$

and χ_I is the characteristic function of I .

To show that $\hat{g}(x)$ exists for each $x \in \mathbf{R}$, it suffices to show that $g \in L_1$.

$$\begin{aligned} \int_{-\infty}^{\infty} |g(t)| dt &\leq \sum_{m=1}^{\infty} m^{1-\gamma} a_m^{-\alpha} \int_{a_m}^{b_m} \frac{|\sin(m(t - c_m))|}{|t - c_m|} dt \\ &\leq 2 \sum_{m=1}^{\infty} \frac{1 + \log(ma_m)}{m^{\gamma-1} a_m^\alpha} < \infty. \end{aligned}$$

Now we are going to show that $\hat{g} \in \text{Lip} \alpha$ for $\gamma \geq (1 - \alpha)^{-1}$. Given h such that $2|h| < 1$, there exists an m so that,

$$(10) \quad 1/a_{m+1} < |h| \leq 1/a_m$$

$$\begin{aligned} \hat{g}(x + h) - \hat{g}(x) &= \sum_{l=1}^{m-1} l^{1-\gamma} a_l^{-\alpha} \int_{a_l}^{b_l} \frac{\sin l(t - c_l)}{t - c_l} e^{-itx} (e^{-iuh} - 1) dt, \\ &\quad + m^{1-\gamma} a_m^{-\alpha} \int_{a_m}^{b_m} \frac{\sin m(t - c_m)}{t - c_m} e^{-itx} (e^{-iuh} - 1) dt \\ (11) \quad &\quad + \left(\sum_{l=m+1}^{\infty} l^{1-\gamma} a_l^{-\alpha} \int_{a_l}^{b_l} \frac{\sin l(t - c_l)}{t - c_l} e^{-u(x+h)} dt \right. \\ &\quad \left. - \sum_{l=m+1}^{\infty} l^{1-\gamma} a_l^{-\alpha} \int_{a_l}^{b_l} \frac{\sin l(t - c_l)}{t - c_l} e^{-itx} dt \right) \\ &= I_1 + I_2 + I_3. \end{aligned}$$

We are going to show that each term on the right hand side of (11) is $O(h^\alpha)$ for $\gamma \geq (1 - \alpha)^{-1}$ separately.

After substituting $u = t - c_i$ by (2) and (10) $I_3 = O(h^\alpha)$.

Since $0 < h < 1/a_m$, by the second mean value theorem for integrals there exists ξ_i ($a_i < \xi_i < b_i$) such that

$$|I_1| \leq \sum_{i=1}^{m-1} \frac{\sin b_i h}{l^{\gamma-1} a_i^\alpha} \left| \int_{\xi_i}^{b_i} \frac{\sin l(t - c_i)}{t - c_i} e^{-itx} dt \right|.$$

After substitution $u = t - c_i$, by (3), (9) and (10) it follows

$$|I_1/h^\alpha| \leq C \sum_{i=1}^{m-1} (a_i h)^{1-\alpha} l^{1-\gamma} (\log a_i + l) = O(1).$$

It remains to show $I_2 = O(h^\alpha)$. By (2) we have,

$$(12) \quad I_2/h^\alpha = O((ha_m)^{-\alpha} m^{1-\gamma}) = O(1)$$

if $(ha_m)^{-\alpha} m^{1-\gamma} \leq 1$.

Since $e^{-ith} - 1 = (\cos th - 1) + i \sin th$, by the second mean value theorem for integrals and (3) we get

$$(13) \quad I_2/h^\alpha = O(m(a_m h)^{1-\alpha}) = O(1), \quad \text{if } m(a_m h)^{1-\alpha} \leq 1.$$

Hence by (12) and (13) we conclude that $I_2 = O(h^\alpha)$ if $\gamma \geq (1 - \alpha)^{-1}$.

Now we are going to show that there exist some sequences $\{h_n\} \rightarrow 0$, $\{x_n\}$, and $\{B_n\} \rightarrow \infty$ such that

$$\left| \frac{1}{h_n^\alpha} \int_{1/h_n}^{2/h_n} g(t) e^{-itx} dt \right| > B_n \quad \text{for } |x - x_n| \leq 1/a_n.$$

Consider $h_m = 1/c_m$ and $x_m = m$.

$$h_m^{-\alpha} \int_{1/h_m}^{2/h_m} g(t) e^{-itx} dt = J_1 + J_2$$

where

$$J_1 = -\frac{ie^{ic_m x}}{h_m^\alpha m^{\gamma-1} a_m^\alpha} \int_0^{1/2a_m} \frac{\sin mv}{v} \sin xvdv,$$

and

$$J_2 = \frac{e^{ic_m x}}{h_m^\alpha m^{\gamma-1} a_m^\alpha} \int_0^{1/2a_m} \frac{\sin mv}{v} \cos xvdv.$$

From (1) of Lemma 1 we can conclude that $J_2 = O(1)$. Hence it suffices to show that $|J_1(x)| \geq m/2$ if $|x - x_m| \leq 1/a_m$.

$$|J_1| = \left(\frac{3}{2}\right)^\alpha m^{1-\gamma} \left| \int_0^1 + \int_1^{\frac{1}{2}a_m} \right|.$$

It is clear that ($\gamma \geq 3$)

$$m^{1-\gamma} \left| \int_0^1 \frac{\sin mv}{v} \sin xvdv \right| = O(1).$$

On the other hand,

$$\begin{aligned} & 2m^{1-\gamma} \int_1^{\frac{1}{2}a_m} \frac{\sin mv}{v} \sin xvdv \\ &= m^{1-\gamma} \int_1^{\frac{1}{2}a_m} \left(\frac{\cos(m-x)v}{v} + \frac{\cos(m+x)v}{v} \right) dv. \end{aligned}$$

We can easily see that for these x 's,

$$m^{1-\gamma} \int_1^{\frac{1}{2}a_m} \frac{\cos(m+x)v}{v} dv = O(1).$$

For the remaining term we note that $\cos u \geq 1 - u^2/2$. Therefore for x satisfying $|x - m| \leq 1/a_m$,

$$\begin{aligned} m^{1-\gamma} \left| \int_1^{\frac{1}{2}a_m} \frac{\cos(m-x)v}{v} dv \right| &\geq m^{1-\gamma} \int_1^{\frac{1}{2}a_m} \left(\frac{1}{v} - (m-x)^2 v \right) dv \\ &\geq m^{1-\gamma} \left(\log a_m / 2 - \frac{(m-x)^2 a_m^2}{8} \right). \end{aligned}$$

Since $(m-x)^2 a_m^2 \leq 1$ we conclude that for m sufficiently large $|J_1| \geq m/2$.

The proof is then complete.

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Susan Jane Zimmerman Andima and W. J. Thron, <i>Order-induced topological properties</i>	297
Gregory Wade Bell, <i>Cohomology of degree 1 and 2 of the Suzuki groups</i> ...	319
Richard Body and Roy Rene Douglas, <i>Rational homotopy and unique factorization</i>	331
Frank Lewis Capobianco, <i>Fixed sets of involutions</i>	339
L. Carlitz, <i>Some theorems on generalized Dedekind-Rademacher sums</i>	347
Mary Rodriguez Embry and Alan Leslie Lambert, <i>The structure of a special class of weighted translation semigroups</i>	359
Steve Ferry, <i>Strongly regular mappings with compact ANR fibers are Hurewicz fiberings</i>	373
Ivan Filippenko and Marvin David Marcus, <i>On the unitary invariance of the numerical radius</i>	383
H. Groemer, <i>On the extension of additive functionals on classes of convex sets</i>	397
Rita Hall, <i>On the cohomology of Kuga's fiber variety</i>	411
H. B. Hamilton, <i>Congruences on N-semigroups</i>	423
Manfred Herrmann and Rolf Schmidt, <i>Regular sequences and lifting property</i>	449
James Edgar Keesling, <i>Decompositions of the Stone-Ćech compactification which are shape equivalences</i>	455
Michael Jay Klass and Lawrence Edward Myers, <i>On stopping rules and the expected supremum of S_n/T_n</i>	467
Ronald Charles Linton, <i>λ-large subgroups of C_λ-groups</i>	477
William Owen Murray, IV and L. Bruce Treybig, <i>Triangulations with the free cell property</i>	487
Louis Jackson Ratliff, Jr., <i>Polynomial rings and H_1-local rings</i>	497
Michael Rich, <i>On alternate rings and their attached Jordan rings</i>	511
Gary Sampson and H. Tuy, <i>Fourier transforms and their Lipschitz classes</i>	519
Helga Schirmer, <i>Effluent and noneffluent fixed points on dendrites</i>	539
Daniel Byron Shapiro, <i>Intersections of the space of skew-symmetric maps with its translates</i>	553
Edwin Spanier, <i>Tautness for Alexander-Spanier cohomology</i>	561
Alan Stein and Ivan Ernest Stux, <i>A mean value theorem for binary digits</i> ...	565
Franklin D. Tall, <i>Normal subspaces of the density topology</i>	579
William Yslas Vélez, <i>Prime ideal decomposition in $F(\mu^{1/p})$</i>	589
James Chin-Sze Wong, <i>Convolution and separate continuity</i>	601