

Pacific Journal of Mathematics

TAUTNESS FOR ALEXANDER-SPANIER COHOMOLOGY

EDWIN SPANIER

TAUTNESS FOR ALEXANDER–SPANIER COHOMOLOGY

E. H. SPANIER

The purpose of this note is to give a straightforward unified proof of the tautness of Alexander–Spanier cohomology in the cases where it is known to be valid and to give a necessary condition that every closed (arbitrary) subspace be taut with respect to zero dimensional cohomology.

Let F denote a contravariant functor from the category of topological spaces to the category of abelian groups. A subspace A of a topological space X is said to be *taut with respect to F* if the canonical map $\varinjlim \{F(U)\} \rightarrow F(A)$ is an isomorphism (the direct limit is taken over the family of all neighborhoods of A in X , the family being directed downward by inclusion). The subspace A is *taut* in X if it is taut with respect to the Alexander–Spanier cohomology theory \bar{H} for every dimension and every coefficient group (for notation and terminology dealing with \bar{H} see [6]).

This concept of tautness has proved to be important. In [6] and [7] it is shown that a closed subspace of a paracompact Hausdorff space is taut, and this is used to deduce a strong excision property for \bar{H} . This tautness property is also used in [6] to derive the continuity property for \bar{H} . In [4] it is shown that an arbitrary subspace of a metric space is taut with respect to Čech cohomology, and this is used to obtain a general duality in spheres. Since the Čech cohomology is isomorphic to \bar{H} [3], every subspace of a metric space is taut. In [2] it is shown that every neighborhood retract of X is taut in X , and this is used to prove a generalized homotopy property for compact spaces. In [1] tautness is considered for sheaf cohomology and used in proving the Vietoris–Begle mapping theorem.

We shall prove a simple lemma which gives a sufficient condition for tautness. This sufficient condition is enough to establish tautness in all the various cases where it is known.

Let \mathcal{U} be a collection of subsets of X and A a subset of X . The *star of A with respect to \mathcal{U}* , denoted by $\text{st}(A, \mathcal{U})$, is defined to be the union of those elements of \mathcal{U} whose intersection with A is nonempty. An *open covering of A in X* is a collection \mathcal{U} of open sets of X such that $A \subset \text{st}(A, \mathcal{U})$.

The following seems to be the main fact underlying tautness (see [2] and [6]).

LEMMA. Let A be a subspace of X and suppose that for every open covering \mathcal{U} of A in X there are an open covering \mathcal{V} of A in X and a function (not necessarily continuous) $f: \text{st}(A, \mathcal{V}) \rightarrow A$ such that:

(1) $f(a) = a$ for all $a \in A$.

(2) For each $V \in \mathcal{V}$ with $V \cap A \neq \emptyset$ there is $U \in \mathcal{U}$ such that $V \cup f(V) \subset U$.

Then A is taut in X .

Proof. (Recall the notation is as in [6].) An arbitrary q -dimensional cohomology class of A is represented by a q -cochain $\varphi \in C^q(A)$ such that $\delta\varphi = 0$ on $\mathcal{U}^{q+2} \cap A^{q+2}$ where \mathcal{U} is an open covering of A in X . Choose \mathcal{V} and f with respect to this \mathcal{U} to satisfy (1) and (2). Then $f^*\varphi \in C^q(\text{st}(A, \mathcal{V}))$ is a q -cochain such that $\delta f^*\varphi = f^*\delta\varphi$, and, by (2), the latter vanishes on $\{V \in \mathcal{V} \mid V \cap A \neq \emptyset\}^{q+2}$. Thus, $f^*\varphi$ represents an element of $\bar{H}^q(\text{st}(A, \mathcal{V}))$, and, by (1), its restriction to A is the element of $\bar{H}^q(A)$ represented by φ . Therefore, the canonical map $\varinjlim \{\bar{H}^q(U)\} \rightarrow \bar{H}^q(A)$ is an epimorphism.

Let U be a neighborhood of A . An element of $\bar{H}^q(U)$ whose restriction to A is 0 is represented by a q -cochain $\varphi \in C^q(U)$ such that $\delta\varphi = 0$ on \mathcal{U}_1^{q+2} where \mathcal{U}_1 is an open covering of U and such that there is a $(q-1)$ -cochain $\varphi' \in C^{q-1}(A)$ with $\varphi|_A = \delta\varphi'$ on $\mathcal{U}_2^{q+1} \cap A^{q+1}$ where \mathcal{U}_2 is an open covering of A in X . Let $\mathcal{U} = \{U_1 \cap U_2 \mid U_1 \in \mathcal{U}_1 \text{ and } U_2 \in \mathcal{U}_2\}$. Then \mathcal{U} is an open covering of A in X such that $\delta\varphi = 0$ on \mathcal{U}^{q+2} and $\varphi|_A = \delta\varphi'$ on $\mathcal{U}^{q+1} \cap A^{q+1}$. Let \mathcal{V} and f satisfy (1) and (2) with respect to this \mathcal{U} . It follows from (1) and (2) using the Fundamental Lemma 9.1 of [5] that $\varphi|_{\text{st}(A, \mathcal{V})}$ and $f^*(\varphi|_A)$ represent the same element of $\bar{H}^q(\text{st}(A, \mathcal{V}))$. Since $f^*(\varphi|_A) = f^*\delta\varphi' = \delta f^*\varphi'$ on $\{V \in \mathcal{V} \mid V \cap A \neq \emptyset\}^{q+1}$, we see that $f^*(\varphi|_A)$ represents 0 in $\bar{H}^q(\text{st}(A, \mathcal{V}))$. Therefore, $\varphi|_{\text{st}(A, \mathcal{V})}$ represents 0 in $\bar{H}^q(\text{st}(A, \mathcal{V}))$, and the canonical map $\varinjlim \{\bar{H}^q(U)\} \rightarrow \bar{H}^q(A)$ is a monomorphism.

THEOREM 1. In each of the following cases A is taut in X .

(1) A is compact and X is Hausdorff.

(2) A is closed and X is paracompact Hausdorff.

(3) A is arbitrary and every open subset of X is paracompact Hausdorff.

(4) A is a neighborhood retract of X .

Proof. In each of the first three cases it is easy to verify that if \mathcal{U} is any open covering of A in X there is an open covering \mathcal{V} of A in X such that the collection $\{\text{st}(V, \mathcal{V}) \mid V \in \mathcal{V} \text{ and } V \cap A \neq \emptyset\}$ is a refinement of \mathcal{U} . If $f: \text{st}(A, \mathcal{V}) \rightarrow A$ is defined so that $f(a) = a$ for $a \in A$ and so that for every $x \in \text{st}(A, \mathcal{V})$ there is $V' \in \mathcal{V}$ with x and $f(x)$ both in V' , then \mathcal{V}

and f satisfy (1) and (2) of the Lemma with respect to \mathcal{U} (see Lemma 1 on p. 316 of [6]). Therefore, A is taut in X .

In the fourth case let $r: N \rightarrow A$ be a retraction of an open neighborhood N of A to A . If \mathcal{U} is an open covering of A in X let $\mathcal{V} = \{U \cap r^{-1}(U \cap A) \mid U \in \mathcal{U}\}$. Then \mathcal{V} is an open covering of A in X . Define $f: \text{st}(A, \mathcal{V}) \rightarrow A$ by $f = r \mid \text{st}(A, \mathcal{V})$. Then \mathcal{V} and f satisfy (1) and (2) of the Lemma with respect to \mathcal{U} and so A is taut in X .

The following result is a necessary condition for tautness of every closed (arbitrary) subspace with respect to \bar{H}^0 . It can be used to provide examples where tautness fails to hold.

THEOREM 2. *If X is a space such that every closed (arbitrary) subspace is taut with respect to \bar{H}^0 , then X is normal (completely normal).*

Proof. We present the proof in the completely normal case, the normal case being analogous. To show X is completely normal it suffices to show that if E and F are subsets of X such that $\bar{E} \cap F = \emptyset = E \cap \bar{F}$ then E and F can be separated by open sets in X . Given such E and F let $A = E \cup F$. Then A is a subspace of X and E and F are both open and closed in A . Let φ be the 0-cocycle on A which is 0 on E and 1 on F . Assuming A is taut in X , there is an open neighborhood W of A in X and a 0-cocycle ψ on W such that $\psi \mid A = \varphi$. Since a 0-cocycle is a locally constant function, $U = \{x \in W \mid \psi(x) = 0\}$ and $V = \{x \in W \mid \psi(x) = 1\}$ are disjoint open sets in W , hence in X , which separate E and F .

REFERENCES

1. G. E. Bredon, *Sheaf Theory*, McGraw-Hill Book Company, Inc., New York, 1967.
2. Satya Deo, *On the tautness property of Alexander-Spanier cohomology*, Proc. Amer. Math. Soc., **52** (1975), 441-444.
3. C. H. Dowker, *Homology groups of relations*, Ann. of Math., (2) **56** (1952), 84-95.
4. K. Sitnikov, *Combinatorial topology of nonclosed sets I. The first duality law; spectral duality*, Mat. Sb. N. S. **34** (76) (1954), 3-54 (Russian). Amer. Math. Soc. Transl. (2) **15** (1960), 245-295.
5. E. H. Spanier, *Cohomology theory for general spaces*, Ann. of Math., (2) **49** (1948), 407-427.
6. ———, *Algebraic Topology*, McGraw-Hill Book Company, Inc., New York, 1966.
7. A. D. Wallace, *The map excision theorem*, Duke Math. J., **19** (1952), 177-182.

Received May 6, 1977.

UNIVERSITY OF CALIFORNIA
BERKELEY, CA 94720

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, CA 90024

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, CA 90007

R. A. BEAUMONT
University of Washington
Seattle, WA 98105

R. FINN AND J. MILGRAM
Stanford University
Stanford, CA 94305

C. C. MOORE
University of California
Berkeley, CA 94720

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Copyright © 1978 Pacific Journal of Mathematics
All Rights Reserved

Susan Jane Zimmerman Andima and W. J. Thron, <i>Order-induced topological properties</i>	297
Gregory Wade Bell, <i>Cohomology of degree 1 and 2 of the Suzuki groups</i> ...	319
Richard Body and Roy Rene Douglas, <i>Rational homotopy and unique factorization</i>	331
Frank Lewis Capobianco, <i>Fixed sets of involutions</i>	339
L. Carlitz, <i>Some theorems on generalized Dedekind-Rademacher sums</i>	347
Mary Rodriguez Embry and Alan Leslie Lambert, <i>The structure of a special class of weighted translation semigroups</i>	359
Steve Ferry, <i>Strongly regular mappings with compact ANR fibers are Hurewicz fiberings</i>	373
Ivan Filippenko and Marvin David Marcus, <i>On the unitary invariance of the numerical radius</i>	383
H. Groemer, <i>On the extension of additive functionals on classes of convex sets</i>	397
Rita Hall, <i>On the cohomology of Kuga's fiber variety</i>	411
H. B. Hamilton, <i>Congruences on N-semigroups</i>	423
Manfred Herrmann and Rolf Schmidt, <i>Regular sequences and lifting property</i>	449
James Edgar Keesling, <i>Decompositions of the Stone-Ćech compactification which are shape equivalences</i>	455
Michael Jay Klass and Lawrence Edward Myers, <i>On stopping rules and the expected supremum of S_n/T_n</i>	467
Ronald Charles Linton, <i>λ-large subgroups of C_λ-groups</i>	477
William Owen Murray, IV and L. Bruce Treybig, <i>Triangulations with the free cell property</i>	487
Louis Jackson Ratliff, Jr., <i>Polynomial rings and H_1-local rings</i>	497
Michael Rich, <i>On alternate rings and their attached Jordan rings</i>	511
Gary Sampson and H. Tuy, <i>Fourier transforms and their Lipschitz classes</i>	519
Helga Schirmer, <i>Effluent and noneffluent fixed points on dendrites</i>	539
Daniel Byron Shapiro, <i>Intersections of the space of skew-symmetric maps with its translates</i>	553
Edwin Spanier, <i>Tautness for Alexander-Spanier cohomology</i>	561
Alan Stein and Ivan Ernest Stux, <i>A mean value theorem for binary digits</i> ...	565
Franklin D. Tall, <i>Normal subspaces of the density topology</i>	579
William Yslas Véllez, <i>Prime ideal decomposition in $F(\mu^{1/p})$</i>	589
James Chin-Sze Wong, <i>Convolution and separate continuity</i>	601