TAUTNESS FOR ALEXANDER-SPANIER COHOMOLOGY

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The purpose of this note is to give a straightforward unified proof of the tautness of Alexander–Spanier cohomology in the cases where it is known to be valid and to give a necessary condition that every closed (arbitrary) subspace be taut with respect to zero dimensional cohomology.

Let $F$ denote a contravariant functor from the category of topological spaces to the category of abelian groups. A subspace $A$ of a topological space $X$ is said to be taut with respect to $F$ if the canonical map $\lim \{ F(U) \} \rightarrow F(A)$ is an isomorphism (the direct limit is taken over the family of all neighborhoods of $A$ in $X$, the family being directed downward by inclusion). The subspace $A$ is taut in $X$ if it is taut with respect to the Alexander–Spanier cohomology theory $\tilde{H}$ for every dimension and every coefficient group (for notation and terminology dealing with $\tilde{H}$ see [6]).

This concept of tautness has proved to be important. In [6] and [7] it is shown that a closed subspace of a paracompact Hausdorff space is taut, and this is used to deduce a strong excision property for $\tilde{H}$. This tautness property is also used in [6] to derive the continuity property for $\tilde{H}$. In [4] it is shown that an arbitrary subspace of a metric space is taut with respect to Čech cohomology, and this is used to obtain a general duality in spheres. Since the Čech cohomology is isomorphic to $\tilde{H}$ [3], every subspace of a metric space is taut. In [2] it is shown that every neighborhood retract of $X$ is taut in $X$, and this is used to prove a generalized homotopy property for compact spaces. In [1] tautness is considered for sheaf cohomology and used in proving the Vietoris–Begle mapping theorem.

We shall prove a simple lemma which gives a sufficient condition for tautness. This sufficient condition is enough to establish tautness in all the various cases where it is known.

Let $\mathcal{U}$ be a collection of subsets of $X$ and $A$ a subset of $X$. The star of $A$ with respect to $\mathcal{U}$, denoted by $\text{st}(A, \mathcal{U})$, is defined to be the union of those elements of $\mathcal{U}$ whose intersection with $A$ is nonempty. An open covering of $A$ in $X$ is a collection $\mathcal{U}$ of open sets of $X$ such that $A \subseteq \text{st}(A, \mathcal{U})$.

The following seems to be the main fact underlying tautness (see [2] and [6]).
LEMMA. Let $A$ be a subspace of $X$ and suppose that for every open covering $\mathcal{U}$ of $A$ in $X$ there are an open covering $\mathcal{V}$ of $A$ in $X$ and a function (not necessarily continuous) $f: \text{st}(A, \mathcal{V}) \rightarrow A$ such that:

1. $f(a) = a$ for all $a \in A$.
2. For each $V \in \mathcal{V}$ with $V \cap A \neq \emptyset$ there is $U \in \mathcal{U}$ such that $V \cup f(V) \subset U$.

Then $A$ is taut in $X$.

Proof. (Recall the notation is as in [6].) An arbitrary $q$-dimensional cohomology class of $A$ is represented by a $q$-cochain $\varphi \in C^q(A)$ such that $\delta \varphi = 0$ on $\mathcal{U}^{q+2} \cap A^{q+2}$ where $\mathcal{U}$ is an open covering of $A$ in $X$. Choose $\mathcal{V}$ and $f$ with respect to this $\mathcal{U}$ to satisfy (1) and (2). Then $f^* \varphi \in C^q(\text{st}(A, \mathcal{V}))$ is a $q$-cochain such that $\delta f^* \varphi = f^* \delta \varphi$, and, by (2), the latter vanishes on $\{V \in \mathcal{V} \mid V \cap A \neq \emptyset\}^{q+2}$. Thus, $f^* \varphi$ represents an element of $\tilde{H}^q(\text{st}(A, \mathcal{V}))$, and, by (1), its restriction to $A$ is the element of $\tilde{H}^q(A)$ represented by $\varphi$. Therefore, the canonical map $\lim \{\tilde{H}^q(U)\} \rightarrow \tilde{H}^q(A)$ is an epimorphism.

Let $U$ be a neighborhood of $A$. An element of $\tilde{H}^q(U)$ whose restriction to $A$ is $0$ is represented by a $q$-cochain $\varphi \in C^q(U)$ such that $\delta \varphi = 0$ on $\mathcal{U}^{q+2} \cap A^{q+2}$ where $\mathcal{U}$ is an open covering of $U$ and such that there is a $(q-1)$-cochain $\varphi' \in C^{q-1}(A)$ with $\varphi | A = \delta \varphi'$ on $\mathcal{U}^{q+1} \cap A^{q+1}$ where $\mathcal{U}$ is an open covering of $A$ in $X$. Let $\mathcal{U} = \{U_1 \cap U_2 \mid U_1, U_2 \in \mathcal{U}_1\}$ and $U_2 \in \mathcal{U}_2$. Then $\mathcal{U}$ is an open covering of $A$ in $X$ such that $\delta \varphi = 0$ on $\mathcal{U}^{q+2}$ and $\varphi | A = \delta \varphi'$ on $\mathcal{U}^{q+1} \cap A^{q+1}$. Let $\mathcal{V}$ and $f$ satisfy (1) and (2) with respect to this $\mathcal{U}$. It follows from (1) and (2) using the Fundamental Lemma 9.1 of [5] that $\varphi | \text{st}(A, \mathcal{V})$ and $f^*(\varphi | A)$ represent the same element of $\tilde{H}^q(\text{st}(A, \mathcal{V}))$. Since $f^*(\varphi | A) = f^* \delta \varphi' = \delta f^* \varphi'$ on $\{V \in \mathcal{V} \mid V \cap A \neq \emptyset\}^{q+1}$, we see that $f^*(\varphi | A)$ represents $0$ in $\tilde{H}^q(\text{st}(A, \mathcal{V}))$. Therefore, $\varphi | \text{st}(A, \mathcal{V})$ represents $0$ in $\tilde{H}^q(\text{st}(A, \mathcal{V}))$, and the canonical map $\lim \{\tilde{H}^q(U)\} \rightarrow \tilde{H}^q(A)$ is a monomorphism.

THEOREM 1. In each of the following cases $A$ is taut in $X$.

1. $A$ is compact and $X$ is Hausdorff.
2. $A$ is closed and $X$ is paracompact Hausdorff.
3. $A$ is arbitrary and every open subset of $X$ is paracompact Hausdorff.
4. $A$ is a neighborhood retract of $X$.

Proof. In each of the first three cases it is easy to verify that if $\mathcal{U}$ is any open covering of $A$ in $X$ there is an open covering $\mathcal{V}$ of $A$ in $X$ such that the collection $\{\text{st}(V, \mathcal{V}) \mid V \in \mathcal{V} \text{ and } V \cap A \neq \emptyset\}$ is a refinement of $\mathcal{U}$. If $f: \text{st}(A, \mathcal{V}) \rightarrow A$ is defined so that $f(a) = a$ for $a \in A$ and so that for every $x \in \text{st}(A, \mathcal{V})$ there is $V' \in \mathcal{V}$ with $x$ and $f(x)$ both in $V'$, then $\mathcal{V}$
and $f$ satisfy (1) and (2) of the Lemma with respect to $\mathcal{U}$ (see Lemma 1 on p. 316 of [6]). Therefore, $A$ is taut in $X$.

In the fourth case let $r: N \to A$ be a retraction of an open neighborhood $N$ of $A$ to $A$. If $\mathcal{U}$ is an open covering of $A$ in $X$ let $\mathcal{V} = \{ U \cap r^{-1}(U \cap A) | U \in \mathcal{U} \}$. Then $\mathcal{V}$ is an open covering of $A$ in $X$. Define $f: st(A, \mathcal{V}) \to A$ by $f = r | st(A, \mathcal{V})$. Then $\mathcal{V}$ and $f$ satisfy (1) and (2) of the Lemma with respect to $\mathcal{U}$ and so $A$ is taut in $X$.

The following result is a necessary condition for tautness of every closed (arbitrary) subspace with respect to $\overline{H}^0$. It can be used to provide examples where tautness fails to hold.

**Theorem 2.** If $X$ is a space such that every closed (arbitrary) subspace is taut with respect to $\overline{H}^0$, then $X$ is normal (completely normal).

**Proof.** We present the proof in the completely normal case, the normal case being analogous. To show $X$ is completely normal it suffices to show that if $E$ and $F$ are subsets of $X$ such that $\overline{E} \cap \overline{F} = \emptyset = E \cap F$ then $E$ and $F$ can be separated by open sets in $X$. Given such $E$ and $F$ let $A = E \cup F$. Then $A$ is a subspace of $X$ and $E$ and $F$ are both open and closed in $A$. Let $\varphi$ be the 0-cocycle on $A$ which is 0 on $E$ and 1 on $F$. Assuming $A$ is taut in $X$, there is an open neighborhood $W$ of $A$ in $X$ and a 0-cocycle $\psi$ on $W$ such that $\psi | A = \varphi$. Since a 0-cocycle is a locally constant function, $U = \{ x \in W | \psi(x) = 0 \}$ and $V = \{ x \in W | \psi(x) = 1 \}$ are disjoint open sets in $W$, hence in $X$, which separate $E$ and $F$.

**References**


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