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**The purpose of this note is to give a straightforward unified proof of the tautness of Alexander–Spanier cohomology in the cases where it is known to be valid and to give a necessary condition that every closed (arbitrary) subspace be taut with respect to zero dimensional cohomology.**

Let $F$ denote a contravariant functor from the category of topological spaces to the category of abelian groups. A subspace $A$ of a topological space $X$ is said to be *taut with respect to* $F$ if the canonical map $\varinjlim\{F(U)\} \to F(A)$ is an isomorphism (the direct limit is taken over the family of all neighborhoods of $A$ in $X$, the family being directed downward by inclusion). The subspace $A$ is *taut* in $X$ if it is taut with respect to the Alexander–Spanier cohomology theory $\bar{H}$ for every dimension and every coefficient group (for notation and terminology dealing with $\bar{H}$ see [6]).

This concept of tautness has proved to be important. In [6] and [7] it is shown that a closed subspace of a paracompact Hausdorff space is taut, and this is used to deduce a strong excision property for $\bar{H}$. This tautness property is also used in [6] to derive the continuity property for $\bar{H}$. In [4] it is shown that an arbitrary subspace of a metric space is taut with respect to Čech cohomology, and this is used to obtain a general duality in spheres. Since the Čech cohomology is isomorphic to $\bar{H}$ [3], every subspace of a metric space is taut. In [2] it is shown that every neighborhood retract of $X$ is taut in $X$, and this is used to prove a generalized homotopy property for compact spaces. In [1] tautness is considered for sheaf cohomology and used in proving the Vietoris–Begle mapping theorem.

We shall prove a simple lemma which gives a sufficient condition for tautness. This sufficient condition is enough to establish tautness in all the various cases where it is known.

Let $\mathcal{U}$ be a collection of subsets of $X$ and $A$ a subset of $X$. The *star of* $A$ *with respect to* $\mathcal{U}$, denoted by $\mathrm{st}(A, \mathcal{U})$, is defined to be the union of those elements of $\mathcal{U}$ whose intersection with $A$ is nonempty. An *open covering of* $A$ *in* $X$ is a collection $\mathcal{U}$ of open sets of $X$ such that $A \subset \mathrm{st}(A, \mathcal{U})$.

The following seems to be the main fact underlying tautness (see [2] and [6]).

LEMMA. *Let $A$ be a subspace of $X$ and suppose that for every open covering $\mathcal{U}$ of $A$ in $X$ there are an open covering $\mathcal{V}$ of $A$ in $X$ and a function (not necessarily continuous) $f\colon \mathrm{st}(A,\mathcal{V})\to A$ such that:*

*(1) $f(a)=a$ for all $a\in A$.*

*(2) For each $V\in\mathcal{V}$ with $V\cap A\neq\emptyset$ there is $U\in\mathcal{U}$ such that $V\cup f(V)\subset U$.*

*Then $A$ is taut in $X$.*

*Proof.* (Recall the notation is as in **[6]**.) An arbitrary $q$-dimensional cohomology class of $A$ is represented by a $q$-cochain $\varphi\in C^q(A)$ such that $\delta\varphi=0$ on $\mathcal{U}^{q+2}\cap A^{q+2}$ where $\mathcal{U}$ is an open covering of $A$ in $X$. Choose $\mathcal{V}$ and $f$ with respect to this $\mathcal{U}$ to satisfy (1) and (2). Then $f^*\varphi\in C^q(\mathrm{st}(A,\mathcal{V}))$ is a $q$-cochain such that $\delta f^*\varphi=f^*\delta\varphi$, and, by (2), the latter vanishes on $\{V\in\mathcal{V}\mid V\cap A\neq\emptyset\}^{q+2}$. Thus, $f^*\varphi$ represents an element of $\bar{H}^q(\mathrm{st}(A,\mathcal{V}))$, and, by (1), its restriction to $A$ is the element of $\bar{H}^q(A)$ represented by $\varphi$. Therefore, the canonical map $\varinjlim\{\bar{H}^q(U)\}\to\bar{H}^q(A)$ is an epimorphism.

Let $U$ be a neighborhood of $A$. An element of $\bar{H}^q(U)$ whose restriction to $A$ is 0 is represented by a $q$-cochain $\varphi\in C^q(U)$ such that $\delta\varphi=0$ on $\mathcal{U}_1^{q+2}$ where $\mathcal{U}_1$ is an open covering of $U$ and such that there is a $(q-1)$-cochain $\varphi'\in C^{q-1}(A)$ with $\varphi\mid A=\delta\varphi'$ on $\mathcal{U}_2^{q+1}\cap A^{q+1}$ where $\mathcal{U}_2$ is an open covering of $A$ in $X$. Let $\mathcal{U}=\{U_1\cap U_2\mid U_1\in\mathcal{U}_1$ and $U_2\in\mathcal{U}_2\}$. Then $\mathcal{U}$ is an open covering of $A$ in $X$ such that $\delta\varphi=0$ on $\mathcal{U}^{q+2}$ and $\varphi\mid A=\delta\varphi'$ on $\mathcal{U}^{q+1}\cap A^{q+1}$. Let $\mathcal{V}$ and $f$ satisfy (1) and (2) with respect to this $\mathcal{U}$. It follows from (1) and (2) using the Fundamental Lemma 9.1 of **[5]** that $\varphi\mid\mathrm{st}(A,\mathcal{V})$ and $f^*(\varphi\mid A)$ represent the same element of $\bar{H}^q(\mathrm{st}(A,\mathcal{V}))$. Since $f^*(\varphi\mid A)=f^*\delta\varphi'=\delta f^*\varphi'$ on $\{V\in\mathcal{V}\mid V\cap A\neq\emptyset\}^{q+1}$, we see that $f^*(\varphi\mid A)$ represents 0 in $\bar{H}^q(\mathrm{st}(A,\mathcal{V}))$. Therefore, $\varphi\mid\mathrm{st}(A,\mathcal{V})$ represents 0 in $\bar{H}^q(\mathrm{st}(A,\mathcal{V}))$, and the canonical map $\varinjlim\{\bar{H}^q(U)\}\to\bar{H}^q(A)$ is a monomorphism.

THEOREM 1. *In each of the following cases $A$ is taut in $X$.*

*(1) $A$ is compact and $X$ is Hausdorff.*

*(2) $A$ is closed and $X$ is paracompact Hausdorff.*

*(3) $A$ is arbitrary and every open subset of $X$ is paracompact Hausdorff.*

*(4) $A$ is a neighborhood retract of $X$.*

*Proof.* In each of the first three cases it is easy to verify that if $\mathcal{U}$ is any open covering of $A$ in $X$ there is an open covering $\mathcal{V}$ of $A$ in $X$ such that the collection $\{\mathrm{st}(V,\mathcal{V})\mid V\in\mathcal{V}$ and $V\cap A\neq\emptyset\}$ is a refinement of $\mathcal{U}$. If $f\colon\mathrm{st}(A,\mathcal{V})\to A$ is defined so that $f(a)=a$ for $a\in A$ and so that for every $x\in\mathrm{st}(A,\mathcal{V})$ there is $V'\in\mathcal{V}$ with $x$ and $f(x)$ both in $V'$, then $\mathcal{V}$

and $f$ satisfy (1) and (2) of the Lemma with respect to $\mathcal{U}$ (see Lemma 1 on p. 316 of [6]). Therefore, $A$ is taut in $X$.

In the fourth case let $r: N \to A$ be a retraction of an open neighborhood $N$ of $A$ to $A$. If $\mathcal{U}$ is an open covering of $A$ in $X$ let $\mathcal{V} = \{U \cap r^{-1}(U \cap A) \mid U \in \mathcal{U}\}$. Then $\mathcal{V}$ is an open covering of $A$ in $X$. Define $f: \operatorname{st}(A, \mathcal{V}) \to A$ by $f = r \mid \operatorname{st}(A, \mathcal{V})$. Then $\mathcal{V}$ and $f$ satisfy (1) and (2) of the Lemma with respect to $\mathcal{U}$ and so $A$ is taut in $X$.

The following result is a necessary condition for tautness of every closed (arbitrary) subspace with respect to $\bar{H}^0$. It can be used to provide examples where tautness fails to hold.

THEOREM 2. *If $X$ is a space such that every closed (arbitrary) subspace is taut with respect to $\bar{H}^0$, then $X$ is normal (completely normal).*

*Proof.* We present the proof in the completely normal case, the normal case being analogous. To show $X$ is completely normal it suffices to show that if $E$ and $F$ are subsets of $X$ such that $\bar{E} \cap F = \emptyset = E \cap \bar{F}$ then $E$ and $F$ can be separated by open sets in $X$. Given such $E$ and $F$ let $A = E \cup F$. Then $A$ is a subspace of $X$ and $E$ and $F$ are both open and closed in $A$. Let $\varphi$ be the 0-cocycle on $A$ which is 0 on $E$ and 1 on $F$. Assuming $A$ is taut in $X$, there is an open neighborhood $W$ of $A$ in $X$ and a 0-cocycle $\psi$ on $W$ such that $\psi \mid A = \varphi$. Since a 0-cocycle is a locally constant function, $U = \{x \in W \mid \psi(x) = 0\}$ and $V = \{x \in W \mid \psi(x) = 1\}$ are disjoint open sets in $W$, hence in $X$, which separate $E$ and $F$.

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