

Pacific Journal of Mathematics

A MEAN VALUE THEOREM FOR BINARY DIGITS

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This paper continues the investigation of the dyadically additive function α defined by $\alpha(n)$ = the number of 1's in the binary expansion of n .

Previously, Bellman and Shapiro (cf. "On a problem in additive number theory." *Annals of Mathematics*, 49 (1948) 333-340) showed that $\sum_{k=1}^x \alpha(k) \sim x \log x / 2 \log 2$. They then considered the iterates of α defined by $\alpha_q = \alpha_{q-1} \circ \alpha$ and observed that $A_r(x) = \sum_{k=1}^x \alpha_r(k)$ is not asymptotic to any elementary function for $r \geq 2$.

In this paper the function $A_2(x)$ will be examined more closely. Defining $c(x)$ by $A_2(x) = c(x)x \log \log x / 2 \log 2$, we will prove the following theorems.

THEOREM 1. *As x ranges over the positive integers, $c(x)$ ranges densely over $[1/2, 3/2]$. Furthermore, given any $c \in [1/2, 3/2]$, there is an explicit way to construct a sequence of integers x for which $c(x) \rightarrow c$ as $x \rightarrow \infty$.*

THEOREM 2.

$$(1.1) \quad \begin{aligned} 1/2 + O(\log \log \log x / \log \log x) &\leq c(x) \\ &\leq 3/2 + O(\log \log \log x / \log \log x). \end{aligned}$$

THEOREM 3.

$$(1.2) \quad \liminf c(x) = 1/2, \quad \limsup c(x) = 3/2.$$

Note. Theorem 3 is an immediate consequence of Theorems 1 and 2.

2. The proof of Theorem 1 is obtained by considering a special set of integers.

Let $\mathcal{M} = \{x : x = 2^M - 1, M \text{ even}, M/2 = \sum_{i=1}^r 2^{a_i} - 1, a_1 > a_2 > \dots > a_r \text{ positive integers, and } a_r/a_1 \geq 1/2 + \log \log \log x / \log \log x\}$.

LEMMA 1. *If $x \in \mathcal{M}$, then*

$$(2.1) \quad A_2(x) = \left(r + \frac{a_r}{2} + O(1) \right) x.$$

Proof. If $x \in \mathcal{M}$, then (cf. [1])

$$(2.2) \quad A_2(x) = \sum_{k < 2^M} \alpha(\alpha(k)) = \sum_{n \leq M} \binom{M}{n} \alpha(n).$$

We can then write

$$(2.3) \quad A_2(x) = \sum_1 \binom{M}{n} \alpha(n) + \sum_2 \binom{M}{n} \alpha(n)$$

where Σ_1 is the sum over $\{n: |M/2 - n| < 2a_r\}$ and Σ_2 is the sum over $\{n: |M/2 - n| \geq 2a_r\}$.

Chebyshev's inequality yields

$$(2.4) \quad \begin{aligned} \sum_2 \binom{M}{n} \alpha(n) &\ll 2^M M \cdot 2^{-2a_r} \log M \\ &\ll 2^M \cdot M \cdot 2^{-a_1(1+2\log \log \log x / \log \log x)} \log M \end{aligned}$$

which implies

$$(2.5) \quad \sum_2 \binom{M}{n} \alpha(n) = O(x).$$

(Here and further on, inequalities such as $M = O(\log x)$, $a_1 = O(\log M)$, $\alpha(n) = O(\log n)$ and $r = O(\log M)$ will be used without comment.)

We will use the symmetry of the binomial coefficients to estimate Σ_1 .

$$(2.6) \quad \begin{aligned} \sum_1 \binom{M}{n} \alpha(n) &= \frac{1}{2} \sum_{0 \neq |t| < 2^r} \binom{M}{M/2+t} \left\{ \alpha\left(\frac{M}{2}-t\right) + \alpha\left(\frac{M}{2}+t\right) \right\} \\ &\quad + \binom{M}{M/2} \alpha(M/2). \end{aligned}$$

Writing $t = \sum_{j=1}^w 2^j$, we obtain

$$(2.7) \quad \alpha\left(\frac{M}{2}+t\right) = \alpha\left(\sum_{i=1}^r 2^i + \sum_{j=1}^w 2^j - 1\right) = r + w - 1 + b_w$$

and

$$(2.8) \quad \alpha\left(\frac{M}{2} - t\right) = \alpha\left(\sum_{i=1}^r 2^{a_i} - 1 - \sum_{j=1}^w 2^{b_j}\right) = r - 1 + a_r - w$$

so that

$$(2.9) \quad \alpha\left(\frac{M}{2} + t\right) + \alpha\left(\frac{M}{2} - t\right) = 2r - 2 + a_r + b_w.$$

We can now rewrite (2.6), obtaining

$$(2.10) \quad \sum_1 \binom{M}{n} \alpha(n) = \sum_{0 < |t| < 2^r} \binom{M}{M/2 + t} \left(r - 1 + \frac{a_r}{2} + \frac{b_w}{2}\right) + \binom{M}{M/2} a_r.$$

Chebyshev's inequality implies that

$$\sum_{|t| \geq 2^r} \binom{M}{M/2 + t} \left(r + \frac{a_r}{2}\right) = O(x)$$

as in the analysis of Σ_2 . Since

$$\binom{M}{M/2} \left(r + a_r/2\right) = O(x \log \log x / \sqrt{\log x}) = O(x),$$

we obtain

$$(2.11) \quad \sum_{|t| < 2^r} \binom{M}{M/2 + t} \left(r + \frac{a_r}{2}\right) = 2^M \left(r + \frac{a_r}{2}\right) + O(x) = \left(r + \frac{a_r}{2}\right) + O(x).$$

Thus it remains only to show that each remaining term is $O(x)$. We have already seen that

$$(2.12) \quad \binom{M}{M/2} a_r/2 = O(x)$$

and easily obtain

$$(2.13) \quad \sum_{|t| < 2^r} \binom{M}{M/2 + t} (-1) = O(2^M) = O(x).$$

We estimate the remaining term by observing that $b_w = b_w(t)$ is the largest exponent such that $2^{b_w} | t$. Thus we can write

$$\begin{aligned}
 \sum_{0 < |t| < 2^r} \binom{M}{M/2 + t} b_w &\leq \sum_{\substack{t \\ 2^i | t}} \binom{M}{M/2 + t} \leq \sum_{i \geq 0} \sum_{q > 0} \binom{M}{M/2 + 2^i q} \\
 (2.14) \qquad \qquad \qquad &\leq \sum_{i \geq 0} \frac{1}{2^i} \sum_q \binom{M}{q} = O(x).
 \end{aligned}$$

This completes the proof of Lemma 1.

Lemma 1 implies that

$$(2.15) \qquad c(x) = \frac{r + a_r/2}{\log \log x/2 \log 2} + o(1).$$

Since $a_1 = \log \log x/2 \log 2 + O(1)$, we obtain

$$(2.16) \qquad c(x) = \frac{r + a_r/2}{a_1} + o(1).$$

We now complete the proof of Theorem 1 by showing that if $\epsilon > 0$ then there exist arbitrarily large q such that if $(1/2 + \epsilon)q < z < (3/2 - \epsilon)q$ is an integer, then there exists $x \in \mathcal{M}$ such that $a_1 = q$ and $r + a_r/2 = z$.

Suppose we choose $(1/2 + \epsilon)q - 4 \leq s < (1/2 + \epsilon)q - 2$, s even. As t takes on all possible integer values between 2 and $q - s$, $t + s/2$ certainly takes on all integer values between $(1/2 + \epsilon)q$ and $(3/2 - \epsilon)q$.

If q is large enough, it is certainly possible to find $x \in \mathcal{M}$ such that $a_1 = q$, $r = t$ and $a_r = s$, completing the proof of Theorem 1.

3. We carry out the proof of Theorem 2 in a series of steps.

Let $\mathcal{M}^1 = \{x: x = 2^M - 1, M \text{ even}, M/2 = \sum_{i=1}^r 2^{a_i} - 1, a_1 > a_2 > \dots > a_r, \text{ integers and } a_r/a_1 \geq (1/2) \log \log \log x / (\log \log x + \log \log 2)\}$.

We begin by proving the conclusion of Theorem 2 holds for element of \mathcal{M}^1 .

LEMMA 2. *If $x \in \mathcal{M}^1$ then*

$$(3.1) \qquad \frac{1}{2} + O\left(\frac{\log \log \log x}{\log \log x}\right) < c(x) < \frac{3}{2} + O\left(\frac{\log \log \log x}{\log \log x}\right).$$

Proof. We begin as in Lemma 1, writing

$$(3.2) \qquad A_2(x) = \sum_1 \binom{M}{n} \alpha(n) + \sum_2 \binom{M}{n} \alpha(n)$$

except where Σ_1 is the sum over $\{n: |M/2 - n| < 2^{(\frac{1}{2} + \epsilon)a_1}\}$ and Σ_2 is the sum over $\{n: |M/2 - n| \geq 2^{(1/2 + \epsilon)a_1}\}$, where $\epsilon = \epsilon(x) = \log \log \log x / (\log \log x + \log \log 2)$.

The second term can be estimated as the corresponding term was in Lemma 1, yielding

$$(3.3) \quad \sum_2 \binom{M}{n} \alpha(n) = O(x).$$

We estimate the first sum by considering two cases.

Case 1. $a_r \geq (1/2 + \epsilon)a_1$. We can treat this case as we treated Lemma 1, obtaining $\Sigma_1 \binom{M}{n} \alpha(n) = (r + a_r/2)x + O(x)$ and hence

$$(3.4) \quad A_2(x) = (r + a_r/2)x + O(x).$$

Since $0 \leq r \leq a_1 - a_r + 1$, we obtain $a_r/2 \leq r + a_r/2 \leq a_1 - a_r/2 + 1$. Since $(1/2 + \epsilon)a_1 \leq a_r \leq a_1$, we obtain

$$(3.5) \quad \left(\frac{1}{4} + \frac{\epsilon}{2}\right) a_1 \leq r + a_r/2 \leq \left(\frac{3}{4} - \frac{\epsilon}{2}\right) a_1 + 1.$$

But $a_1 = (\log \log x / \log 2) + O(1)$, so

$$(3.6) \quad \left(\frac{1}{4} + \frac{\epsilon}{2}\right) \frac{\log \log x}{\log 2} + O(1) \leq r + a_r/2 \leq \left(\frac{3}{4} - \frac{\epsilon}{2}\right) \frac{\log \log x}{\log 2} + O(1)$$

which implies

$$(3.7) \quad \frac{1}{4} \frac{\log \log x}{\log 2} + O(\log \log \log x) \leq r + a_r/2 \leq \frac{3}{4} \frac{\log \log x}{\log 2} + O(\log \log \log x).$$

Thus

$$(3.8) \quad \begin{aligned} & \left(\frac{1}{2} + O\left(\frac{\log \log \log x}{\log \log x}\right)\right) \left(\frac{x \log \log x}{2 \log 2}\right) \leq A_2(x) \\ & \leq \left(\frac{3}{2} + O\left(\frac{\log \log \log x}{\log \log x}\right)\right) \left(\frac{x \log \log x}{2 \log 2}\right) \end{aligned}$$

which proves Lemma 2 for this case.

Case 2. $(1/2 - \epsilon)a_1 < a_r < (1/2 + \epsilon)a_1$.

As in Lemma 1, we write

$$\begin{aligned}
 \sum_1 \binom{M}{n} \alpha(n) &= \frac{1}{2} \sum_{0 \neq t < 2^{(1/2 + \epsilon)a_1}} \binom{M}{M/2 + t} \left\{ \alpha\left(\frac{M}{2} - t\right) + \alpha\left(\frac{M}{2} + t\right) \right\} \\
 (3.9) \qquad \qquad \qquad &+ \binom{M}{M/2} \alpha\left(\frac{M}{2}\right).
 \end{aligned}$$

Here, however, an overlap of the nonzero digits in the binary representations of t and $M/2 + 1$ forces us to use the subadditive properties of α . Writing $M/2$ and t as before, we obtain

$$\begin{aligned}
 \frac{M}{2} + t &= 2^{a_1} + \dots + 2^{a_r} + 2^{b_1} + \dots + 2^{b_w} - 1 \\
 (3.10) \qquad \qquad \qquad & \\
 \frac{M}{2} - t &= 2^{a_1} + \dots + 2^{a_r} - 2^{b_1} - \dots - 2^{b_w} - 1.
 \end{aligned}$$

The subadditivity of α implies $\alpha(M/2 + t) \leq \alpha(M/2 + 1) + \alpha(t - 1)$ so that

$$(3.11) \qquad \qquad \qquad \alpha\left(\frac{M}{2} + t\right) \leq r + w + b_w.$$

Also, $\alpha(M/2 + t)$ is at least $\alpha(t)$ minus the overlap between the binary expansions of $M/2$ and t , so that

$$(3.12) \qquad \qquad \qquad \alpha\left(\frac{M}{2} + t\right) \geq w - 2\epsilon a_1.$$

Since $\alpha(M/2 - t)$ is no greater than the number of places available, less $\alpha(t)$, plus the overlap, we obtain

$$(3.13) \qquad \qquad \qquad \alpha\left(\frac{M}{2} - t\right) \leq a_1 + 1 - w + 2\epsilon a_1.$$

Also, $\alpha(M/2 - t)$ must be at least the number of 1's that $M/2$ ends with less $\alpha(t)$, so that

$$(3.14) \qquad \qquad \qquad \alpha\left(\frac{M}{2} - t\right) \geq a_r - w.$$

Combining (3.11)–(3.14) we obtain

$$(3.15) \quad a_r - 2\epsilon a_1 \leq \alpha \left(\frac{M}{2} + t \right) + \alpha \left(\frac{M}{2} - t \right) \leq a_1 + r + b_w + 2\epsilon a_1 + 1.$$

Since $a_r > (1/2 - \epsilon)a_1$ and $r \leq a_1 - a_r + 1 < a_1 - (1/2 - \epsilon)a_1 + 1 = (1/2 + \epsilon)a_1 + 1$ we obtain

$$(3.16) \quad \left(\frac{1}{2} - 3\epsilon \right) a_1 \leq \alpha \left(\frac{M}{2} + t \right) + \alpha \left(\frac{M}{2} - t \right) \leq \left(\frac{3}{2} + 3\epsilon \right) a_1 + b_w + 1.$$

Plugging the first inequality of (3.16) into (3.9) yields

$$\sum_1 \binom{M}{n} \alpha(n) \geq \frac{1}{2} \left(\frac{1}{2} - 3\epsilon \right) a_1 - \sum_{0 \neq t < 2^{(1/2 + \epsilon)a_1}} \binom{M}{M/2 + t} + \binom{M}{M/2} \alpha \left(\frac{M}{2} \right).$$

Chebyshev's inequality yields

$$\sum_{0 \neq t < 2^{(1/2 + \epsilon)a_1}} \binom{M}{M/2 + t} = x + O \left(\frac{x}{2^{2\epsilon a_1}} \right)$$

which implies

$$(3.17) \quad \sum_1 \binom{M}{n} \alpha(n) \geq \frac{1}{4} a_1 x - \frac{3}{2} \epsilon a_1 x + O(x).$$

Recalling $a_1 = (\log \log x) / \log 2 + O(1)$ and combining (3.17) with (3.3) yields

$$(3.18) \quad A_2(x) \geq \frac{1}{2} \frac{x \log \log x}{2 \log 2} + O(x \log \log \log x)$$

which implies

$$(3.19) \quad c(x) \geq \frac{1}{2} + O \left(\frac{\log \log \log x}{\log \log x} \right).$$

Plugging the second inequality of (3.16) into (3.9) yields

$$(3.20) \quad \begin{aligned} \sum_1 \binom{M}{n} \alpha(n) &\leq \frac{1}{2} \left(\frac{3}{2} + 3\epsilon \right) a_1 - \sum_{0 \neq t < 2^{(1/2 + \epsilon)a_1}} \binom{M}{M/2 + t} \\ &\quad + \frac{1}{2} \sum_{0 < |t| < 2^{(1/2 + \epsilon)a_1}} \binom{M}{M/2 + t} (b_w + 1) \\ &\quad + \binom{M}{M/2} \alpha \left(\frac{M}{2} \right). \end{aligned}$$

As in (2.14) we see that

$$\sum_{0 < |t| < 2^{(1/2+\epsilon)a_1}} \binom{M}{M/2+t} (b_w + 1) = O(x)$$

to obtain

$$(3.21) \quad \sum_1 \binom{M}{n} \alpha(n) \leq \frac{3}{4} a_1 x + \frac{3}{2} \epsilon a_1 x + O(x).$$

Repeating the reasoning of (3.17)–(3.19) yields

$$(3.22) \quad c(x) \leq \frac{3}{2} + O\left(\frac{\log \log \log x}{\log \log x}\right).$$

Combining (3.19) with (3.22) completes the proof of Lemma 2.

We now consider a lemma which will enable us to extend the conclusion of Theorem 2 to all integers of the form $2^n - 1$.

LEMMA 3. *If $x = 2^N - 1$, then there exists an even integer $M \geq N$ such that $M - N \leq \sqrt{N/\log N}$, $M/2 = \sum_{i=1}^r 2^{a_i} - 1$ with $a_r/a_1 \geq 1/2 - \epsilon$ and*

$$(3.23) \quad A_2(x) = \frac{A_2(2^M - 1)}{2^{M-N}} + O(x),$$

where

$$\epsilon = \epsilon(x) = \frac{\log \log \log x}{\log \log x + \log \log 2}.$$

Proof. Let $N = \sum_{i=1}^l 2^{a_i}$, $a_1 > a_2 > \dots > a_l$. Define n by $n + 1 = \sum_j 2^{c_j}$, where $\{c_j\}$ runs over all integer values in the interval $[1, (1/2)a_1(1 - 2\epsilon) + 1]$ not equal to any of the a_i 's. If no such c 's exist, let $n = 0$ if N is even, $n = 1$ if N is odd. Let $M = N + n$. Clearly $n = M - N \ll 2^{(1/2)a_1(1-2\epsilon)} \ll N^{1/2-\epsilon} \ll \sqrt{N/\log N}$ and only (3.23) requires further analysis.

As before, $A_2(2^M - 1) = \sum_{s \leq M} \binom{M}{s} \alpha(s)$.

We rewrite this as

$$(3.24) \quad A_2(2^M - 1) = s_1 + s_2$$

where

$$(3.25) \quad s_1 = 2^n \sum_s \binom{N}{s} \alpha(s) = 2^{M-N} A_2(2^N - 1)$$

$$(3.26) \quad s_2 = \sum_s \left\{ \binom{M}{s} - 2^n \binom{N}{2} \right\} \alpha(s).$$

We bound s_2 from above by writing

$$(3.27) \quad s_2 \leq \log M \sum_s \left| \binom{M}{s} - 2^n \binom{N}{2} \right|.$$

But

$$\begin{aligned} \sum_s \left| \binom{M}{s} - 2^n \binom{N}{s} \right| &= \sum_s \left| \sum_q \binom{N}{s-q} \binom{n}{q} - \binom{N}{s} \binom{n}{q} \right| \\ &\leq \sum_q \binom{n}{q} \sum_s \left| \binom{N}{s-q} - \binom{N}{s} \right| \\ &\leq \sum_q \binom{n}{q} \cdot 2q \max_s \binom{N}{s} \leq n \cdot 2^n \cdot 2^N / \sqrt{N} \\ &\leq \frac{\sqrt{N}}{\log N} \cdot 2^n \cdot 2^N / \sqrt{N} = \frac{2^{N+n}}{\log N} \end{aligned}$$

so $s_2 \leq 2^{N+n} \leq 2^n x = 2^{M-N} x$ and

$$(3.28) \quad A_2(2^M - 1) = 2^{M-N} \{A_2(x) + O(x)\},$$

proving the lemma.

COROLLARY 1. *If $x = 2^N - 1$, then*

$$\frac{1}{2} + O\left(\frac{\log \log \log x}{\log \log x}\right) \leq c(x) \leq \frac{3}{2} + O\left(\frac{\log \log \log x}{\log \log x}\right).$$

Proof. Find an M as in Lemma 3 so that

$$A_2(x) = \frac{A_2(2^M - 1)}{2^{M-N}} + O(x).$$

Applying Lemma 2 to $2^M - 1$ immediately yields this result.

LEMMA 4. *Let $x = \sum_{i=1}^r 2^{s_i}$, $s_1 > s_2 > \dots > s_r$. Then*

$$(3.29) \quad A_2(x) = \sum_{i=1}^r A_2(2^i - 1) + O(x \log \log x / \sqrt{\log x}).$$

Proof.

$$A_2(x) = \sum_{i=1}^r \sum_{n < 2^i} \alpha_2 \left(\sum_{j=1}^{i-1} 2^j + n \right).$$

Since $\alpha(\sum_{j=1}^{i-1} 2^j + n) = \alpha(n) + i - 1$ we obtain

$$A_2(x) = \sum_{i=1}^r \sum_{n < 2^i} \alpha(\alpha(n) + i - 1).$$

Letting $E_i = \sum_{n < 2^i} \{\alpha(\alpha(n) + i - 1) - \alpha(\alpha(n))\}$, we obtain

$$(3.30) \quad A_2(x) = \sum_{i=1}^r A_2(2^i - 1) + \sum_{i=2}^r E_i.$$

We now must merely show that $\sum_{i=2}^r E_i = O(x \log \log x / \sqrt{\log x})$. Rewrite

$$\begin{aligned} E_i &= \sum_{l \leq s_i} \sum_{\substack{n < 2^{i-1} \\ \alpha(n)=l}} \{\alpha(l + i - 1) - \alpha(l)\} \\ &= \sum_l \binom{s_i}{l} \{\alpha(l + i - 1) - \alpha(l)\}. \end{aligned}$$

Summing by parts,

$$E_i = \sum_l \alpha(l) \left\{ \binom{s_i}{l-i+1} - \binom{s_i}{l} \right\}.$$

Since $\alpha(l) = O(\log(s_i + i))$ and

$$\sum_l \left| \binom{s_i}{l-i+1} - \binom{s_i}{l} \right| = O \left(i \binom{s_i}{[s_i]/2} \right) = O \left(i \frac{2^{s_i}}{\sqrt{s_i}} \right)$$

we obtain

$$(3.31) \quad E_i \ll i \cdot \log(s_i + i) 2^{s_i} / \sqrt{s_i}.$$

Thus

$$\sum_{i=2}^r E_i \ll \sum_{i=2}^r i \log(s+i) 2^s / \sqrt{s_i}.$$

Since $s_i \leq s_1 - i + 1$ and $s_i + i \ll \log x$, and writing $s = s_1$, we obtain

$$(3.32) \quad \sum_{i=2}^r E_i \ll \log \log x \sum_{i=1}^r i 2^{s-i} / \sqrt{s-i}.$$

Now

$$\sum_{i \leq s/2} i 2^{s-i} / \sqrt{s-i} \ll \frac{2^s}{\sqrt{s}} \sum \frac{i}{2^i} \ll 2^s / \sqrt{s}$$

while

$$\sum_{i > s/2} i 2^{s-i} / \sqrt{s-i} \ll \sum_{i > s/2} i 2^{s-i} \ll s \cdot 2^{s/2} \ll \frac{2^s}{\sqrt{s}}.$$

Since $2^s = O(x)$ and $s = \log x / \log 2 + O(1)$, we obtain

$$(3.33) \quad \sum_{i=2}^r E_i = O(x \log \log x / \sqrt{\log x}),$$

completing the proof of Lemma 4.

We can now easily prove Theorem 2.

Proof of Theorem 2. Let $x = \sum_{i=1}^r 2^i$. By Lemma 4,

$$(3.34) \quad A_2(x) = \sum_{i=1}^r A_2(2^i - 1) + O(x).$$

Corollary 1 implies that

$$A_2(2^s - 1) \leq \frac{3}{2} \frac{2^s \log \log x}{2 \log 2} + O(2^s \log \log \log x),$$

so that

$$A_2(x) \leq \frac{3}{2} \sum_{i=1}^r \left(\frac{2^i \log \log x}{2 \log 2} + O(2^i \log \log \log x) \right) + O(x)$$

and hence

$$(3.35) \quad A_2(x) \leq \frac{3}{2} \frac{x \log \log x}{2 \log 2} + O(x \log \log \log x)$$

which is equivalent to

$$(3.36) \quad c(x) \leq \frac{3}{2} + O\left(\frac{\log \log \log x}{\log \log x}\right).$$

We now obtain a lower bound. Again using Corollary 1, we obtain

$$A_2(x) \geq \frac{1}{2} \sum_{i=1}^r \left(\frac{2^{s_i} \log \log 2^{s_i}}{2 \log 2} + O(2^{s_i} \log \log \log x) \right) + O(x)$$

and hence

$$(3.37) \quad A_2(x) \geq \frac{1}{2} \sum_{i=1}^r \frac{2^{s_i} \log \log 2^{s_i}}{2 \log 2} + O(x \log \log \log x).$$

Since $\log \log 2^{s_i} = \log \log x + O(1)$ if $s_i \geq s_1/2$, we obtain

$$(3.38) \quad A_2(x) \geq \frac{1}{2} \sum_{s_i \geq s_1/2} \frac{2^{s_i} \log \log x}{2 \log 2} + O(x \log \log \log x).$$

But $\sum_{s_i \geq s_1/2} 2^{s_i} = x + O(2^{s_1/2}) = x + O(\sqrt{x})$ yielding

$$(3.39) \quad A_2(x) \geq \frac{1}{2} \frac{x \log \log x}{2 \log 2} + O(x \log \log \log x)$$

which implies

$$(3.40) \quad c(x) \geq \frac{1}{2} + O\left(\frac{\log \log \log x}{\log \log x}\right),$$

completing the proof of Theorem 2.

REFERENCES

1. R. Bellman and H. N. Shapiro, *On a problem in additive number theory*, *Annals of Math.*, **49** (1948), 333–340.
2. L. Mirsky, *A theorem of representations of integers in the scale of r* , *Scripta Math.*, **15** (1949), 11–12.

3. I. Shiokawa, *On a problem in additive number theory*, Math. J. Okayama University, **16** (1973–74), 167–176.
4. S. C. Tang, *An improvement and generalization of Bellman–Shapiro’s theorem on a problem in additive number theory*, Proc. Amer. Math. Soc., **12** (1963), 199–204.

Received September 14, 1976 and in revised form May 16, 1977. Work by the first author was partially supported by The University of Connecticut Faculty Summer Fellowship, 1975, and work by the second author was partially supported by the National Science Foundation, Grant NSF-MPS-75-08545.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

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PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

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