A MEAN VALUE THEOREM FOR BINARY DIGITS

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This paper continues the investigation of the dyadically additive function \( \alpha \) defined by \( \alpha(n) = \) the number of 1's in the binary expansion of \( n \).

Previously, Bellman and Shapiro (cf. "On a problem in additive number theory." Annals of Mathematics, 49 (1948) 333–340) showed that \( \sum_{k=1}^{x} \alpha(k) \sim x \log x / 2 \log 2 \). They then considered the iterates of \( \alpha \) defined by \( \alpha_{q} = \alpha_{q-1} \circ \alpha \) and observed that \( A_{r}(x) = \sum_{k=1}^{x} \alpha_{r}(k) \) is not asymptotic to any elementary function for \( r \geq 2 \).

In this paper the function \( A_{2}(x) \) will be examined more closely. Defining \( c(x) \) by \( A_{2}(x) = c(x) x \log \log x / \log 2 \), we will prove the following theorems.

**Theorem 1.** As \( x \) ranges over the positive integers, \( c(x) \) ranges densely over \([1/2, 3/2]\). Furthermore, given any \( c \in [1/2, 3/2] \), there is an explicit way to construct a sequence of integers \( x \) for which \( c(x) \to c \) as \( x \to \infty \).

**Theorem 2.**

\[
1/2 + O(\log \log \log x / \log \log x) \leq c(x) \leq 3/2 + O(\log \log \log x / \log \log x).
\]

**Theorem 3.**

\[
\lim \inf c(x) = 1/2, \quad \lim \sup c(x) = 3/2.
\]

**Note.** Theorem 3 is an immediate consequence of Theorems 1 and 2.

2. The proof of Theorem 1 is obtained by considering a special set of integers.

Let \( \mathcal{M} = \{ x : x = 2^{M} - 1, M \) even, \( M/2 = \sum_{i=1}^{a_{r} - 1} 2^{a_{i}} - 1, a_{1} > a_{2} > \cdots > a_{r} \) positive integers, and \( a_{r} / a_{1} \geq 1/2 + \log \log \log x / \log \log x \} \).

**Lemma 1.** If \( x \in \mathcal{M} \), then
Proof. If $x \in \mathcal{M}$, then (cf. [1])

\begin{equation}
A_2(x) = \sum_{k \leq M} \alpha(\alpha(k)) = \sum_{n \leq M} \binom{M}{n} \alpha(n).
\end{equation}

We can then write

\begin{equation}
A_2(x) = \sum_1 \binom{M}{n} \alpha(n) + \sum_2 \binom{M}{n} \alpha(n)
\end{equation}

where $\Sigma_1$ is the sum over $\{n: |M/2 - n| < 2a_r\}$ and $\Sigma_2$ is the sum over $\{n: |M/2 - n| \geq 2a_r\}$.

Chebyshev's inequality yields

\begin{equation}
\sum_2 \binom{M}{n} \alpha(n) \ll 2^M M \cdot 2^{-2a_r} \log M
\end{equation}

\begin{equation}
\ll 2^M \cdot M \cdot 2^{-a_r(1+2 \log \log \log x / \log \log x)} \log M
\end{equation}

which implies

\begin{equation}
\sum_2 \binom{M}{n} \alpha(n) = O(x).
\end{equation}

(Here and further on, inequalities such as $M = O(\log x)$, $a_1 = O(\log M)$, $\alpha(n) = O(\log n)$ and $r = O(\log M)$ will be used without comment).

We will use the symmetry of the binomial coefficients to estimate $\Sigma_1$.

\begin{equation}
\sum_1 \binom{M}{n} \alpha(n) = \frac{1}{2} \sum_{0 \leq t \leq M/2} \binom{M}{M/2 + t} \left\{ \alpha \left( \frac{M}{2} - t \right) + \alpha \left( \frac{M}{2} + t \right) \right\}
\end{equation}

\begin{equation}
+ \binom{M}{M/2} \alpha(M/2).
\end{equation}

Writing $t = \Sigma_{h=1}^w 2^h$, we obtain

\begin{equation}
\alpha \left( \frac{M}{2} + t \right) = \alpha \left( \sum_{i=1}^r 2^a + \sum_{j=1}^w 2^b - 1 \right) = r + w - 1 + b_w
\end{equation}
and
\[(2.8) \quad \alpha \left( \frac{M}{2} - t \right) = \alpha \left( \sum_{i=1}^{r} 2^a - 1 - \sum_{j=1}^{r} 2^b \right) = r - 1 + a + w \]
so that
\[(2.9) \quad \alpha \left( \frac{M}{2} + t \right) + \alpha \left( \frac{M}{2} - t \right) = 2r - 2 + a + b.

We can now rewrite (2.6), obtaining
\[(2.10) \quad \sum_{1}^{(*)} \alpha(n) = \sum_{0 < |z| < 2^\nu} \left( \frac{M}{M/2 + 1} \right) \left( r - 1 + \frac{a}{2} + \frac{b}{2} \right) + \left( \frac{M}{M/2} \right) a_r.\]

Chebyshev's inequality implies that
\[\sum_{|z| < 2^\nu} \left( \frac{M}{M/2 + t} \right) \left( r + \frac{a}{2} \right) = O(x)\]
as in the analysis of \(\Sigma_2\). Since
\[\left( \frac{M}{M/2} \right) \left( r + a_r/2 \right) = O(x \log \log x / \sqrt{\log x}) = O(x),\]
we obtain
\[(2.11) \quad \sum_{|z| < 2^\nu} \left( \frac{M}{M/2 + t} \right) \left( r + \frac{a}{2} \right) = 2^m \left( r + \frac{a}{2} \right) + O(x) = \left( r + \frac{a}{2} \right) + O(x).\]

Thus it remains only to show that each remaining term is \(O(x)\). We have already seen that
\[(2.12) \quad \left( \frac{M}{M/2} \right) a_r/2 = O(x)\]
and easily obtain
\[(2.13) \quad \sum_{|z| < 2^\nu} \left( \frac{M}{M/2 + t} \right) (-1) = O(2^m) = O(x).\]

We estimate the remaining term by observing that \(b_w = b_w(t)\) is the largest exponent such that \(2^{b_w} | t\). Thus we can write
\[
\sum_{0 < |t| < 2^s} \left( \frac{M}{M/2 + t} \right) b_n \leq \sum_{t \in 2i} \left( \frac{M}{M/2 + t} \right) \leq \sum_{i \geq 0} \sum_{q > 0} \left( \frac{M}{M/2 + 2^q} \right)
\]

(2.14)

\[
\leq \sum_{i \geq 0} \frac{1}{2^i} \sum_{q > 0} \left( \frac{M}{q} \right) = O(x).
\]

This completes the proof of Lemma 1.

Lemma 1 implies that

(2.15)

\[
c(x) = \frac{r + a_r / 2}{\log \log x/2 \log 2} + o(1).
\]

Since \(a_1 = \log \log x/2 \log 2 + O(1)\), we obtain

(2.16)

\[
c(x) = \frac{r + a_r / 2}{a_1} + o(1).
\]

We now complete the proof of Theorem 1 by showing that if \(\epsilon > 0\) then there exist arbitrarily large \(q\) such that if \((1/2 + \epsilon)q < z < (3/2 - \epsilon)q\) is an integer, then there exists \(x \in \mathcal{M}\) such that \(a_1 = q\) and \(r + a_r/2 = z\).

Suppose we choose \((1/2 + \epsilon)q - 4 \leq s < (1/2 + \epsilon)q - 2, s\) even. As \(t\) takes on all possible integer values between \(2\) and \(q - s\), \(t + s/2\) certainly takes on all integer values between \((1/2 + \epsilon)q\) and \((3/2 - \epsilon)q\).

If \(q\) is large enough, it is certainly possible to find \(x \in \mathcal{M}\) such that \(a_1 = q, r = t\) and \(a_s = s\), completing the proof of Theorem 1.

3. We carry out the proof of Theorem 2 in a series of steps.

Let \(\mathcal{M}^1 = \{x: x = 2^M - 1, M\) even, \(M/2 = \Sigma_{i=1}^r 2^n - 1, a_1 > a_2 > \cdots > a_r\) integers and \(a_r/a_1 \geq (1/2)\log \log \log x / (\log \log x + \log \log 2)\}).

We begin by proving the conclusion of Theorem 2 holds for element of \(\mathcal{M}^1\).

**Lemma 2.** If \(x \in \mathcal{M}^1\) then

(3.1)

\[
\frac{1}{2} + O \left( \frac{\log \log \log x}{\log \log x} \right) < c(x) < \frac{3}{2} + O \left( \frac{\log \log \log x}{\log \log x} \right).
\]

**Proof.** We begin as in Lemma 1, writing

(3.2)

\[
A_2(x) = \sum_{n} \left( \frac{M}{n} \right) \alpha(n) + \sum_{n} \left( \frac{M}{n} \right) \alpha(n)
\]
except where \( \Sigma_1 \) is the sum over \( \{ n : |M/2 - n| < 2^{(d+\epsilon)a_1} \} \) and \( \Sigma_2 \) is the sum over \( \{ n : |M/2 - n| \geq 2^{(d+\epsilon)a_1} \} \), where \( \epsilon = \epsilon(x) = \log \log \log x/(\log \log x + \log \log 2) \).

The second term can be estimated as the corresponding term was in Lemma 1, yielding

\[
\sum_1 \binom{M}{n} \alpha(n) = O(x).
\]

We estimate the first sum by considering two cases.

**Case 1.** \( a_i \geq (1/2 + \epsilon)a_1 \). We can treat this case as we treated Lemma 1, obtaining \( \Sigma_1 \binom{M}{n} \alpha(n) = (r + a_i/2)x + O(x) \) and hence

\[
A_2(x) = (r + a_i/2)x + O(x).
\]

Since \( 0 \leq r \leq a_i - a_i + 1 \), we obtain \( a_i/2 \leq r + a_i/2 \leq a_i - a_i/2 + 1 \). Since \( (1/2 + \epsilon)a_i \leq a_i \leq a_i \), we obtain

\[
\left( \frac{3}{4} - \frac{\epsilon}{2} \right) a_i + 1.
\]

But \( a_i = (\log \log x/\log 2) + O(1) \), so

\[
\left( \frac{1}{4} + \frac{\epsilon}{2} \right) \frac{\log \log x}{\log 2} + O(1) \leq r + a_i/2 \leq \left( \frac{3}{4} - \frac{\epsilon}{2} \right) \frac{\log \log x}{\log 2} + O(1)
\]

which implies

\[
\frac{1}{4} \frac{\log \log x}{\log 2} + O(\log \log \log x) \leq r + a_i/2 \leq \frac{3}{4} \frac{\log \log x}{\log 2} + O(\log \log \log x).
\]

Thus

\[
\left( \frac{1}{2} + O\left( \frac{\log \log \log x}{\log \log x} \right) \right) \left( \frac{x \log \log x}{\log x} \right) \leq A_2(x)
\]

\[
\leq \left( \frac{3}{2} + O\left( \frac{\log \log \log x}{\log \log x} \right) \right) \left( \frac{x \log \log x}{\log x} \right)
\]

which proves Lemma 2 for this case.
Case 2. \((1/2 - \epsilon)a_1 < a < (1/2 + \epsilon)a_1\).

As in Lemma 1, we write

\[
\sum_{t} \binom{M}{n} \alpha(n) = \frac{1}{2} \sum_{0 \neq t < 2^{(1/2+\epsilon)n}} \binom{M}{M/2 + t} \left\{ \alpha\left(\frac{M}{2} - t\right) + \alpha\left(\frac{M}{2} + t\right) \right\}
\]

(3.9)

\[+ \binom{M}{M/2} \alpha\left(\frac{M}{2}\right).\]

Here, however, an overlap of the nonzero digits in the binary representations of \(t\) and \(M/2 + 1\) forces us to use the subadditive properties of \(\alpha\). Writing \(M/2\) and \(t\) as before, we obtain

\[
\frac{M}{2} + t = 2^{a_1} + \cdots + 2^{a_r} + 2^{b_1} + \cdots + 2^{b_w} - 1
\]

(3.10)

\[
\frac{M}{2} - t = 2^{a_1} + \cdots + 2^{a_r} - 2^{b_1} - \cdots - 2^{b_w} - 1.
\]

The subadditivity of \(\alpha\) implies \(\alpha(M/2 + t) \leq \alpha(M/2 + 1) + \alpha(t - 1)\) so that

\[
\alpha\left(\frac{M}{2} + t\right) \leq r + w + b_w.
\]

(3.11)

Also, \(\alpha(M/2 + t)\) is at least \(\alpha(t)\) minus the overlap between the binary expansions of \(M/2\) and \(t\), so that

\[
\alpha\left(\frac{M}{2} + t\right) \geq w - 2\epsilon a_1.
\]

(3.12)

Since \(\alpha(M/2 - t)\) is no greater than the number of places available, less \(\alpha(t)\), plus the overlap, we obtain

\[
\alpha\left(\frac{M}{2} - t\right) \leq a_1 + 1 - w + 2\epsilon a_1.
\]

(3.13)

Also, \(\alpha(M/2 - t)\) must be at least the number of 1's that \(M/2\) ends with less \(\alpha(t)\), so that

\[
\alpha\left(\frac{M}{2} - t\right) \geq a_r - w.
\]

(3.14)

Combining (3.11)–(3.14) we obtain
(3.15) \( a_r - 2\varepsilon a_1 \leq \alpha \left( \frac{M}{2} + t \right) + \alpha \left( \frac{M}{2} - t \right) \leq a_1 + r + b_w + 2\varepsilon a_1 + 1. \)

Since \( a_r > (1/2 - \varepsilon)a_1 \) and \( r \leq a_1 - a_r + 1 < a_1 - (1/2 - \varepsilon)a_1 + 1 = (1/2 + \varepsilon)a_1 + 1 \) we obtain

(3.16) \( \left( \frac{1}{2} - 3\varepsilon \right) a_1 \leq \alpha \left( \frac{M}{2} + t \right) + \alpha \left( \frac{M}{2} - t \right) \leq \left( \frac{3}{2} + 3\varepsilon \right) a_1 + b_w + 1. \)

Plugging the first inequality of (3.16) into (3.9) yields

\[
\sum_1 \left( \frac{M}{n} \right) \alpha(n) \geq \frac{1}{2} \left( \frac{1}{2} - 3\varepsilon \right) a_1 \sum_{0 < d_i < \sqrt{2^{a_i}}} \left( \frac{M}{M/2 + t} \right) + \left( \frac{M}{M/2} \right) \alpha \left( \frac{M}{2} \right) \leq a_1 + b_w + 1.
\]

Chebyshev’s inequality yields

\[
\sum_{0 < d_i < \sqrt{2^{a_i}}} \left( \frac{M}{M/2 + t} \right) = x + O \left( \frac{x}{2^{2\varepsilon a_1}} \right)
\]

which implies

(3.17) \( \sum_1 \left( \frac{M}{n} \right) \alpha(n) \geq \frac{1}{4} a_1 x - \frac{3}{2} \varepsilon a_1 x + O(x). \)

Recalling \( a_1 = (\log \log x) / \log 2 + O(1) \) and combining (3.17) with (3.3) yields

(3.18) \( A_2(x) \geq \frac{1}{2} \frac{x \log \log x}{2 \log 2} + O(x \log \log \log x) \)

which implies

(3.19) \( c(x) \geq \frac{1}{2} + O \left( \frac{\log \log x}{\log x} \right). \)

Plugging the second inequality of (3.16) into (3.9) yields

\[
\sum_1 \left( \frac{M}{n} \right) \alpha(n) \leq \frac{1}{2} \left( \frac{3}{2} + 3\varepsilon \right) a_1 \sum_{0 < d_i < \sqrt{2^{a_i}}} \left( \frac{M}{M/2 + t} \right) + \frac{1}{2} \sum_{0 < \|d\| < \sqrt{2^{a_2}}} \left( \frac{M}{M/2 + t} \right) (b_w + 1) + \left( \frac{M}{M/2} \right) \alpha \left( \frac{M}{2} \right).
\]
As in (2.14) we see that

\[ \sum_{0<|x|<2^{1/2+\epsilon}a_1} \binom{M}{M/2 + t} (b_v + 1) = O(x) \]

to obtain

\[ \sum_1 (\binom{M}{n} \alpha(n)) \approx \frac{3}{4} a_1 x + \frac{3}{2} \epsilon a_1 x + O(x). \]

Repeating the reasoning of (3.17)-(3.19) yields

\[ c(x) \leq \frac{3}{2} + O \left( \frac{\log \log \log x}{\log \log x} \right). \]

Combining (3.19) with (3.22) completes the proof of Lemma 2.

We now consider a lemma which will enable us to extend the conclusion of Theorem 2 to all integers of the form $2^n - 1$.

**Lemma 3.** If $x = 2^N - 1$, then there exists an even integer $M \geq N$ such that $M - N \leq \sqrt{N}/\log N$, $M/2 = \sum_{i=1}^\ell a_i - 1$ with $a_i/a_1 \geq 1/2 - \epsilon$ and

\[ A_2(x) = \frac{A_2(2^M - 1)}{2^{M-N}} + O(x), \]

where

\[ \epsilon = \epsilon(x) = \frac{\log \log \log x}{\log \log x + \log \log 2}. \]

**Proof.** Let $N = \sum_{i=1}^\ell 2^{a_i}$, $a_1 > a_2 > \cdots > a_{\ell}$. Define $n$ by $n + 1 = \sum_{i=1}^\ell 2^{c_i}$, where $\{c_i\}$ runs over all integer values in the interval $[1, (1/2)a_1(1-2\epsilon)+1]$ not equal to any of the $a_i$'s. If no such $c$'s exist, let $n = 0$ if $N$ is even, $n = 1$ if $N$ is odd. Let $M = N + n$. Clearly $n = M - N \leq 2(1/2)a_1(1-2\epsilon) \leq N^{1/2 - \epsilon} \leq \sqrt{N}/\log N$ and only (3.23) requires further analysis.

As before, $A_2(2^M - 1) = \sum_{s \leq M} \binom{M}{s} \alpha(s)$.

We rewrite this as

\[ A_2(2^M - 1) = s_1 + s_2 \]

where
(3.25) \[ s_1 = 2^n \sum_s \binom{N}{s} \alpha(s) = 2^{M-N} A_2(2^N - 1) \]

(3.26) \[ s_2 = \sum_s \left\{ \binom{M}{s} - 2^n \binom{N}{2} \right\} \alpha(s). \]

We bound \( s_2 \) from above by writing

(3.27) \[ s_2 \ll \log M \sum_s \left| \binom{M}{s} - 2^n \binom{N}{2} \right|. \]

But

\[
\sum_s \left| \binom{M}{s} - 2^n \binom{N}{s} \right| = \sum_s \left| \sum_q \binom{N}{s-q} \binom{n}{q} - \binom{N}{s} \binom{n}{q} \right|
\]

\[
= \sum_q \binom{n}{q} \sum_s \left| \binom{N}{s-q} - \binom{N}{s} \right|
\]

\[
= \sum_q \binom{n}{q} \cdot 2q \max \binom{N}{s} \ll n \cdot 2^n \cdot 2^N/\sqrt{N}
\]

\[
\ll \frac{\sqrt{N}}{\log N} \cdot 2^n \cdot 2^N/\sqrt{N} = \frac{2^{N+n}}{\log N}
\]

so \( s_2 \ll 2^{N+n} \ll 2^n x = 2^{M-N} x \) and

(3.28) \[ A_2(2^M - 1) = 2^{M-N} \{ A_2(x) + O(x) \}, \]

proving the lemma.

**Corollary 1.** If \( x = 2^N - 1 \), then

\[
\frac{1}{2} + O \left( \frac{\log \log \log x}{\log \log x} \right) \leq c(x) \leq \frac{3}{2} + O \left( \frac{\log \log \log x}{\log \log x} \right).
\]

**Proof.** Find an \( M \) as in Lemma 3 so that

\[
A_2(x) = \frac{A_2(2^M - 1)}{2^{M-N}} + O(x).
\]

Applying Lemma 2 to \( 2^M - 1 \) immediately yields this result.

**Lemma 4.** Let \( x = \sum_{i=1}^{s_r} 2^r, \ s_1 > s_2 > \cdots > s_r \). Then
Proof.

\[ A_2(x) = \sum_{i=1}^{r} \sum_{n \leq 2^i} \alpha_2 \left( \sum_{j=1}^{i-1} 2^n + n \right). \]

Since \( \alpha(\sum_{j=1}^{i-1} 2^n + n) = \alpha(n) + i - 1 \) we obtain

\[ A_2(x) = \sum_{i=1}^{r} \sum_{n \leq 2^i} \alpha(\alpha(n) + i - 1). \]

Letting \( E_i = \sum_{n \leq 2^i} \{ \alpha(\alpha(n) + i - 1) - \alpha(\alpha(n)) \} \), we obtain

\[ A_2(x) = \sum_{i=1}^{r} A_2(2^i - 1) + \sum_{i=2}^{r} E_i. \]

We now must merely show that \( \sum_{i=2}^{r} E_i = O(x \log \log x / \sqrt{\log x}) \).

Rewrite

\[ E_i = \sum_{l \leq n} \sum_{n \leq 2^i - 1} \{ \alpha(l + i - 1) - \alpha(l) \} \]

\[ = \sum_{l} \binom{s_i}{l} \{ \alpha(l + i - 1) - \alpha(l) \}. \]

Summing by parts,

\[ E_i = \sum_{l} \alpha(l) \left\{ \binom{s_i}{l} - \binom{s_i}{l-i+1} \right\}. \]

Since \( \alpha(l) = O(\log(s_i + i)) \) and

\[ \sum_{l} \left| \binom{s_i}{l-i+1} - \binom{s_i}{l} \right| = O \left( i \left( \frac{s_i}{[s_i]/2} \right) \right) = O \left( i \frac{2^i}{\sqrt{s_i}} \right) \]

we obtain

\[ E_i \ll i \cdot \log(s_i + i) 2^i / \sqrt{s_i}. \]

Thus

\[ (3.31) \]
\[ \sum_{i=2}^{r} E_i \leq \sum_{i=2}^{r} i \log (s_i + i) 2^{i} / \sqrt{s_i}. \]

Since \( s_i \leq s_1 - i + 1 \) and \( s_i + i \leq \log x \), and writing \( s = s_1 \), we obtain

\[ (3.32) \quad \sum_{i=2}^{r} E_i \leq \log \log x \sum_{i=1}^{r} i 2^{s_i} / \sqrt{s} - i. \]

Now

\[ \sum_{i \leq s/2} i 2^{s_i} / \sqrt{s} \leq \frac{2^s}{\sqrt{s}} \sum_{i \leq s/2} i \leq 2^s / \sqrt{s} \]

while

\[ \sum_{i > s/2} i 2^{s_i} / \sqrt{s} \leq \sum_{i > s/2} i 2^{s_i} \leq s \cdot 2^{s/2} \leq \frac{2^s}{\sqrt{s}}. \]

Since \( 2^s = O(x) \) and \( s = \log x / \log 2 + O(1) \), we obtain

\[ (3.33) \quad \sum_{i=2}^{r} E_i = O(x \log \log x / \sqrt{\log x}), \]

completing the proof of Lemma 4.

We can now easily prove Theorem 2.

**Proof of Theorem 2.** Let \( x = \Sigma_{i=1}^{r} 2^i \). By Lemma 4,

\[ (3.34) \quad A_2(x) = \sum_{i=1}^{r} A_2(2^i - 1) + O(x). \]

Corollary 1 implies that

\[ A_2(2^i - 1) \leq \frac{3}{2} \frac{2^i \log \log x}{2 \log 2} + O(2^i \log \log \log x), \]

so that

\[ A_2(x) \leq \frac{3}{2} \sum_{i=1}^{r} \left( \frac{2^i \log \log x}{2 \log 2} + O(2^i \log \log \log x) \right) + O(x) \]

and hence
(3.35) \[ A_2(x) \leq \frac{3}{2} \frac{x \log \log x}{2 \log 2} + O(x \log \log \log x) \]

which is equivalent to

(3.36) \[ c(x) \geq \frac{3}{2} + O\left(\frac{\log \log \log x}{\log x}\right). \]

We now obtain a lower bound. Again using Corollary 1, we obtain

\[ A_2(x) \geq \frac{1}{2} \sum_{i=1}^{r} \left(\frac{2^s \log \log 2^s}{2 \log 2} + O(2^s \log \log \log x)\right) + O(x) \]

and hence

(3.37) \[ A_2(x) \geq \frac{1}{2} \sum_{s \geq s_i/2} \frac{2^s \log \log 2^s}{2 \log 2} + O(x \log \log \log x). \]

Since \( \log \log 2^s = \log \log x + O(1) \) if \( s \geq s_i/2 \), we obtain

(3.38) \[ A_2(x) \geq \frac{1}{2} \sum_{s \geq s_i/2} \frac{2^s \log \log x}{2 \log 2} + O(x \log \log \log x). \]

But \( \sum_{s \geq s_i/2} 2^s = x + O(2^{s_i/2}) = x + O(\sqrt{x}) \) yielding

(3.39) \[ A_2(x) \geq \frac{1}{2} \frac{x \log \log x}{2 \log 2} + O(x \log \log \log x) \]

which implies

(3.40) \[ c(x) \geq \frac{1}{2} + O\left(\frac{\log \log \log x}{\log x}\right), \]

completing the proof of Theorem 2.

References


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