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It is proved that for arbitrary locally compact groups G the automorphism group $\mathrm{Aut}(G)$ is a complete topological group. Several conditions equivalent to closedness of the group $\mathrm{Int}(G)$ of inner automorphisms are given, such as G admits no nontrivial central sequences. It is shown that $\mathrm{Aut}(G)$ is topologically embedded in the automorphism group $\mathrm{Aut}\,\mathscr{R}(G)$ of the group von Neumann algebra. However, closedness of $\mathrm{Int}\,\mathscr{R}(G)$ does not imply closedness of $\mathrm{Int}(G)$, nor conversely.

1. Let G be a locally compact group and $\mathrm{Aut}(G)$ the group of all its topological automorphisms with the Birkhoff topology. A neighborhood basis of the identity automorphism consists of sets $N(C, V) = \{\alpha \in \mathrm{Aut}(G): \alpha(x) \in Vx \text{ and } \alpha^{-1}(x) \in Vx, \text{ all } x \in C\}$, where C is compact and V is a neighborhood of the identity e of G. As is well known, $\mathrm{Aut}(G)$ is a Hausdorff topological group but not generally locally compact [1; p. 57]. In this article we are mainly concerned with the topological properties of $\mathrm{Aut}(G)$ and its subgroup $\mathrm{Int}(G)$ of inner automorphisms. We prove that for G arbitrary locally compact $\mathrm{Aut}(G)$ is a complete topological group. In particular, if G is also separable $\mathrm{Aut}(G)$ is a Polish group. Furthermore, we give two new characterizations of the topology for $\mathrm{Aut}(G)$, (1.1 and 1.6). In §2 we turn to the question of when certain subgroups (among them $\mathrm{Int}(G)$) are closed in $\mathrm{Aut}(G)$, and several equivalent conditions are given; for instance, $\mathrm{Int}(G)$ is closed iff G admits no nontrivial central sequences (2.2). Applications to more special classes of groups are also given, as well as to the question of unimodularity of $\mathrm{Int}(G)$, (2.7). We remark that there is no separability assumption on the groups before 1.11.

LEMMA 1.1. *The sets*

$$W_{\phi_1,\ldots,\phi_n;\varepsilon} = \{\tau \in \mathrm{Aut}(G);\ \|\phi_j \circ \tau - \phi_j\|_\infty < \varepsilon,\ 1 \leqq j \leqq n\}$$

where $\phi_j \in C_c(G)$ and $\varepsilon > 0$, form a basis for the neighborhoods of the identity in $\mathrm{Aut}(G)$.

Proof. Let $\phi_1, \cdots, \phi_n \in C_c(G)$ and $\varepsilon > 0$ be given. Note that $\|\phi_j \circ \tau - \phi_j\|_\infty < \varepsilon$ implies $\|\phi_j \circ \tau^{-1} - \phi_j\|_\infty < \varepsilon$, $\tau \in \mathrm{Aut}(G)$. Set $F = \bigcup_{i=1}^n$ support (ϕ_i), and let W be a symmetric neighborhood of e in G such that $|\phi_i(x) - \phi_i(wx)| < \varepsilon$ for all $x \in G, w \in W, 1 \leqq i \leqq n$. We claim

$N(F, W) \subseteq W_{\phi_1,\ldots,\phi_n;\varepsilon}$. Let $\tau \in N(F, W)$. Then for $x \in F$, $\tau(x)x^{-1} \in W$, so

$$(*) \qquad |\phi_i(x) - \phi_i(\tau(x))| < \varepsilon , \qquad 1 \leqq i \leqq n .$$

If $\tau(x) \in F$, then $\tau^{-1}(\tau(x))\tau(x)^{-1} \in W$, i.e., $x\tau(x)^{-1} \in W$, so $(*)$ holds. If $x \notin F$ and $\tau(x) \notin F$ then $\phi_i(x) = \phi_i(\tau(x)) = 0$, so again $(*)$ is satisfied.

Conversely, let $F \subset G$ be compact and W a neighborhood of e in G. Let U be a compact neighborhood of e in G such that $U^2 \cdot U^{-1} \subset W$. Let $\psi \in C_c(G)$ be such that $0 \leqq \psi \leqq 1$, support $(\psi) \subset U^2$, and $\psi(u) \geqq 1/2\ \forall u \in U$. (The existence of such a ψ is clear.) Let $\{x_1, \cdots, x_n\}$ be a finite subset of F such that $\{U_{x_i}: 1 \leqq i \leqq n\}$ covers F. Define $\psi_i \in C_c(G)$ by $\psi_i(x) = \psi(xx_i^{-1})$, $1 \leqq i \leqq n$. It is now routine to verify that $W_{\psi_1,\ldots,\psi_n;1/2} \subset N(F, W)$.

1.2. By Braconnier [1] there is a continuous (modular) homomorphism Δ: Aut $(G) \rightarrow R^+$ with the property

$$\Delta(\alpha)^{-1} \int_G f \circ \alpha^{-1}(x)dx = \int_G f(x)dx , \qquad \text{for} \quad f \in C_c(G) ,$$

where dx is a fixed Haar measure. Defining

$$\tilde{\alpha}(f) = \Delta(\alpha)^{-1} f \circ \alpha^{-1},\ f \in L^1(G),\ \alpha \in \text{Aut}\,(G) ,$$

it is easy to see that $\tilde{\alpha}$ becomes an automorphism of the group algebra $L^1(G)$. Denote by λ the left regular representation of G as well as the left regular representation of $L^1(G)$ on $L^2(G)$. Viewing $\tilde{\alpha}$, $\alpha \in$ Aut (G), as an automorphism of $\lambda(L^1(G))$, we show that $\tilde{\alpha}$ can be extended to an automorphism of the von Neumann algebra of the left regular representation, $\mathscr{R}(G) = \lambda(L^1(G))'' = \lambda(G)''$. We define a unitary operator U^α, $\alpha \in$ Aut (G), by

$$U^\alpha g = \Delta(\alpha)^{-1/2} g \circ \alpha^{-1} , \quad g \in L^2(G) .$$

A straight forward calculation shows

$$\lambda(\tilde{\alpha}(f)) = U^\alpha \lambda(f) U^{\alpha^{-1}} .$$

The unitary implementation $\alpha \mapsto U^\alpha$ allows us to define $\tilde{\alpha}(T)$ for $T \in \mathscr{R}(G)$ by

$$\tilde{\alpha}(T) = U^\alpha T U^{\alpha^{-1}} .$$

LEMMA 1.3. *The map* $\alpha \in$ Aut $(G) \mapsto U^\alpha g \in L^2(G)$ *is continuous* $(g \in L^2(G))$.

Proof. This follows from Proposition 2, page 78 of [1].

1.4. Our next aim is to study Aut (G) by embedding it in Aut $\mathscr{R}(G)$, and we shall prove that the embedding is topological if

Aut $\mathscr{R}(G)$ is provided with the appropriate topology, namely the uniform-weak topology. A neighborhood base at the identity $\iota \in$ Aut $\mathscr{R}(G)$ is given by

$$\{\alpha \in \operatorname{Aut} \mathscr{R}(G): | < (\alpha - \iota)\mathscr{R}_1, \phi_i > | < \varepsilon, \phi_i \in \mathscr{R}(G)_*, 1 \leqq i \leqq n\} ,$$

where $\varepsilon > 0$ and $\mathscr{R}_1$ denotes the unit ball in $\mathscr{R}(G)$. Recall that the predual, $\mathscr{R}(G)_*$, is the Fourier algebra $A(G)$ (see [5]). Let

$$W_{\phi_1 .. \phi_n;\varepsilon} = \{\alpha \in \operatorname{Aut}(G): ||\phi_i - \phi_i \circ \alpha|| < \varepsilon, 1 \leqq i \leqq n\} , \quad \phi_i \in A(G) ,$$

where $||\cdot||$ denotes the norm in $A(G)$.

LEMMA 1.5.

$$W_{\phi_1,\dots,\phi_n;\varepsilon} = \{\alpha \in \operatorname{Aut}(G): | < (\tilde{\alpha} - \iota)\mathscr{R}_1, \phi_i > | < \varepsilon , \qquad 1 \leqq i \leqq n .$$

Proof. First note $\langle\tilde{\alpha}(T), \phi\rangle = \langle T, \phi\circ\alpha\rangle$ for $T \in \mathscr{R}(G)$, $\phi \in A(G)$ and $\alpha \in \operatorname{Aut}(G)$; i.e., $\tilde{\alpha}^t(\phi) = \phi\circ\alpha$: If $T = \lambda(f)$, $f \in L^1(G)$, we have

$$\langle\tilde{\alpha}(\lambda(f)), \phi\rangle = \Delta(\alpha)^{-1}\int_G f\circ\alpha^{-1}(x)\phi(x)dx = \langle\lambda(f), \phi\circ\alpha\rangle .$$

Since $\{\lambda(f): f \in L^1(G)\}$ is dense in $\mathscr{R}(G)$, the claim follows. Now $\langle(\tilde{\alpha} - \iota)T, \phi\rangle = \langle T, \phi\circ\alpha - \phi\rangle$, $T \in \mathscr{R}_1$. Taking the supremum over all $T \in \mathscr{R}_1$ we get

$$\sup_{T \in \mathscr{R}_1} \langle(\tilde{\alpha} - \iota)T, \phi\rangle = ||\phi\circ\alpha - \alpha|| , \qquad \phi \in A(G) ,$$

and the lemma follows.

PROPOSITION 1.6. *The sets* $W_{\phi_1,\dots,\phi_n;\varepsilon}$, $\phi_i \in A(G)$ *and* $\varepsilon > 0$, *form a base at the identity* $\iota \in \operatorname{Aut}(G)$ *for the Birkhoff topology. Hence the embedding* $\operatorname{Aut}(G) \hookrightarrow \operatorname{Aut}\mathscr{R}(G)$ *is topological.*

Proof. We show first that the topology generated by the sets $W_{\phi_1,\dots,\phi_n;\varepsilon}$ is weaker then that of Aut(G). The proof of Lemma 1.5 shows that for $\phi \in A(G)$, $\alpha \in \operatorname{Aut}(G)$.

$$||\phi - \phi\circ\alpha|| = \sup_{T \in \mathscr{R}_1} | \langle T - \tilde{\alpha}(T), \phi\rangle | .$$

Writing $\phi = (f\circ\tilde{g})^\vee$, $f, g \in L^2(G)$, we have

$$\begin{aligned} ||\phi - \phi\circ\alpha|| &= \sup_{T \in \mathscr{R}_1} |\langle(T - \tilde{\alpha}(T))f, g\rangle| \\ &= \sup_{T \in \mathscr{R}_1} |\langle(T - U^\alpha T U^{\alpha^{-1}})f, g\rangle| \\ &= \sup_{T \in \mathscr{R}_1} |\langle(U^{\alpha^{-1}}T - TU^{\alpha^{-1}})f, U^{\alpha^{-1}}g\rangle| \\ &\leqq \sup_{T \in \mathscr{R}_1} |\langle(U^{\alpha^{-1}}T - T)f, U^{\alpha^{-1}}g\rangle| \\ &\quad + \sup_{T \in \mathscr{R}_1} | \langle(T - U^{\alpha^{-1}})f, U^{\alpha^{-1}}g\rangle | . \end{aligned}$$

Now

$$\begin{aligned}
|\langle (T - TU^{\alpha^{-1}})f, U^{\alpha^{-1}}g\rangle| &\leqq \|T(f - U^{\alpha^{-1}}f)\|_2 \|U^{\alpha^{-1}}g\|_2 \\
&\leqq \|f - U^{\alpha^{-1}}f\|_2 \|g\|_2, \qquad \text{all} \quad T \in \mathscr{R}_1 . \\
|\langle (U^{\alpha^{-1}}T - T)f, U^{\alpha^{-1}}g\rangle| & \\
&= |\langle U^{\alpha^{-1}}Tf, U^{\alpha^{-1}}g\rangle - \langle Tf, U^{\alpha^{-1}}g\rangle| \\
&= |\langle Tf, g\rangle - \langle Tf, U^{\alpha^{-1}}g\rangle| = |\langle Tf, g - U^{\alpha^{-1}}g\rangle| \\
&\leqq \|Tf\|_2 \|g - U^{\alpha^{-1}}g\|_2 \leqq \|f\|_2 \|g - U^{\alpha^{-1}}g\|_2 , \\
&\qquad\qquad \text{all} \quad T \in \mathscr{R}_1 .
\end{aligned}$$

Let N be a neighborhood of $\iota \in \mathrm{Aut}(G)$ such that $\|f - U^{\alpha^{-1}}f\|_2 \|g\|_2 < \varepsilon/2$ and $\|f\|_2 \|g - U^{\alpha^{-1}}g\|_2 < \varepsilon/2$. Then $\|\phi - \phi\circ\alpha\| < \varepsilon$.

Conversely, let $F \subset G$ be compact and W a neighborhood of e in G. Let U be a compact neighborhood of e such that $U^2 \cdot U^{-1} \subset W$.

Since $A(G)$ is a regular algebra, there exists $\psi \in A(G)$ with $0 \leqq \psi \leqq 1$, $\psi(u) = 1$ for $u \in U$, and support $(\psi) \subset U^2$ [5; Lemma 3.2]. Let $\{x_1, \cdots, x_n\} \subset F$ be so that $\{Ux_i: 1 \leqq i \leqq n\}$ covers F. Define $\psi_i(y) = \psi(yx_i^{-1})$, $1 \leqq i \leqq n$. We claim $W_{\psi_1,\ldots,\psi_n;1} \subset N(F, W)$. Indeed, suppose $\tau \in W_{\psi_1,\ldots,\psi_n;1}$ and let $x \in F$. Then $x \in Ux_j$ for some j. Now $\|\psi_j\circ\tau - \psi_j\| < 1$ implies $\|\psi_j\circ\tau - \psi_j\|_\infty < 1$, so that $|\psi_j\circ\tau(x) - \psi_j(x)| < 1$. But for $x \in Ux_j$, $\psi_j(x) = \psi(xx_j^{-1}) = 1$. Hence $\tau(x) \in$ support (ψ_j), or $\tau(x) \in U^2x_j$. But then

$$\tau(x)x^{-1} \in U^2x_jx^{-1} \in U^2U^{-1} \subset W .$$

In addition

$$\|\psi_j\circ\tau^{-1} - \psi_j\|_\infty = \|\psi_j\circ\tau - \psi_j\|_\infty < 1 ,$$

so the same argument as above yields $\tau^{-1}(x) \in Wx$.

COROLLARY 1.7. *Suppose G has small neighborhoods of the identity, invariant under inner automorphisms* (i.e., $G \in$ [SIN]). *Then viewing the group* Int (G) *as a subgroup of* Aut $\mathscr{R}(G)$, *the pointwise-weak and uniform-weak topologies coincide on* Int (G).

Proof. $G \in$ [SIN] if and only if $\mathscr{R}(G)$ is a finite von Neumann algebra, [4; 13. 10.5]. The conclusion follows from [10; Proposition 3.7].

Note that the above can just as well be stated for $[\mathrm{SIN}]_B$-groups where $B \subset \mathrm{Aut}(G)$ is a subgroup. Also, the corollary is not too surprising in view of the fact that for [SIN]-groups the point-open and Birkhoff topologies of Aut (G) agree on Int (G) [9; Satz 1.6].

1.8. We say that G is an $[\mathrm{FIA}]_B^-$-group if B is a relatively compact subgroup of Aut (G) (see [7]). It is now a trivial consequence of 1.6 that $G \in [\mathrm{FIA}]_B^-$ if and only if B, viewed as a subgroup of Aut $\mathscr{B}(G)$ endowed with the uniform-weak topology, is relatively compact. Cf. [6; Theorem 2.4]. By [6; Corollary 1.6], the pointwise-weak topology may be substituted for the uniform-weak topology.

We mention another consequence of Proposition 1.6 which was suggested to us by Kenneth Ross. An important tool in harmonic analysis on $[\mathrm{FIA}]_B^-$-groups is the "sharp operator," which is defined as follows: if f is a continuous function on $G \in [\mathrm{FIA}]_B^-$, then

$$f^\#(x) = \int_{B^-} f \circ \beta(x) d\beta ,$$

where $d\beta$ is normalized Haar measure on the compact group $B^- \subset$ Aut (G). $f^\#$ is a continuous, B-invariant function on G. We show that if f is in the Fourier algebra $A(G)$, so is $f^\#$. By Proposition 1.6 the map $\beta \to f \circ \beta$, Aut $(G) \to A(G)$, is continuous. Viewing $f^\#$ as a vector valued integral, we can then adapt [14; Lemma 1.4] to show that $f^\# \in A(G)$.

1.9. Next we show in an elementary way that for an arbitrary locally compact group G, Aut (G) is a complete topological group (in its two-sided uniformity).

THEOREM. *Let G be a locally compact group; then* Aut (G) *is complete with respect to its two-sided uniformity.*

Proof. Let (α_ν) be a Cauchy net in Aut (G). Since $\alpha \mapsto U^\alpha$, Aut $(G) \to \mathscr{L}(L^2(G))$ is continuous in the strong operator topology, it is also weakly continuous. Now $U^\alpha \in \mathscr{L}(L^2(G))_1$ (=unit ball of $\mathscr{L}(L^2(G))$); also the weak and ultraweak topology coincide on $\mathscr{L}(L^2(G))_1$ and $\mathscr{L}(L^2(G))_1$ is compact in this topology. Thus (U^{α_ν}) has a point of accumulation $U \in \mathscr{L}(L^2(G))_1$; let (α_μ) be a subnet such that $U^{\alpha_\mu} \underset{\mu}{\to} U$ weakly. Then for $f, g \in L^2(G)$

$$\begin{aligned} \langle (U^{\alpha_\nu} - U)f, g\rangle &= \langle (U^{\alpha^1_\nu} - U^{\alpha_\mu})f, g\rangle + \langle (U^{\alpha_\mu} - U)f, g\rangle \\ &= \langle f - U^{\alpha_\nu^{-1}\alpha_\mu} f, U^{\alpha_\nu^{-1}} g\rangle + \langle (U^{\alpha_\mu} - U)f, g\rangle \\ &\leqq \| f - U^{\alpha_\nu^{-1}\alpha_\mu} f \|_2 \| g \|_2 + \langle (U^{\alpha_\mu} - U)f, g\rangle \underset{\mu,\,\nu}{\longrightarrow} 0 \end{aligned}$$

since $\alpha_\nu^{-1}\alpha_\mu \underset{(\nu,\,\mu)}{\longrightarrow} \iota$ in Aut (G). Thus $U^{\alpha_\nu} \underset{\nu}{\to} U$ in the weak operator topology. Similarly $U^{\alpha_\nu^{-1}}$ converges weakly to some $V \in \mathscr{L}(L^2(G))_1$. We claim $V = U^{-1}$. Let $f, g \in L^2(G)$, $\varepsilon > 0$. Let ν_0 be such that for $\nu > \nu_0$

$$|\langle U^{\alpha_\nu}Vf - UVf, g\rangle| < \varepsilon\,, \quad \text{and} \quad \|U^{\alpha_\nu^{-1}}g - U^{\alpha_{\nu_0}^{-1}}g\|_2 < \frac{\varepsilon}{2\|f\|_2}\,.$$

Choose ν_1 such that $\nu > \nu_1$ implies

$$|\langle U^{\alpha_\nu^{-1}}f - Vf, U^{\alpha_{\nu_0}^{-1}}g\rangle| < \varepsilon\,.$$

Then for $\nu, \mu > \nu_0$ and ν_1, we have

$$\begin{aligned}&|\langle U^{\alpha_\mu}U^{\alpha_\nu^{-1}}f - UVf, g\rangle| \\ &\qquad \leqq |\langle U^{\alpha_\mu}U^{\alpha_\nu^{-1}}f - U^{\alpha_\mu}Vf, g\rangle| + |\langle U^{\alpha_u}Vf - UVf, g\rangle|\,,\end{aligned}$$

where $|\langle U^{\alpha_\mu}Vf - UVf, g\rangle| < \varepsilon$. Also

$$\begin{aligned}&|< U^{\alpha_\mu}U^{\alpha}_{\nu^{-1}}f - U^{\alpha_\mu}Vf, g\rangle| = |\langle U^{\alpha-1}_{\nu}f - Vf, U^{\alpha}_{\mu^{-1}}g\rangle| \\ &\leqq |\langle U^{\alpha_\nu^{-1}}f - Vf, U^{\alpha_{\nu_0}^{-1}}g\rangle| + |\langle U^{\alpha_\nu^{-1}}f - Vf, U^{\alpha_\mu^{-1}}g - U^{\alpha_{\nu_0}^{-1}}g\rangle| \\ &< \varepsilon + \|U^{\alpha_\nu^{-1}}f - Vf\|_2\|U^{\alpha_\mu^{-1}}g - U^{\alpha_{\nu_0}^{-1}}g\|_2 \\ &< \varepsilon + 2\|f\|_2\|U^{\alpha_\mu^{-1}}g - U^{\alpha_{\nu_0}^{-1}}g\|_2 < 2\varepsilon\,,\end{aligned}$$

so that

$$|\langle U^{\alpha_\mu}U^{\alpha_\nu^{-1}}f - UVf, g\rangle| < 3\varepsilon\,.$$

But

$$\langle U^{\alpha_\mu}U^{\alpha_\nu^{-1}}f, g\rangle = \langle U^{\alpha_\mu\alpha_\nu^{-1}}f, g\rangle \xrightarrow[(\mu,\nu)]{} \langle f, g\rangle\,,$$

hence

$$\langle UVf, g\rangle = \langle f, g\rangle\,, \qquad \text{all} \quad f, g \in L^2(G);$$

thus $V = U^{-1}$. In addition,

$$\langle Uf, g\rangle = \lim_\nu < U^{\alpha_\nu}f, g\rangle = \lim_\nu \langle f, U^{\alpha_\nu^{-1}}g\rangle = \langle f, Vg\rangle\,,$$

so $V = U^*$, and we have $U^{-1} = U^*$, so U is unitary. A standard argument shows U^{α_ν} converges strongly to U:

$$\begin{aligned}&\|U^{\alpha_\nu}f - Uf\|_2^2 = \langle U^{\alpha_\nu}f, U^{\alpha_\nu}f\rangle - \langle Uf, U^{\alpha_\nu}f\rangle \\ &\quad - \langle U^{\alpha_\nu}f, Uf\rangle + \langle Uf, Uf\rangle = 2\langle f, f\rangle - \langle Uf, U^{\alpha_\nu}f\rangle \\ &\quad - \langle U^{\alpha_\nu}f, Uf\rangle \xrightarrow[\nu]{} 0\,.\end{aligned}$$

It remains to show that $\lambda(x) \mapsto U\lambda(x)U^{-1}$ defines an automorphism of $\lambda(G)$ (and thus of G). Fix $x \in G$; clearly $(\alpha_\nu(x))$ is a Cauchy net in G and (since G is complete) converges to an element, say $\alpha(x) \in G$. Then

$$U^{\alpha_\nu}\lambda(x)U^{\alpha_\nu^{-1}} = \lambda(\alpha_\nu(x)) \xrightarrow[\nu]{} \lambda(\alpha(x)) \quad \text{weakly},$$

and

$$U^{\alpha_\nu}\lambda(x)U^{\alpha_\nu^{-1}} \xrightarrow[\nu]{} U\lambda(x)U^{-1} \quad \text{weakly}.$$

So $\lambda(\alpha(x)) = U\lambda(x)U^{-1}$. To prove α is a homomorphism,

$$\lambda(\alpha(xy)) = U\lambda(xy)U^{-1} = (U\lambda(x)U^{-1})(U\lambda(y)U^{-1}) = \lambda(\alpha(x))\lambda(\alpha(y)) \\ = \lambda(\alpha(x)\alpha(y));$$

so $\alpha(xy) = \alpha(x)\alpha(y)$. Also $\lambda(\alpha(x^{-1})) = U\lambda(x^{-1})U^{-1} = U\lambda(x)^{-1}U^{-1} = (U\lambda(x)U^{-1})^{-1} = \lambda(\alpha(x))^{-1} = \lambda(\alpha(x)^{-1})$ i.e., $\alpha(x^{-1}) = \alpha(x)^{-1}$. To prove continuity of α, let $(x_\mu) \to x_0$ in G. Then

$$\lambda(\alpha(x_\mu)) = U\lambda(x_\mu)U^{-1} \xrightarrow[\mu]{} U\lambda(x_0)U^{-1} = \lambda(\alpha(x_0))$$

in the weak operator topology. But $x \mapsto \lambda(x)$ is a homeomorphism of G onto $\lambda(G)$, where $\lambda(G) \subset \mathscr{L}(L^2(G))$ carries the weak topology ([6; Lemma 2.2]). Thus $\alpha(x_\mu) \to \alpha(x_0)$. Similarly, α^{-1} is continuous, and we have $\alpha \in \operatorname{Aut}(G)$, so that $\operatorname{Aut}(G)$ is complete.

REMARK 1.10. Since by 1.6 Aut (G) is topologically embedded in the complete group Aut $\mathscr{R}(G)$, [10; Proposition 3.5], it would be natural to prove completeness of Aut (G) by showing it is closed in Aut $\mathscr{R}(G)$. Actually, such a proof can be given, utilizing the profound duality theory in [16]. We sketch the argument. Consider a net (α_ν) in Aut (G) such that $\tilde{\alpha}_\nu \to \gamma \in \operatorname{Aut}\mathscr{R}(G)$ in the uniform weak topology. By duality theory $\mathscr{R}(G)$ is a Hopf-von Neumann algebra with comultiplication $\delta: \mathscr{R}(G) \to \mathscr{R}(G) \otimes \mathscr{R}(G)$ which is a σ-weakly continuous isomorphism given by $\delta(T) = W^{-1}(T \otimes 1)W$, $T \in \mathscr{R}(G)$, where $Wk(s, t) = k(s, st)$, $k \in L^2(G \times G)$, $s, t \in G$, [16; Section 4]. Furthermore, one has

$$\{T \in \mathscr{R}(G): \delta(T) = T \otimes T\}\backslash\{0\} \\ = \{T \in \mathscr{R}(G): T = \lambda(s), \text{ for some } s \in G\}.$$

Notice that Aut (G) corresponds to the subgroup

$$\{\beta \in \operatorname{Aut}\mathscr{R}(G): \delta(\beta\lambda(s)) = \beta\lambda(s) \otimes \alpha\lambda(s), \quad \text{all} \quad s \in G\}.$$

Since $\tilde{\alpha}_\nu \to \gamma \in \operatorname{Aut}(\mathscr{R}(G))$ and $\delta(\tilde{\alpha}_\nu\lambda(s)) = \tilde{\alpha}_\nu\lambda(s) \otimes \tilde{\alpha}_\nu\lambda(s)$, all $s \in G$, continuity of δ gives

$$\delta(\gamma(\lambda(s))) = \gamma(\lambda(s)) \otimes \gamma(\lambda(s)), \quad \text{all} \quad s \in G.$$

Thus $\gamma = \tilde{\alpha}$ for some $\alpha \in \operatorname{Aut}(G)$.

COROLLARY 1.11. *If G is a separable locally compact group, then* Aut(G) *is a Polish topological group.*

Proof. Indeed, if $G = \bigcup_{n=1}^{\infty} F_n$, F_n compact, and if $\{U_m\}_{m \in N}$ is a neighborhood base at $e \in G$, then $\{N(F_n, U_m)\}_{n,m}$ is a neighborhood base at $\iota \in \text{Aut}(G)$, so that Aut(G) is metrizable [11; 8.3] and by 1.9. It is complete.

2. We proceed now to applications of the Theorem in 1.9 First we turn to the question of when certain subgroups of Aut(G) are closed. The following result contains a group theoretical analog to [2; Theorem 3.1]. We thank Erling Stormer for showing us Connes' paper [2], and for helpful discussions concerning central sequences of vov Neumann algebras.

PROPOSITION 2.1. *Let G be a separable locally compact group, and B a subgroup of* Aut(G). *Suppose there is a separable locally compact group H and a continuous surjective homomorphism $\omega: H \to B$. Then the following are equivalent.*

(a) *B is closed in* Aut(G).

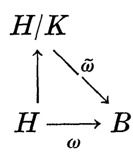
(b) *$\omega: H \to B$ is open onto its range B.*

(c) *For any neighborhood V of the identity in H there exist $\phi_1, \cdots, \phi_n \in C_c(G)$ and $\varepsilon > 0$ such that, for all $h \in H$,*

$$\|\phi_i \circ \omega(h) - \phi_i\|_\infty < \varepsilon\ , \qquad 1 \leqq i \leqq n\ , \quad \textit{implies} \quad h \in V \cdot (\ker \omega)\ .$$

(d) *Same statement as* (c) *with $C_c(G)$ replaced by the Fourier algebra $A(G)$ (and its norm $\|\cdot\|$).*

Proof. (a) $\Rightarrow$ (b). If B is closed in Aut(G) then H and B are both Polish. Observe then that a continuous homomorphism between two Polish groups is open [12; Corollary 3, p. 98]. (b) $\Rightarrow$ (c). Put $K = \ker \omega$. Since ω is open it follows from Lemma 1.1. that given a neighborhood V of the identity in H there are functions $\phi_1, \cdots, \phi_n \in C_c(G)$ and $\varepsilon > 0$ so that $W_{\phi_1,\ldots,\phi_n;\varepsilon} \cap B \subset \omega(V)$. Now ω can be lifted to a map $\tilde{\omega}$ of $H/K \to B$, so that the diagram commutes and $\tilde{\omega}$ is a homeomorphism.



Thus $\omega(h) \in W_{\phi_1,\ldots,\phi_n;\varepsilon}$ implies $\omega(h) \in \omega(V) = \tilde{\omega}(VK)$; hence $\tilde{\omega}(hK) \in \tilde{\omega}(VK)$, so that $h \in hK \subset VK$.

(c) $\Rightarrow$ (d) is clear in view of Proposition 1.6.

(d) $\Rightarrow$ (a). By 1.6 and 1.11 there is a sequence (ϕ_n) from $A(G)$ such that the sets $W_n = W_{\phi_1,\dots,\phi_n;1/n}$ form a base for the identity in Aut(G). Let $\{V_n\}$ be a countable base for the identity in H. By (d), given n there is an $m(n)$ so that $\omega(h) \in W_{m(n)}$ implies $h \in V_nK$. Let $\theta \in B^-$ and choose a sequence (α_n) from B so that $\alpha_n \to \theta$ and $\alpha_{n+j}^{-1}\alpha_n \in W_{m(n)}$ for $j \geqq 0$. Setting $\tilde{\omega}^{-1}(\alpha_n) = h_nK$, we have $h_{n+j}^{-1}h_n \cdot K \subset V_nK$, $j \geqq 0$. This says that (h_nK) is Cauchy in the left uniformity of H/K. Since H/K is locally compact, it is complete, and $h_nK \underset{n}{\to} hK \in H/K$, hence $\omega(h) = \tilde{\omega}(hK) = \theta$ by continuity of $\tilde{\omega}$, and thus $\theta \in B$.

2.2. Define a homomorphism Ad: $G \to \mathrm{Int}(G)$ by $\mathrm{Ad}(g)(x) = gxg^{-1}$. A sequence (x_n) from G is said to be *central* if $\mathrm{Ad}(x_n) \underset{n}{\to} \iota$: in Aut$(G)$. (x_n) is *trivial* if there is a sequence (z_n) from the center $Z(G)$ of G such that $x_nz_n^{-1} \underset{n}{\to} e$.

COROLLARY. *Let G be separable locally compact. Then* Int(G) *is closed if and only if all central sequences are trivial.*

Proof. If Int(G) is closed, let (x_n) be a central sequence and $\{V_n\}$ a nested neighborhood base for the identity in G. By (d) of 2.1 for each n we can find a set $\{\phi, \cdots, \phi_{i_n}\} \subset A(G)$ and $\varepsilon_n > 0$ so that for $x \in G$, $\|\phi_j \circ \mathrm{Ad}(x) - \phi_j\| < \varepsilon_n$, $1 \leqq j \leqq i_n$, implies $x \in V_nZ(G)$. Note that if $\omega = \mathrm{Ad}$ in 2.1, ker ω is just $Z(G)$. Choosing a sequence (k_j) from N such that $k \geqq k_j \Rightarrow \|\phi_j \circ \mathrm{Ad}(x_k) - \phi_j\| < \varepsilon_n$, $1 \leqq j \leqq i_n$, we have $x_k \in V_nZ(G)$, hence $x_kz_k^{-1} \in V_n$ for some $z_k \in Z(G)$. Then $x_kz_k^{-1} \to e$, and (x_n) is trivial. The converse is shown the same way as (d) $\Rightarrow$ (a) in 2.1.

2.3. We remark that the class of groups for which Aut(G) is locally compact includes the compactly generated Lie groups [9; Satz 2.2]. For Int(G) we have the following

COROLLARY. *Let G be separable and locally compact. Then* Int(G) *is locally compact* $\Leftrightarrow$ Int(G) *is closed.*

Proof. If Int(G) is locally compact, it is necessarily closed [9; Theorem 5.11]. On the other hand if Int(G) is closed, take $G = H$ and $\omega = \mathrm{Ad}$ in 2.1. Then by continuity of Ad, Int(G) is homeomorphic with $G/Z(G)$.

2.4. If Int (G) is not closed it is still reasonable to ask if Int$(G)^-$ will be locally compact.

COROLLARY. *Let G be a separable, connected locally compact group. Then the closure* $\mathrm{Int}(G)^-$ *in* $\mathrm{Aut}(G)$ *is locally compact.*

Proof. By [17; Lemma 2.2] there is a locally compact connected group P and a continuous map $\rho_G: P \to \mathrm{Aut}(G)$ with $\rho_G(P) = \mathrm{Int}(G)^-$. Since G is separable, it follows from the construction of P in [17] that P is also separable. Thus by Corollary 1.11 and [12; Corollary 3] ρ_G is a homeomorphism and hence $\mathrm{Int}(G)^-$ is locally compact.

We now give an example that shows that for nonconnected groups, $\mathrm{Int}(G)^-$ need not be locally compact. Let G be the countable weak direct sum of the free group on two generators with the discrete topology: $G = \sum_{n=1}^{\infty} G_n$, where G_n is generated by $\{a_n, b_n\}$. The neutral element of G_n is the empty word, Φ_n, and $e = (\Phi_1, \Phi_2, \cdots)$ is the neutral element of G. If $\mathrm{Int}(G)^-$ were locally compact there would be a relatively compact open neighborhood N of the identity ι in $\mathrm{Int}(G)$. If N_1 is another open neighborhood of ι, since $\bigcup_{x \in G} N_1^- \mathrm{Ad}(x)$ covers $\mathrm{Int}(G)^-$, there would be a finite subcover, $N^- \subset \bigcup_{i=1}^{n} N_1^- \mathrm{Ad}(x_i)$ of N^-. Thus

$$(*) \qquad N = N^- \cap \mathrm{Int}(G) \subset \left[\bigcup_{i=1}^{n} N_1^- \mathrm{Ad}(x_i)\right] \cap \mathrm{Int}(G) = \bigcup_{i=1}^{n} N_1 \mathrm{Ad}(x_i)\,.$$

We may assume $N = N(C, \{e\})$, where $C = \{a_1, b_1\} \times \{a_2, b_2\} \times \cdots \times \{a_n, b_n\} \times \{\Phi_{n+1}\} \times \cdots$, since N must contain a neighborhood of this form. It is then easy to see $\mathrm{Ad}(g) \in N$ if and only if $g = (\Phi_1, \Phi_2, \cdots \Phi_n, g_{n+1}, \cdots)$, $g_{n+j} \in G_{n+j}$, $j \geqq 1$. Let $N_1 = N(C', \{e\})$, $C' = \{a_1, b_1\} \times \cdots \times \{a_{n+1}, b_{n+1}\} \times \{\Phi_{n+2}\} \times \cdots$. Then N and N_1 are subgroups, $\mathrm{Ad}(g) \in N_1$ iff $g = (\Phi_1, \cdots, \Phi_n, \Phi_{n+1}, g_{n+2}, \cdots) g_{n+j} \in G_{n+j}$, $j \geqq 2$. N_1 is normal in N and $N/N_1 \cong G_{n+1}$. This contradicts $(*)$.

2.5. Let G_F be the closed normal subgroup of elements x in G having relatively compact conjugacy classes $\{gxg^{-1}: g \in G\}$. If $G \in$ [SIN], G_F is open since any compact $\mathrm{Int}(G)$-invariant neighborhood of e is contained in G_F. Let $\omega: G \to \mathrm{Aut}(G_F)$ be the continuous homomorphism $\omega(g) = \mathrm{Ad}(g)|_{G_F}$, and let B be the subgroup $\omega(G) \subset \mathrm{Aut}(G_F)$. Clearly G_F is an $[\mathrm{SIN}]_B$-group, and we have

COROLLARY. *Let G be separable. Then, with notation as above, B is closed $\Leftrightarrow$ B is compact $\Leftrightarrow$ $G/\ker\omega$ is compact.*

Proof. The first equivalence is proved in [7]. If B is closed, B is homomorphic with $G/\ker\omega$ (the proposition in 2.1, (a) $\Rightarrow$ (b)) so by compactness of B, $G/\ker\omega$ must be compact. Conversely, if $G/\ker\omega$ is compact then so is $B = \tilde{\omega}(G/\ker\omega)$ by continuity of the lifted map $\tilde{\omega}$.

Specializing the preceding corollary even further we obtain

COROLLARY 2.6. *Let* G *be a locally compact group and suppose* $\mathrm{Int}(G)^-$ *is compact. Then* $\mathrm{Int}(G)$ *is closed* $\Leftrightarrow G/Z(G)$ *is compact* ($Z(G)$ = *the center of* (G)).

Proof. This follows immediately from the Corollary in 2.5 if G is separable. From [7] $\mathrm{Int}(G)$ is closed $\Leftrightarrow \mathrm{Int}(G)$ is compact. But $\mathrm{Int}(G)$ compact implies $\mathrm{Ad}\colon G \to \mathrm{Int}(G)$ is open [11; Theorem 5.29], hence $\mathrm{Int}(G) \cong G/Z(G)$, and so $G/Z(G)$ is compact. Conversely if $G/Z(G)$ is compact, lifting Ad to a continuous map $G/Z(G) \to \mathrm{Int}(G)$ we see that $\mathrm{Int}(G)$ is compact, hence closed.

COROLLARY 2.7. *Let* G *be a separable locally compact group. Then* $\mathrm{Int}(G)$ *is unimodular* $\Leftrightarrow G$ *is unimodular and* $\mathrm{Int}(G)$ *is closed.*

Proof. If $\mathrm{Int}(G)$ is unimodular, in particular it is closed, so by the proposition in 2.1 it is topologically isomorphic with $G/Z(G)$, so that the latter is unimodular. It is then easy to see G is unimodular; we give a proof for completeness. Let dz and $d\dot{x}$ be Haar measures on $Z(G)$ and $G/Z(G)$ respectively, and $x \mapsto \dot{x}$, $G \mapsto G/Z(G)$ the canonical map. Let

$$\mu(\phi) = \int_{G/Z(G)}\int_{Z/(G)}\phi(xz)dz\, d\dot{x}\ , \qquad \phi \in C_c(G)\ .$$

By the Weil integration formula μ is a left Haar measure on G. Using right-invariance of $d\dot{x}$ and the fact that $Z(G)$ is the center, one verifies easily that μ is even right-invariant. Thus G is unimodular. Conversely, if G is unimodular and $\mathrm{Int}(G)$ is closed we show that $G/Z(G)$ is unimodular. It will then follow that $\mathrm{Int}(G)$ is unimodular, since $\mathrm{Int}(G) \cong G/Z(G)$.

Define μ as above. By assumption μ is right-invariant. The mapping $C_c(G) \to C_c(G/Z(G))$, $\phi \mapsto \tilde{\phi}$, $\tilde{\phi}(\dot{x}) = \int_{Z(G)}\phi(xz)dz$ is surjective [11, Theorem 15.21]. $\mu(\phi) = \mu(\phi_y)$ for all $\phi \in C_c(G)$, $y \in G$, then implies $d\dot{x}$ is right-invariant:

$$\int_{G/Z(G)}\tilde{\phi}_{\dot{y}}(\dot{x})d\dot{x} = \mu(\phi_{\dot{y}}) = \mu(\phi) = \int_{G/Z(G)}\tilde{\phi}(\dot{x})d\dot{x}$$

(here $\phi_y(x) = \phi(yx)$). Thus $\mathrm{Int}(G)$ is unimodular.

Finally we show that closedness of $\mathrm{Int}(G)$ does not imply closedness of $\mathrm{Int}\,\mathscr{R}(G)$, nor conversely.

PROPOSITION 2.8. *There is a group G such that* $\mathrm{Int}(G)$ *is closed and* $\mathrm{Int}\,\mathscr{R}(G)$ *is nonclosed. On the other hand, there is a group G with* $\mathrm{Int}(G)$ *nonclosed and* $\mathrm{Int}\,\mathscr{R}(G)$ *closed.*

Before proving the proposition we need a fact, the proof of which we include for the sake of completeness. If Q and Q^* represent the rationals and nonzero rationals respectively, let $G=\{(p, q): p \in Q^*, q \in Q\}$ with multiplication $(p, q)(p', q') = (pp', q + pq')$. Provide G with the discrete topology. Then $\mathrm{Aut}(G) = \mathrm{Int}(G)$. To see this, let $\alpha \in \mathrm{Aut}(G)$ and set $\alpha(1, q) = (\alpha_1(q), \alpha_2(q))$, $q \in Q$. Now $\alpha(1, q)\alpha(1, q') = (\alpha_1(q)\alpha_1(q'), \alpha_2(q) + \alpha_1(q)\alpha_2(q'))$. Also, $\alpha[(1, q)(1, q')] = (\alpha_1(q + q'), \alpha_2(q + q'))$. This forces $\alpha_1(q + q') = \alpha_1(q)\alpha_1(q')$ and thus $\alpha_1(q) = 1$ for all $q \in Q$, since the only homomorphism of the additive group $(Q, +)$ into the multiplicative group $(Q^*, \cdot)$ is the trivial one. Thus $\alpha_2(q + q') = \alpha_2(q) + \alpha_2(q')$, so $\alpha_2 \in \mathrm{Aut}(Q, +)$, and so $\alpha_2(q) = aq$, $a \in Q^*$. Set $\alpha(q, 0) = (\beta_1(p), \beta_2(p))$, $p \in Q^*$. We calculate $\alpha(p, q) = \alpha[(p, 0)(1, q/p)] = \alpha(p, 0)\alpha(1, q/p) = (\beta_1(p), \beta_2(p) + \beta_1(p)\cdot(aq/p))$. But also

$$\begin{aligned}\alpha(p, q) &= \alpha[(1, q)(p, 0)] = \alpha(1, q)\alpha(p, 0)\\ &= (\beta_1(p), aq + \beta_2(p)) .\end{aligned}$$

We have $\beta_2(p) + (aq/p)\beta_1(p) = aq + \beta_2(p)$, and hence $\beta_1(p) = p$. Furthermore, equating $\alpha(p, 0)\alpha(p', 0)$ with $\alpha(p', 0)\alpha(p, 0)$, $(p, p' \in Q^*)$, we arrive at $\beta_2(p)(1 - p') = \beta_2(p')(1 - p)$. If $p, p' \neq 1$, then $\beta_2(p)/(1 - p) = \beta_2(p')/(1 - p') = b \in Q$, a constant. Thus $\beta_2(p) = b(1 - p)$, $p \neq 1$, $p \in Q^*$. But since $\alpha(1, 0) = (1, 0)$, $\beta_2(1) = 0$, so the equation holds for all $p \in Q^*$. Now α has been completely determined:

$$\begin{aligned}\alpha(p, q) &= \alpha[(1, q)(p, 0)(p, 0)]\\ &= (p, aq + b(1 - p)) .\end{aligned}$$

But $(a, b)(p, q)(a, b)^{-1} = (p, aq + b(1 - p))$, which means $\alpha \in \mathrm{Int}(G)$.

Proof of Proposition 2.8. Let G be the group described above. Since all the nontrivial conjugacy classes of G are infinite, $\mathscr{R}(G)$ is a type II_1 factor. Since G is amenable, $\mathscr{R}(G)$ must be the hyperfinite factor [3; Corollary 7.2], hence $\mathrm{Int}\,\mathscr{R}(G)$ is nonclosed.

For the other direction, let $A = (\prod_1^\infty Z_2) \oplus (\sum_1^\infty Z_2)$, where $\prod_1^\infty Z_2$ has the product topology and the weak direct sum $\sum_1^\infty Z_2$ the discrete topology. Define $\alpha: A \to A$ as follows

$$\alpha((z_i), (w_i)) = ((z_i + w_i), (w_i)),\ (z_i) \in \prod_1^\infty Z_2,\ (w_i) \in \sum_1^\infty Z_2 .$$

Then α is a continuous homomorphism and $\alpha^2 =$ identity, so that $\alpha \in \mathrm{Aut}(A)$. Let G be the semidirect product $G = Ax_\eta Z_2$, where

$\eta(m) = \alpha^m$, $m \in Z_2^*$. Since α leaves the elements of $\sum_1^\infty Z_2$ fixed, it follows that $G/\prod_1^\infty Z_2$ is abelian so that the commutator $[G, G]$ is compact. In particular all the conjugacy classes of G are precompact. Furthermore one sees that the center $Z(G)$ is equal to $\prod_1^\infty Z_2$ so $G/Z(G)$ is noncompact. Since $Z/(G)$ is open it is clear that G has small invariant neighborhoods of the identity, and by the Ascoli theorem for groups [7; Satz 1.7], $\text{Int}\,(G)^-$ is compact. According to Corollary 2.6, $\text{Int}\,(G)$ is not closed in $\text{Aut}\,(G)$. This can also be seen directly: let $\tau((x_i), (y_i), 0) = ((x_i), (y_i), 0)$ and $\tau((x_i), (y_i), 1) = ((x_i + 1), (y_i), 0)$, where $(x_i) \in \prod_1^\infty Z_2$, $(y_i) \in \sum_1^\infty Z_2$.

Then

$$\tau \in \text{Int}\,(G)^- \backslash \text{Int}\,(G)\,.$$

Observe next that G is type I, containing a normal abelian subgroup A of finite index, thus $\text{Int}\,\mathscr{R}(G) = \{\alpha \in \text{Aut}\,\mathscr{R}(G)\colon \alpha$ leaves the center of $\mathscr{R}(G)$ pointwise fixed$\}$ is closed [15; Corollary 2.9. 32].

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* This example has appeared in [**13**; p. 104].

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