DECOMPOSING MODULES INTO PROJECTIVES AND INJECTIVES

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A ring $R$ is called a right PCI-ring if and only if for any cyclic right $R$-module $C$ either $C \cong R$ or $C$ is injective. Faith has shown that right PCI-rings are either semiprime Artinian or simple right semihereditary right Ore domains. Thus if $R_1$ and $R_2$ are right PCI-rings then $R = R_1 \oplus R_2$ is not a right PCI-ring unless $R_1$ and $R_2$ are both semiprime Artinian but $R$ has the property that every cyclic right $R$-module is the direct sum of a projective right $R$-module and an injective right $R$-module, and rings with this property on cyclic right $R$-modules will be called right CDPI-rings. On the other hand, if $S$ is a semiprime Artinian ring then the ring of $2 \times 2$ upper triangular matrices with entries in $S$ is also a right CDPI-ring. The structure of right Noetherian right CDPI-rings is discussed. These rings are finite direct sums of right Artinian rings and simple rings. A classification of right Artinian right CDPI-rings is given. However the structure of simple right Noetherian right CDPI-rings is more difficult to determine precisely and the problem of finding it reduces to a conjecture of Faith.

1. Introduction. Recall that if $X$ is a nonempty subset of a ring $R$ (and by a ring we shall always mean a ring with identity element) then the left annihilator of $X$ is the set of all elements $r$ of $R$ such that $rx = 0$ for every element $x$ of $X$, and is denoted by $l(X)$. Similarly the right annihilator of $X$ is $r(X) = \{r \in R: xr = 0 \text{ for all } x \text{ in } X\}$. A subset $A$ of $R$ is called a left (respectively right) annihilator in case $A = l(X)$ ($A = r(X)$) for some nonempty subset $X$ of $R$. A ring $R$ is a Baer ring if and only if for every right annihilator $A$ in $R$ there exists an idempotent element $e$ such that $A = eR$, equivalently for every left annihilator $B$ in $R$ there exists an idempotent element $f$ such that $B = Rf$. Examples of Baer rings can be found in [6]. Baer rings are examples of right $PP$-rings, that is rings such that every principal right ideal is projective. On the other hand, Small [9], Theorem 1, showed that if $R$ is a right $PP$-ring and $R$ does not contain an infinite collection of orthogonal idempotents then $R$ is a Baer ring.

A right CDPI-ring $R$ is a right $PP$-ring (in fact it is right semihereditary, see [10], Lemma 2.4) and has the property that $R/E$ is an injective right $R$-module for every essential right ideal $E$ of $R$ (see Corollary 2.2). Rings with this latter property we shall call right $RIC$-rings ("RIC" for restricted injective condition). If a ring
$R$ is a Baer ring, then $R$ is a right CDPI-ring if and only if $R/E$ is an injective right $R$-module for every right ideal $E$ of $R$ with zero left annihilator (Theorem 2.4). Recall that Osofsky [8] proved that a ring $R$ is semiprime Artinian if and only if every cyclic right $R$-module is injective.

A ring $R$ is a right CEPI-ring provided every cyclic right $R$-module is the extension of a projective right $R$-module by an injective right $R$-module. The class of right CEPI-rings coincides with the class of right PP-right RIC-rings (Theorem 2.9) but strictly contains the class of right CDPI-rings since there is an example in [10] of a right and left Artinian right and left CEPI-ring which is not a right CDPI-ring.

Let us call a ring $R$ a right PCI-domain provided $R$ is a right PCI-ring and a domain. Goodearl [5] called a ring $R$ a right SI-ring in case every singular right $R$-module is injective. By [10], Corollary 4.8, if $R$ is a right Noetherian right CDPI-ring then $R$ is a right SI-ring and hence by [5], Theorem 3.11, and [3], Theorems 14 and 17, $R$ is a finite direct sum $A \oplus B_1 \oplus B_2 \oplus \cdots \oplus B_n$ where $A$ is a right Artinian right CDPI-ring and for each integer $1 \leq i \leq n$, the ring $B_i$ is a right CDPI-ring Morita equivalent to a right Noetherian simple right PCI-domain, and conversely. The ring $A$ can be characterized as a certain ring $(S, M, 0, T)$ of $2 \times 2$ "matrices"

$$\begin{bmatrix} s & m \\ 0 & t \end{bmatrix}$$

with $s$ in a semiprime Artinian ring $S$, $t$ in a semiprime Artinian ring $T$ and $m$ in a certain left $S^∗$, right $T$-bimodule $M$, under the usual matrix addition and multiplication (Corollary 3.8).

When it comes to the rings $B_i (1 \leq i \leq n)$ the natural question which arises is the following one.

**Question 1.1.** Given a right Noetherian simple right PCI-domain $D$, is any ring $S$ Morita equivalent to $D$ a right CDPI-ring?

This question is related to a conjecture of Faith [3], p. 111, and to show the connection between them we make the following definitions. Let $m$ be a positive integer. A ring $R$ is a right FGDPI-ring (right FGDPI$_m$-ring) if and only if every finitely generated ($m$-generator) right $R$-module is the direct sum of a projective right $R$-module and an injective right $R$-module. Right Noetherian semiprime right FGDPI$_m$-rings are right FGDPI-rings and are left Goldie (Theorem 5.7). It follows that (see Corollary 4.12) the answer to 1.1 is "yes" if and only if $D$ is a left Ore domain and this is precisely Faith's conjecture, and in this case the rings $B_i (1 \leq i \leq n)$ are just the rings Morita
equivalent to right Noetherian simple right PCI-domains. Recall that if the ring $D$ is a left Ore domain then Faith [3], Theorem 22 and subsequent remarks, proved that $D$ is a left Noetherian left PCI-domain and we call such rings Noetherian simple PCI-domains. Examples of these rings can be found in [2]. Faith's conjecture can be expressed in yet another way (see Theorems 4.11 and 5.7):

**Conjecture 1.2.** If $D$ is a right Noetherian simple right PCI-domain then the ring $D_2$ is a right CDPI-ring where $D_2$ is the complete ring of $2 \times 2$ matrices with entries in $D$.

We shall call a ring $R$ a Noetherian simple PCI-domain if and only if $R$ is a right and left Noetherian simple right and left PCI-domain. Examples of Noetherian simple PCI-domains have been produced by Cozzens [2]. For any positive integer $m$ a ring $R$ is a right Noetherian right FGDPI$_m$-ring if and only if $R$ is a finite direct sum $A \oplus B_1 \oplus B_2 \oplus \cdots \oplus B_n$ where $A$ is a right Artinian right FGDPI$_m$-ring and for each integer $1 \leq i \leq n$ the ring $B_i$ is a simple right and left Noetherian ring Morita equivalent to a Noetherian simple PCI-domain (see Corollary 5.8). There is a corresponding structure theorem for right Noetherian right FGDPI-rings. We have not been able to find explicitly the structure of right Artinian right FGDPI$_m$-rings ($m$ an integer greater than 1) or right Artinian right FGDPI-rings.

We mention one further interesting fact about semiprime rings. If $R$ is a semiprime ring then the following statements are equivalent:

(i) $R$ is a right Noetherian right FGDPI$_2$-ring,
(ii) $R$ is a left Noetherian left FGDPI$_2$-ring,
(iii) $R$ is a right Noetherian right FGDPI-ring, and
(iv) $R$ is a left Noetherian left FGDPI-ring (see Corollary 5.9).

Note also that if $R$ is a right Noetherian right FGDPI$_2$-ring then $R$ is a left SI-ring and in particular $R$ is left hereditary (see Corollary 5.10).

**2. Right CDPI-rings.** In this section we first look at characterizations of right CDPI-rings, we then examine the relationship between right CEPI-rings and right RIC-rings and finally we generalize the theorem of Osofsky mentioned in the Introduction.

**Lemma 2.1** (See [10], Lemma 5.1). A ring $R$ is a right CDPI-ring if and only if for every right ideal $E$ of $R$ there exists an idempotent element $e$ such that $E$ is contained in the right ideal $eR$ and the right $R$-module $eR/E$ is injective.

**Corollary 2.2.** Let $R$ be a right CDPI-ring and $E$ be a right
ideal of $R$ with zero left annihilator. Then the right $R$-module $R/E$ is injective.

If $X$ is a nonempty subset of a ring $R$ then by $rl(X)$ we shall mean $r(l(X))$, the right annihilator of the left annihilator of $X$. The proof of the next result is an easy adaptation of the proof of [10], Lemma 5.7.

**Lemma 2.3.** Let $R$ be a Baer ring. Then $R$ is a right CDPI-ring if and only if $rl(E)/E$ is an injective right $R$-module for each right ideal $E$ of $R$.

**Theorem 2.4.** Let $R$ be a Baer ring. Then $R$ is a right CDPI-ring if and only if $R/E$ is an injective right $R$-module for each right ideal $E$ of $R$ with zero left annihilator.

**Proof.** In view of Corollary 2.2 we need prove only the sufficiency. Suppose that $R$ is a ring such that $R/E$ is injective for every right ideal $E$ with $l(E) = 0$. Let $A$ be a right ideal of $R$. Since $R$ is a Baer ring there exists an idempotent element $a$ of $R$ such that $rl(A) = aR$. Let $B = \{ r \in R : ar \in A \}$. Since $a$ is idempotent it follows that $A = aA$ and hence $A \subseteq B$. Then $aR = rl(A) \subseteq rl(B)$. But $(1 - a) R \subseteq B \subseteq rl(B)$ and hence $Rrl(B)$. Thus $l(B) = 0$ and by hypothesis $R/B$ is injective. Since the mapping $\varphi : R/B \rightarrow aR/A$ defined by $\varphi(r + B) = ar + A$ ($r \in R$) is an $R$-isomorphism it follows that $aR/A$ is injective. By Lemma 2.1 $R$ is a right CDPI-ring.

**Corollary 2.5.** Let $R$ be a ring which does not contain an infinite collection of orthogonal idempotent elements. Then $R$ is a right CDPI-ring if and only if $R$ is a right PP-ring and $R/E$ is an injective right $R$-module for every right ideal $E$ of $R$ with zero left annihilator.

**Proof.** The necessity is a consequence of Corollary 2.2 and [10], Lemma 2.4. The sufficiency follows by the theorem and [9], Theorem 1.

An immediate consequence of Corollary 2.5 is the next result.

**Corollary 2.6.** Let $R$ be a semiprimary ring. Then $R$ is a right CDPI-ring if and only if $R$ is a right PP-ring such that $R/E$ is an injective right $R$-module for every right ideal $E$ of $R$ with zero left annihilator.

**Corollary 2.7.** Let $R$ be a ring which does not contain an
infinite direct sum of nonzero right ideals. Then $R$ is a right CDPI-ring if and only if $R$ is a right nonsingular ring such that $R/E$ is an injective right $R$-module for every right ideal $E$ of $R$ with zero left annihilator.

Proof.} The necessity follows by Corollary 2.2 and [10], Lemma 2.4. Conversely, suppose that $R$ is a right nonsingular ring such that $R/E$ is an injective right $R$-module for each right ideal $E$ with $l(E) = 0$. Since $R$ is right nonsingular it follows that $R$ is a right RIC-ring. Also by [4], Lemma 1.4 and Theorem 2.3 (iii), $R$ is a right Goldie ring. By [10], Corollary 4.3 and Lemma 4.4, $R$ is a right PP-ring. Finally by Corollary 2.5 $R$ is a right CDPI-ring.

Next we consider briefly right CEPI-rings. Let $E$ be a right ideal of a right CEPI-ring $R$. There exists a right ideal $F$ of $R$ containing $E$ such that $F/E$ is projective and $R/F$ is injective. Since $F/E$ is projective there exists a right ideal $G$ of $R$ such that $E \cap G = 0$ and $F = E \oplus G$. Moreover, $G \cong F/E$ is projective. We have proved:

**Lemma 2.8.** A ring $R$ is a right CEPI-ring if and only if for every right ideal $E$ of $R$ there exists a projective right ideal $G$ of $R$ such that $E \cap G = 0$ and $R/(E \oplus G)$ is an injective right $R$-module.

In [10], Lemma 2.4, we proved that a right CEPI-ring is a right semihereditary right RIC-ring. Now we have the following result.

**Theorem 2.9.** A ring $R$ is a right CEPI-ring if and only if $R$ is a right PP-right RIC-ring.

Proof.} As we have just remarked the necessity is proved in [10], Lemma 2.4. Conversely, suppose that $R$ is a right PP-right RIC-ring. Let $E$ be a right ideal of $R$. By Zorn's lemma there exists a maximal collection $S$ of nonzero elements $x_\lambda (\lambda \in A)$ of $R$ such that if $H = \sum x_\lambda R$ then $H = \bigoplus x_\lambda R$ and $E \cap H = 0$. Since $R$ is a right PP-ring, $H$ is projective. Let $a$ be a nonzero element of $R$. If $a \notin S$ then either $aR \cap H \neq 0$ or $E \cap (aR \oplus H) \neq 0$. It follows that $E \oplus H$ is an essential right ideal of $R$. Since $R$ is a right RIC-ring, the right $R$-module $R/(E \oplus H)$ is injective. By Lemma 2.8 $R$ is a right CEPI-ring.

Finally in this section we give the following generalization of Osofsky's theorem [8].
THEOREM 2.10. A ring $R$ is semiprime Artinian if and only if $R$ is a right self-injective right RIC-ring.

Proof. The necessity is a consequence of Osofsky's theorem. Conversely, let $R$ be a right self-injective right RIC-ring. Since $R$ is right self-injective, given any right ideal $A$ of $R$ there exists an idempotent element $e$ of $R$ such that $A$ is an essential submodule of the right ideal $eR$. Since $R$ is a right RIC-ring it follows that $eR/A$ is injective. By Lemma 2.1 $R$ is a right CDPI-ring. Let $C$ be a cyclic right $R$-module. There exists a projective module $P$ and an injective module $Q$ such that $C = P \oplus Q$. Since $P$ is therefore cyclic it follows that $P$ is isomorphic to a direct summand of $R$ and hence $P$ is injective. Thus $C$ is injective. Thus every cyclic right $R$-module is injective and $R$ is semiprime Artinian by Osofsky's theorem [8].

3. Semiprimary right CDPI-rings. Right CDPI-rings are right RIC-rings (see [10], Lemma 2.4). In addition, by [10], Lemma 2.5 and Theorem 4.1, semiprimary right RIC-rings are right SI-rings. Also, by [5], Proposition 3.5, semiprimary right SI-rings are left SI-rings. Thus we have the following result.

LEMMA 3.1. Semiprimary right CDPI-rings are right and left SI-rings.

Let $R$ be a right SI-ring. By [5], Proposition 3.3, $R$ is right hereditary. If in addition $R$ is semiprimary then $R$ is a Baer ring by [9], Theorem 1. Noting this fact, the next result of this section is proved by adapting the proof of [10], Theorem 5.13.

LEMMA 3.2. A ring $R$ is a semiprimary (right) SI-ring if and only if $R$ is semiprime Artinian or there exist semiprime Artinian rings $S$ and $T$ and a left $S$-, right $T$-bimodule $M$ such that $M$ is a faithful left $S$-module and $R$ is isomorphic to the ring $(S, M, 0, T)$.

For the remainder of this section we shall fix the following notation: $S$ and $T$ are semiprime Artinian rings, $M$ is a left $S$-, right $T$-bimodule (not necessarily faithful as a left $S$-module) and $R$ is the ring $(S, M, 0, T)$. That is, $R$ consists of all "matrices"

$$
(s, m, 0, t) = \begin{bmatrix}
  s & m \\
  0 & t
\end{bmatrix}
$$

with $s$ in $S$, $m$ in $M$ and $t$ in $T$, addition and multiplication in $R$ being the usual matrix addition and multiplication. For each non-empty subset $X$ of $M$ let $\text{Ann}_S(X)$ denote the annihilator of $X$ in
S; that is, \( \text{Ann}_S(X) = \{ s \in S : sX = 0 \} \). Let \( I = \text{Ann}_S(M) \) and let \( q \) be the central idempotent element of \( S \) such that \( I = Sq \). The right socle of \( R \) will be denoted by \( A \). It can easily be checked that \( A = (I, M, 0, T) \) and \( A \) is an essential right ideal of \( R \). By [5], Proposition 3.1, \( R \) is a right SI-ring and in view of Lemma 3.2 we can take \( R \) as a typical semiprimary right SI-ring. The Jacobson radical of \( R \) will be denoted by \( J \). Clearly \( J = (0, M, 0, 0) \). Moreover \( A = J \oplus eR \) where \( e \) is the idempotent \((q, 0, 0, 1)\) of \( R \) (here \( 1 \) is the identity element of the ring \( T \)). Note that \( ej = 0 \) and recall that \( A = \cap \{ E : E \) is an essential right ideal of \( R \} \).

**Lemma 3.3.** Let \( R \) be a semiprimary right SI-ring with Jacobson radical \( J \) and let \( X \) be a right \( R \)-module. Then \( X \) is injective if and only if given any homomorphism \( \varphi : J \rightarrow X \) there exists an element \( x \) of \( X \) such that \( \varphi(j) = xj \) for every element \( j \) of \( J \).

**Proof.** The necessity is an immediate consequence of Baer's criterion for injectivity (see for example [1], Lemma 18.3). Conversely, suppose that \( X \) has the stated property. By Lemma 3.2 we can suppose without loss of generality that in the above notation \( R = (S, M, 0, T) \). Let \( Z = Z(X) \) be the singular submodule of \( X \). Since \( R \) is a right SI-ring it follows that \( Z \) is injective and hence there exists a submodule \( Y \) of \( X \) such that \( X = Z \oplus Y \). Note that \( Y \) is nonsingular. Let \( E \) be an essential right ideal of \( R \) and \( \varphi : E \rightarrow Y \) be an \( R \)-homomorphism. Let \( \alpha \) be the restriction of \( \varphi \) to \( J \). By hypothesis there exists an element \( x \) of \( X \) such that \( \alpha(j) = xj \ (j \in J) \).

If \( x = z + y_1 \) where \( z \in Z, y_1 \in Y \), then clearly \( \alpha(j) = y_1j \ (j \in J) \). Let \( y_2 \) be the element \( \varphi(e) \) of \( Y \), where \( e = (q, 0, 0, 1) \) as above. Let \( y \) be the element \( y_1(1 - e) + y_2e \) of \( Y \). If \( a \in A \) then \( a = j + er \) for some elements \( j \) of \( J \) and \( r \) of \( R \) and

\[
\varphi(a) = \varphi(j) + \varphi(e)er = y_1j + y_2er = ya.
\]

Thus \( \varphi(a) = ya \ (a \in A) \). Now let \( b \in E \). Since \( A \) is an essential submodule of \( E \) there exists an essential right ideal \( K \) of \( R \) such that \( bK \subseteq A \). For any element \( k \) of \( K \), \( \varphi(b)k = \varphi(bk) = ybk \) and hence \( (\varphi(b) - yb)k = 0 \). It follows that \( (\varphi(b) - yb)K = 0 \). Since \( Y \) is nonsingular it follows that \( \varphi(b) = yb \). Hence \( \varphi(b) = yb(b \in E) \), and by Baer's criterion \( Y \), and hence \( X \), is injective.

It is clear from the proof of Lemma 3.3 that in Lemma 3.3 we can replace \( J \) be the right socle \( A \).

In view of Corollary 2.6 interest centres on right ideals of \( R \) with zero right annihilator. Let \( E \) be a right ideal of \( R \). Let \( F = \{ a \in S : (a, 0, 0, 0) \in E \} \). Then \( F \) is a right ideal of \( S \) and there exists
an idempotent element \( f \) of \( S \) such that \( F = fS \). If \( f \) is the element 
\((f, 0, 0, 0)\) of \( R \) then \( fR = (fS, fM, 0, 0) \). If \( N \) is the \( T \)-submodule 
\((1 - f)M \) then \( M = fM \oplus N \) and \( E = fR \oplus E_i \) where \( E_i \) is the right 
ideal \( E \cap (0, N, 0, T) \). For, if \( r = (a, b, 0, c) \in E \) with \( a \) in \( S \), \( b \) in \( M \) 
and \( c \) in \( T \) then \((a, 0, 0, 0) = (a, b, 0, c)(1, 0, 0, 0) \in E \) and hence \( a = fa \) 
and \( r = fr \in E_i \). Now \( E_i = (E_i \cap J) \oplus C \) for some right ideal \( C \) con-
tained in \( E \). Let \( D = \{ t \in T : (0, y, 0, t) \in C \) for some element \( y \) of \( M \} \). 
Then \( D \) is a right ideal of \( T \) and there exists an idempotent 
element \( g \) of \( T \) such that \( D = gT \). Let \( m \) be an element of \( M \) such 
that \( c = (0, m, 0, g) \in C \). For any element \( c_i \) of \( C \) it can easily be 
checked that \( c_i - cc_i \in C \cap J = 0 \). It follows that \( c \) is an idempotent 
element of \( R \) and \( C = cR \). In particular \( c \) idempotent implies that \( m = mg \). 
Thus there exists a \( T \)-submodule \( X \) of \( N \) such that \( E \) 
consists of all "matrices" \((fa, fb + x + mt, 0, gt)\) with \( a \) in \( S \), \( b \) in \( M \), \( x \) in \( X \) and \( t \) in \( T \). Now suppose that \( l(E) = 0 \). It can easily be 
checked that if \( e \) is an idempotent element of \( S \) such that \( Ann_s(x) = Se \) 
then \( X = (1 - f)X \) implies that \( e(1 - f) \in Se \) and 
\[
(e(1 - f), -e(1 - f)m, 0, 1 - g)
\]
belongs to \( l(E) \). Thus \( e(1 - f) = 0 \) and \( g = 1 \). But \( e(1 - f) = 0 \) 
implies that \( e = ef \) and \( Se \subseteq Sf \). This gives the following result 
after a little checking.

**Lemma 3.4.** A right ideal \( E \) of the above ring \( R \) has zero left 
annihilator if and only if there exists a \( T \)-submodule \( X \) of \( M \), an 
idempotent element \( e \) of \( S \) such that \( Se = Ann_s(X) \), an idempotent 
element \( f \) of \( S \) such that \( Se \subseteq Sf \), and an element \( m \) of \( M \) such that 
\( E \) consists of all "matrices" \((fa, fb + x + mt, 0, t)\) with \( a \) in \( S \), \( b \) in 
\( M \), \( x \) in \( X \) and \( t \) in \( T \).

**Lemma 3.5.** If \( X = Ann_M(Ann_s(X)) \) for every \( T \)-submodule \( X \) 
of \( M \) then \( R \) is a right CDPI-ring.

**Proof.** By \( Ann_M(Ann_s(X)) \) we mean the set of elements \( m \) of 
\( M \) such that \( Ann_s(X)m = 0 \). In the notation of the previous lemma 
let \( E \) be the right ideal of all "matrices" \((fa, fb + x + mt, 0, t)\) with 
\( a \) in \( S \), \( b \) in \( M \), \( x \) in \( X \) and \( t \) in \( T \). Let \( s \in Ann_s(fM + X) \); then 
\( sfM = sX = 0 \). But \( sX = 0 \) implies that \( s = se \) and hence \( sf = sef = 
se = s \). It follows that \( sM = 0 \) and hence \( Ann_s(fM + X) = Ann_s(M) \). 
By hypothesis \( fM + X = Ann_M(Ann_s(fM + X)) = M \). It follows that 
the ideal \((0, M, 0, T)\) is contained in \( E \). Let \( \varphi : J \rightarrow R/E \) be an \( R \)-
homomorphism. If \( b = (0, 0, 0, 1) \) then \( j = jb \) for every element \( j \) 
of \( J \) and it follows that \( \varphi = 0 \). By Corollary 2.6 and Lemmas 3.2-3.4 
\( R \) is a right CDPI-ring.
In particular if $S = M = T$ then $R$ is a right CDPI-ring. This special case was proved in [10], Theorem 5.15. Another special case is when $M$ is a simple right $T$-module and again $R$ is a right CDPI-ring. This corresponds to the Jacobson radical $J$ of $R$ being a minimal right ideal (see [10], Theorem 5.9). We can express Lemma 3.5 in terms of $J$ as follows.

**Corollary 3.6.** Let $R$ be a semiprimary right SI-ring such that $F = J \cap \text{rl}(F)$ for every right ideal $F$ contained in the Jacobson radical $J$ of $R$. Then $R$ is a right CDPI-ring.

**Theorem 3.7.** In the above notation let $R$ be the semiprimary right SI-ring $(S, M, 0, T)$. Then $R$ is a right CDPI-ring if and only if for every $T$-submodule $X$ of $M$ such that $\text{Ann}_S(X) = \text{Ann}_S(M)$ and every $T$-homomorphism $\varphi: M \to M/X$ there exists an element $a$ of $S$ such that $\varphi(m) = am + X$ for all $m$ in $M$.

**Proof.** Suppose first that $R$ is a right CDPI-ring. Let $X$ be a $T$-submodule of $M$ such that $\text{Ann}_S(X) = \text{Ann}_S(M) = Sq$ and $\varphi: M \to M/X$ a $T$-homomorphism. Let $E$ be the right ideal $(Sq, X, 0, T)$. It can easily be checked that $l(E) = 0$ and thus, by Corollary 2.2, $R/E$ is an injective right $R$-module. Define $\varphi: J \to R/E$ by $\varphi(0, m, 0, 0) = (0, \tau(m), 0, 0) + E$ ($m \in M$). Since $\varphi$ is an $R$-homomorphism there exists an element $r = (a, b, 0, c)$ of $R$ such that $\varphi(j) = rj + E$ ($j \in J$). It can easily be checked that this gives $\varphi(m) = am + X$ ($m \in M$).

Conversely, in the notation of Lemma 3.4 let $E$ be the right ideal of $R$ consisting of all “matrices” $(fa, fb + x + mt, 0, t)$ with $a$ in $S$, $b$ in $M$, $x$ in $X$ and $t$ in $T$. Let $Y$ be a set of coset representatives of $X$ in $M$ and define a mapping $\tau: M \to V$ by $\varphi(m) = \tau(m) + X$ ($m \in M$). Let $E$ be the right ideal $(Sq, X, 0, T)$. It can easily be checked that $l(E) = 0$ and thus, by Corollary 2.2, $R/E$ is an injective right $R$-module. Define $\varphi: J \to R/E$ by $\varphi(0, m, 0, 0) = (0, \tau(m), 0, 0) + E$ ($m \in M$). Since $\varphi$ is an $R$-homomorphism there exists an element $r = (a, b, 0, c)$ of $R$ such that $\varphi(j) = rj + E$ ($j \in J$). It can easily be checked that this gives $\varphi(m) = am + X$ ($m \in M$).

Conversely, in the notation of Lemma 3.4 let $E$ be the right ideal of $R$ consisting of all “matrices” $(fa, fb + x + mt, 0, t)$ with $a$ in $S$, $b$ in $M$, $x$ in $X$ and $t$ in $T$. Let $Y$ be a set of coset representatives of $X$ in $M$ and define a mapping $\tau: M \to V$ by $\varphi(m) = \tau(m) + X$ ($m \in M$). Let $E$ be the right ideal $(Sq, X, 0, T)$. It can easily be checked that $l(E) = 0$ and thus, by Corollary 2.2, $R/E$ is an injective right $R$-module. Define $\varphi: J \to R/E$ by $\varphi(0, m, 0, 0) = (0, \tau(m), 0, 0) + E$ ($m \in M$). Since $\varphi$ is an $R$-homomorphism there exists an element $r = (a, b, 0, c)$ of $R$ such that $\varphi(j) = rj + E$ ($j \in J$). It can easily be checked that this gives $\varphi(m) = am + X$ ($m \in M$).

Conversely, in the notation of Lemma 3.4 let $E$ be the right ideal of $R$ consisting of all “matrices” $(fa, fb + x + mt, 0, t)$ with $a$ in $S$, $b$ in $M$, $x$ in $X$ and $t$ in $T$. Let $Y$ be a set of coset representatives of $X$ in $M$ and define a mapping $\tau: M \to V$ by $\varphi(m) = \tau(m) + X$ ($m \in M$). Let $E$ be the right ideal $(Sq, X, 0, T)$. It can easily be checked that $l(E) = 0$ and thus, by Corollary 2.2, $R/E$ is an injective right $R$-module. Define $\varphi: J \to R/E$ by $\varphi(0, m, 0, 0) = (0, \tau(m), 0, 0) + E$ ($m \in M$). Since $\varphi$ is an $R$-homomorphism there exists an element $r = (a, b, 0, c)$ of $R$ such that $\varphi(j) = rj + E$ ($j \in J$). It can easily be checked that this gives $\varphi(m) = am + X$ ($m \in M$).

Conversely, in the notation of Lemma 3.4 let $E$ be the right ideal of $R$ consisting of all “matrices” $(fa, fb + x + mt, 0, t)$ with $a$ in $S$, $b$ in $M$, $x$ in $X$ and $t$ in $T$. Let $Y$ be a set of coset representatives of $X$ in $M$ and define a mapping $\tau: M \to V$ by $\varphi(m) = \tau(m) + X$ ($m \in M$). Let $E$ be the right ideal $(Sq, X, 0, T)$. It can easily be checked that $l(E) = 0$ and thus, by Corollary 2.2, $R/E$ is an injective right $R$-module. Define $\varphi: J \to R/E$ by $\varphi(0, m, 0, 0) = (0, \tau(m), 0, 0) + E$ ($m \in M$). Since $\varphi$ is an $R$-homomorphism there exists an element $r = (a, b, 0, c)$ of $R$ such that $\varphi(j) = rj + E$ ($j \in J$). It can easily be checked that this gives $\varphi(m) = am + X$ ($m \in M$).

Conversely, in the notation of Lemma 3.4 let $E$ be the right ideal of $R$ consisting of all “matrices” $(fa, fb + x + mt, 0, t)$ with $a$ in $S$, $b$ in $M$, $x$ in $X$ and $t$ in $T$. Let $Y$ be a set of coset representatives of $X$ in $M$ and define a mapping $\tau: M \to V$ by $\varphi(m) = \tau(m) + X$ ($m \in M$). Let $E$ be the right ideal $(Sq, X, 0, T)$. It can easily be checked that $l(E) = 0$ and thus, by Corollary 2.2, $R/E$ is an injective right $R$-module. Define $\varphi: J \to R/E$ by $\varphi(0, m, 0, 0) = (0, \tau(m), 0, 0) + E$ ($m \in M$). Since $\varphi$ is an $R$-homomorphism there exists an element $r = (a, b, 0, c)$ of $R$ such that $\varphi(j) = rj + E$ ($j \in J$). It can easily be checked that this gives $\varphi(m) = am + X$ ($m \in M$).
(ii) For all elements $x_1, x_2$ in $M$ there exist elements $y_1$ in $Y$ and $t_1$ in $T$ such that $(x_1 + x_2)^M - x_1^M - x_2^M = y_1 + m t_1, (x_1 + x_2)^T - x_1^T - x_2^T = t_1$.

(iii) For all elements $x$ in $M$ and $c$ in $T$ there exist elements $y_2$ in $Y$ and $t_2$ in $T$ such that $(c x)^M - x^M c = y_2 + m t_2, (c x)^T - x^T c = t_2$.

Define $\beta: M \to M/Y$ by $\beta(x) = (x^M - m x^T) + Y$ for every element $x$ of $M$. By (i), (ii) and (iii) $\beta$ is a $T$-homomorphism. But $Y = f M + X$ implies that $\text{Ann}_S(Y) = \text{Ann}_S(M)$. Therefore by hypothesis there exists an element $s_1$ of $S$ such that $\beta(x) = s_1 x + Y (x \in M)$. Let $s$ be the element $(s_1, 0, 0, 0)$ of $R$. Then for each element $j$ of $J$ there exists an element $x$ of $M$ such that $j = (0, x, 0, 0)$ and hence $\alpha(j) = (0, x^M, 0, x^T) + H = sj + H$. Thus $\alpha(j) = sj + H (j \in J)$. By Corollary 2.6 and Lemmas 3.2-3.4 $R$ is a right CDPI-ring. This proves the theorem.

Combining Lemmas 3.1, 3.2 and Theorem 3.7 we have:

**Corollary 3.8.** A ring $R$ is a semiprimary right CDPI-ring if and only if $R$ is semiprime Artinian or there exist semiprime Artinian rings $S$ and $T$ and a left $S$-, right $T$-bimodule $M$ such that $M$ is a faithful left $S$-module and for every $T$-submodule $X$ of $M$ such that $\text{Ann}_S(X) = 0$ and $T$-homomorphism $\varphi: M \to M/X$ there exists an element $a$ of $S$ with $\varphi(m) = a m + X$ for every $m$ in $M$, and $R$ is isomorphic to the ring $(S, M, 0, T)$.

**Corollary 3.9.** In the above notation let $R$ be the semiprimary right SI-ring $(S, M, 0, T)$. Suppose that $R$ is a right CDPI-ring. Then there does not exist a left $S$-, right $T$-sub-bimodule $X$ of $M$ and a nonzero $T$-submodule $Y$ of $M$ such that $\text{Ann}_S(X) = 0$ and $Y \cap X = 0$ and $Y$ can be embedded in $X$.

**Proof.** Suppose that $M$ contains a sub-bimodule $X$ and a submodule $Y$ with the given properties. Let $X_i$ be a $T$-submodule of $X$ such that there is a $T$-isomorphism $\varphi: X_i \to Y$. Since $T$ is semiprime Artinian there exists a $T$-submodule $N$ of $M$ such that $M = X_i \oplus Y \oplus N$. Define $\alpha: M \to M/X$ by $\alpha(x_i + y + n) = \varphi(x_i) + X$ for all $x_i$ in $X_i$, $y$ in $Y$ and $n$ in $N$. If $R$ is a right CDPI-ring then by the theorem there exists an element $s$ of $S$ such that for each element $x_i$ of $X_i$, $\varphi(x_i) + X = \alpha(x_i) = s x_i + X$. It follows that $\varphi(x_i) \in X \cap Y = 0$ for each element $x_i$ of $X_i$, a contradiction. Thus $R$ is not a right CDPI-ring.

**Corollary 3.10.** Suppose that $S$ and $T$ are simple rings and the above ring $R = (S, M, 0, T)$ is a right CDPI-ring. Then $M$ is a simple left $S$-, right $T$-bimodule.
Proof. Let $X$ be a nonzero left $S$, right $T$-sub-bimodule of $M$. Since $S$ is simple it follows that $\text{Ann}_S(X) = \text{Ann}_S(M) = 0$. If $Y$ is a simple $T$-submodule of $M$ then $Y$ can be embedded in $X$, because $T$ is simple and simple right $T$-modules are isomorphic. By Corollary 3.9 $X \cap Y \neq 0$ and hence $Y \subseteq X$. It follows that $X = M$.

We can express Corollary 3.10 in the following form.

**Corollary 3.11.** Let $R$ be a semiprimary right CDPI-ring with Jacobson radical $J$. If $R$ contains precisely two maximal ideals then $J$ is a minimal ideal of $R$.

4. Category equivalence. Let $R$ be a ring and $A, B$ be right $R$-modules. A monomorphism $\varphi: A \rightarrow B$ is called essential if and only if $\text{Im} \varphi$ is an essential submodule of $B$; that is, $\text{Im} \varphi \cap C \neq 0$ for every nonzero submodule $C$ of $B$. The first lemma in this section is elementary and well known but we shall include its proof for completeness.

**Lemma 4.1.** A right $R$-module $C$ is singular if and only if there exists an exact sequence $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ of right $R$-modules such that $\alpha: A \rightarrow B$ is an essential monomorphism.

**Proof.** Suppose that $C$ is singular. For each element $c$ of $C$ let $R_c = R$ and let $F = \bigoplus_c R_c$. Let $\pi: F \rightarrow C$ be the canonical projection. For each element $c$ of $C$ there exists an essential right ideal $E_c$ of $R = R_c$ such that $cE_c = 0$. Let $E = \bigoplus_c E_c$. Then $E$ is an essential submodule of $F$ and $E \subseteq \ker \pi$. If $K = \ker \pi$ and $i: K \rightarrow F$ is inclusion then $0 \rightarrow K \xrightarrow{i} F \xrightarrow{\pi} C \rightarrow 0$ is an exact sequence such that $i$ is an essential monomorphism. Conversely, suppose that there exists an exact sequence $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ of right $R$-modules such that $\alpha$ is an essential monomorphism. Let $c \in C$ and let $b$ be an element of $B$ such that $\beta(b) = c$. It can easily be checked that $\ker \beta = \text{Im} \alpha$ is an essential submodule of $B$ implies that $G = \{ r \in R : br \in \ker \beta \}$ is an essential right ideal of $R$. Also, $cG = \beta(b)G = \beta(bG) = 0$. It follows that $C$ is singular.

**Corollary 4.2.** A right $R$-module $C$ is a finitely generated singular module if and only if there exists an exact sequence $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ of right $R$-modules such that $B$ is finitely generated and $\alpha: A \rightarrow B$ is an essential monomorphism.

**Lemma 4.3.** A ring $R$ is a right RIC-ring if and only if every
finitely generated singular right $R$-module is injective.

Proof. The sufficiency follows from the fact that if $E$ is an essential right ideal of $R$ then $R/E$ is a cyclic singular right $R$-module. Conversely, suppose that $R$ is a right RIC-ring. Let $n$ be a positive integer and $X$ a right $R$-module generated by elements $x_1, x_2, \cdots, x_n$. If $n = 1$ there is nothing to prove. Suppose that $n > 1$ and let $Y = x_1R + x_2R + \cdots + x_{n-1}R$. Then $Y$ is a singular module. If $Y$ is injective then there exists a submodule $Z$ of $X$ such that $X = Y \oplus Z$. It follows that $Z$ is a cyclic singular module and hence $Z$ is injective. Thus $X$ is injective. The result follows by induction on $n$.

COROLLARY 4.4. Any ring Morita equivalent to a right RIC-ring is itself a right RIC-ring.

Proof. By Corollary 4.2 since category equivalence preserves exact sequences, finitely generated modules and essential monomorphisms (see [1], Propositions 21.4, 21.6(5) and 21.8(2)).

THEOREM 4.5. A ring $R$ is a right CEPI-ring if and only if every finitely generated right $R$-module is the extension of a projective right $R$-module by an injective right $R$-module.

Proof. The given condition is clearly sufficient for $R$ to be a right CEPI-ring. Conversely, suppose that $R$ is a right CEPI-ring. Let $n$ be a positive integer and $X$ be a right $R$-module generated by elements $x_1, x_2, \cdots, x_n$. If $n = 1$ there is nothing to prove and so we suppose that $n > 1$. Let $Y = x_1R + x_2R + \cdots + x_{n-1}R$. Suppose there is a submodule $A$ of $Y$ such that $A$ is projective and $Y/A$ is injective. Since $X/Y$ is cyclic and $R$ is a right CEPI-ring it follows that there exists a submodule $B$ of $X$ such that $Y \subseteq B$, $B/Y$ is projective and $X/B$ is injective. Now consider $B/A$. Since $Y/A$ is injective there exists a submodule $C$ of $B$ such that $A \subseteq C$ and $B/A = (Y/A) \oplus (C/A)$. Since $C/A \cong B/Y$ is projective and $A$ is projective it follows that $C \cong A \oplus (C/A)$ is projective. Moreover, $B/C \cong Y/A$ is injective and hence $X/C \cong (B/C) \oplus (X/B)$ is injective. The result follows by induction on $n$.

COROLLARY 4.6. Any ring Morita equivalent to a right CEPI-ring is itself a right CEPI-ring.

Proof. By the theorem since category equivalence preserves exact sequences, finitely generated modules, projective modules and injective modules (see [1], Propositions 21.4, 21.6(2) and 21.8(2)).
It is interesting to compare Theorem 2.5 with the next result.

**Theorem 4.7.** A ring \( R \) is a right SI-ring if and only if every right \( R \)-module is the extension of a projective right \( R \)-module by an injective right \( R \)-module.

**Proof.** Suppose that every right \( R \)-module is the extension of a projective module by an injective module. In particular, this means that \( R \) is a right CEPI-ring. By [10], Lemma 2.4, \( R \) is right nonsingular. Let \( X \) be a singular right \( R \)-module. There exists a submodule \( Y \) of \( X \) such that \( Y \) is projective and \( X/Y \) is injective. Suppose that \( Y \neq 0 \) and let \( y \) be a nonzero element of \( Y \). Since \( Y \) is projective there exists a homomorphism \( \varphi: Y \to R \) such that \( \varphi(y) \neq 0 \). But there exists an essential right ideal \( E \) of \( R \) such that \( yE = 0 \) and hence \( \varphi(y)E = 0 \). This contradicts the fact that \( R \) is right nonsingular. Thus \( Y = 0 \) and \( X \) is injective. It follows that \( R \) is a right SI-ring.

Conversely, suppose that \( R \) is a right SI-ring. Let \( A \) be a right \( R \)-module and \( \mathcal{U} \) the collection of cyclic submodules of \( A \). By Zorn's lemma there is a maximal collection \( \mathcal{B} \) of members of \( \mathcal{U} \) whose sum is direct. Let \( \Lambda \) be an index set and \( x_\lambda \) elements of \( A \) such that \( \mathcal{B} \) is the collection of submodules \( x_\lambda R (\lambda \in \Lambda) \). Let \( B = \bigoplus_{\lambda \in \Lambda} x_\lambda R \). The choice of \( B \) ensures that \( B \) is an essential submodule of \( A \). Since \( R \) is a right SI-ring it follows that \( R \) is right hereditary (see [5], Proposition 3.3) and hence \( B \) is projective. Moreover \( A/B \) is a singular right \( R \)-module and is injective because \( R \) is a right SI-ring. It follows that every right \( R \)-module is the extension of a projective module by an injective module.

**Corollary 4.8.** If \( R \) is a right Noetherian right RIC-ring then every right \( R \)-module is the extension of a projective right \( R \)-module by an injective right \( R \)-module.

**Proof.** By the theorem and [10], Theorem 4.1.

In particular Corollary 4.8 tells us that any right Noetherian right CDPI-ring \( R \) has the property that every right \( R \)-module is the extension of a projective module by an injective module.

Next we consider right FGDPI-rings. The proof of Corollary 4.6 gives immediately:

**Lemma 4.9.** Any ring Morita equivalent to a right FGDPI-ring is itself a right FGDPI-ring.
Before examining the relationship between right FGDPI-rings and right CDPI-rings we first introduce some notation. Let \( R \) be a ring, \( n \) a positive integer and \( R_n \) the complete ring of \( n \times n \) matrices with entries in \( R \). Let \((r_{ij})\) denote the \( n \times n \) matrix whose \((i, j)\)th entry is the element \( r_{ij} \) or \( R \). For any right \( R \)-module \( X \) let \( X^{(n)} \) denote the right \( R \)-module \( X \oplus X \oplus \cdots \oplus X \) (\( n \) copies). Then \( X^{(n)} \) can be made into an \( R_n \)-module by defining:

\[
(x_1, x_2, \ldots, x_n)(r_{ij}) = \left( \sum_{k=1}^{n} x_k r_{ik}, \sum_{k=1}^{n} x_k r_{kj}, \ldots, \sum_{k=1}^{n} x_k r_{kn} \right),
\]

where \( x_i \in X \) and \( r_{ij} \in R \) (\( 1 \leq i, j \leq n \)). Let \( e_{ij} \) denote the matrix unit in \( R_n \) with 1 in the \((i, j)\)th position and zeros elsewhere. For any right \( R_n \)-module \( Y \), \( Ye_{11} \) is a right \( R \)-module. It is easy to check that for any right \( R \)-module \( X \) the right \( R \)-modules \( X \) and \( X^{(n)}e_{11} \) are isomorphic. Recall the following result.

**Lemma 4.10** (See [7], Corollary 2.3). With the above notation, a right \( R_n \)-module \( X \) is projective (respectively injective) if and only if the right \( R \)-module \( Xe_{11} \) is projective (respectively injective).

**Theorem 4.11.** Let \( n \) be a positive integer. A ring \( R \) is a right FGDPI\(_n\)-ring if and only if \( R_n \) is a right CDPI-ring.

**Proof.** Suppose that \( R_n \) is a right CDPI-ring. Let \( X \) be a right \( R \)-module generated by elements \( x_1, x_2, \ldots, x_n \). If \( Y = X^{(n)} \) then \( Y \) is the cyclic right \( R_n \)-module \((x_1, x_2, \ldots, x_n)R_n \). There exists a projective right \( R_n \)-module \( P \) and an injective right \( R_n \)-module \( Q \) such that \( Y = P \oplus Q \). Then \( Ye_{11} = (Pe_{11}) \oplus (Qe_{11}) \), as \( R \)-modules. Since the right \( R \)-modules \( X \) and \( Ye_{11} \) are isomorphic it follows that \( X \) is the direct sum of a projective module and an injective module by Lemma 4.10. Thus \( R \) is a right FGDPI\(_n\)-ring.

Conversely, suppose that \( R \) is a right FGDPI\(_n\)-ring. Let \( A = aR_n \) be a cyclic right \( R_n \)-module. Then \( Ae_{11} = aR_ne_{11} = \sum_{k=1}^{n} ae_{kk}R = \text{an generator right } R \text{-module. By hypothesis there exists a projective right } R \text{-module } B \text{ and an injective right } R \text{-module } C \text{ such that } Ae_{11} = B \oplus C \). Now \( R_n = Re_{11}R_n \) implies that \( Ae_{11}R_n = AR_ne_{11}R_n = A \) and hence \( A = (BR_n) + (CR_n) \). Since \( B = Be_{11} \) and \( C = Ce_{11} \) it follows that

\[
BR_n = \sum_{k=1}^{n} Be_{1k} \quad \text{and} \quad CR_n = \sum_{k=1}^{n} Ce_{1k}.
\]

It can easily be checked that \( B \cap C = 0 \) implies that \((BR_n) \cap (CR_n) = 0\). That is \( A = (BR_n) \oplus (CR_n) \). Moreover, \((BR_n)e_{11} = B \) and \((CR_n)e_{11} = C \). By Lemma 4.10 \( BR_n \) is a projective right \( R_n \)-module and \( CR_n \) is
an injective right $R_n$-module. It follows that $R_n$ is a right CDPI-ring.

**Corollary 4.12.** A ring $R$ is a right FGDPI-ring if and only if $R_n$ is a right CDPI-ring for every positive integer $n$.

It is interesting to contrast Theorem 4.11 with the next result.

**Theorem 4.13.** Let $R$ be a right CDPI-ring and $e$ be an idempotent element of $R$ such that $R = ReR$. Then the subring $eRe$ of $R$ is a right CDPI-ring.

**Proof.** Let $S$ denote the ring $eRe$ and let $I$ be a right ideal of $S$. If $J$ is the right ideal $IR$ of $R$ then $J \subseteq eR$ since $I = eI$. By hypothesis there exist right ideals $F$ and $G$ of $R$ such that $J \subseteq F \subseteq eR$, $J \subseteq G \subseteq eR$, $F/J$ is a projective right $R$-module, $G/J$ is an injective right $R$-module and $eR/J = (F/J) \oplus (G/J)$. Since $eR/G \cong F/J$ is projective there exists a right ideal $H$ of $R$ such that $eR = G \oplus H$. Then $Ge$ and $He$ are right ideals of $S$, $S = (Ge) \oplus (He)$ and hence $S/(Ge)$ is a projective right $S$-module. Moreover, $eR = F + G$, $F \cap G = J$ together imply $S = (Fe) + (Ge)$ and $(Fe) \cap (Ge) = Je = IRe = IeRe = I$. Thus $S/I$ is the direct sum $(Fe)/I \oplus (Ge)/I$ of the right $S$-modules $(Fe)/I$ and $(Ge)/I$. Also, $(Fe)/I \cong S/(Ge)$ is a projective right $S$-module. It remains to prove that $(Ge)/I$ is an injective right $S$-module. Note that $G = GR = GReR = GeR$. Thus it is sufficient to prove the following result.

**Lemma 4.14.** Let $R$ be a ring and $e$ be an idempotent element of $R$ such that $R = ReR$. Let $A \subseteq B$ be right ideals of the ring $S = eRe$ and $\bar{A} = AR$, $\bar{B} = BR$. If $\bar{B}/\bar{A}$ is an injective right $R$-module then $B/A$ is an injective right $S$-module.

**Proof.** Let $C$ be a right ideal of $S$ and $\varphi : C \to B/A$ an $S$-homomorphism. Let $V$ be a set of coset representatives of $A$ in $B$ and define a mapping $\alpha : C \to V$ by $\alpha(e) + A = \varphi(e)$ ($e \in C$). Define $\bar{\varphi} : CR \to \bar{B}/\bar{A}$ by

$$\bar{\varphi}\left(\sum_{i=1}^{n} c_i r_i\right) = \sum_{i=1}^{n} \alpha(c_i) e r_i + \bar{A}$$

for all positive integers $n$ and elements $c_i$ of $C$ and $r_i$ of $R$ ($1 \leq i \leq n$). Clearly $\bar{\varphi}$ is independent of the choice of $V$. Suppose $n$ is a positive integer, $r_i \in R$ and $c_i \in C$ ($1 \leq i \leq n$) and

$$\sum_{i=1}^{n} c_i r_i = 0.$$
For any element $x$ of $R$,

$$\sum_{i=1}^{n} c_i er_i x e = 0$$

and hence

$$\sum_{i=1}^{n} \varphi(c_i) er_i x e = 0 .$$

That is, for all $x$ in $R$,

$$\sum_{i=1}^{n} \alpha(c_i) er_i x e \in A .$$

Since $R = ReR$ it follows that $1 \in ReR$ and hence

$$\sum_{i=1}^{n} \alpha(c_i) er_i \in AR = \bar{A} .$$

Thus $\varphi$ is well defined and clearly $\varphi$ is an $R$-homomorphism. By hypothesis there exists an element $b$ of $B$ such that $\varphi(r) = br + \bar{A}$ ($r \in C$). It follows that $be \in \bar{Be} = BRe = BeRe = B$. Let $c \in C$. Then $c = cc = ec$ and $\varphi(c) = \alpha(c) + A = \alpha(c)e + A$ and $\bar{\varphi}(c) = \bar{\alpha}(c)e + \bar{A} = bec + \bar{A} = be \bar{c} + \bar{A}$. This implies that $\alpha(c)e - bec \in \bar{A} \cap S = A$ and hence $\varphi(c) = bec + A$. Thus $\varphi(c) = bec + A$ ($c \in C$). It follows that $B/A$ is an injective right $S$-module. This completes the proof of Lemma 4.14 and hence also of Theorem 4.13.

5. Right FGDPI-rings. Let $R$ be a semiprime right Goldie ring. Goldie [4], Theorems 4.1 and 4.4, proved that $R$ has a (classical) right quotient ring $Q$ and $Q$ is semiprime Artinian. Levy [7], Theorem 5.3, proved that if $R$ has the additional property that every finitely generated torsion-free right $R$-module is a submodule of a free right $R$-module then $Q$ is the left quotient ring of $R$ and hence by [4], Theorem 4.4, $R$ is a left Goldie ring. In actual fact to prove that $Q$ was the left quotient ring of $R$ all Levy needed was the fact that every 2-generator right $R$-submodule of $Q$ is contained in a free right $R$-module. Thus we can state Levy’s result in the following form.

**Lemma 5.1.** Let $R$ be a semiprime ring Goldie ring with right quotient ring $Q$ such that every 2-generator right $R$-submodule of $Q$ is contained in a free right $R$-module. Then $R$ is a left Goldie ring.

Next we restate [7], Theorem 6.1, as follows.

**Lemma 5.2.** Let $R$ be a semiprime right and left Goldie right
(and left) semihereditary ring. Then every finitely generated right $R$-module $X$ is the direct sum of its singular submodule $Z(X)$ and a projective $R$-submodule $P$.

**COROLLARY 5.3.** Let $R$ be a semiprime right and left Goldie ring. Then $R$ is a right FGDPI-ring if and only if $R$ is a right RIC-ring.

**Proof.** The necessity follows by [10], Lemma 2.4. Conversely, suppose that $R$ is a right RIC-ring. Let $X$ be a finitely generated right $R$-module with singular submodule $Z$. By [10] Corollary 4.3 and Lemma 4.4, $R$ is right semihereditary. By Lemma 5.2 there exists a projective submodule $P$ of $X$ such that $X = Z \oplus P$. By Lemma 4.3 $Z$ is injective. It follows that $R$ is a right FGDPI-ring.

Let $R$ be a semiprime right Noetherian ring with right quotient ring $Q$ and suppose $Q$ is a finitely generated right $R$-module. Let $a$ be a regular element of $R$ and consider the ascending chain $a^{-1}R \subseteq a^{-2}R \subseteq a^{-3}R \subseteq \cdots$ of $R$-submodules of $Q$. Since $Q$ is a Noetherian right $R$-module there exists a positive integer $n$ such that $a^{-n}R = a^{-n-1}R$. Then $a^{-n-1} = a^{-n}b$ for some element $b$ of $R$ and hence $1 = ab = ba$. It follows that $R = Q$.

**LEMMA 5.4.** Let $R$ be a prime right Noetherian right FGDPI-ring. Then $R$ is a left Goldie ring.

**Proof.** Let $Q$ be the right quotient ring of $R$. In view of Lemma 5.1 it is sufficient to prove that every 2-generator right $R$-submodule of $Q$ is contained in a free right $R$-module. Let $X$ be a 2-generator right $R$-submodule of $Q$. By hypothesis there exists a projective $R$-submodule $P$ of $X$ and an injective $R$-submodule $I$ of $X$ such that $X = P \oplus I$. Suppose that $I \neq 0$. For any regular element $c$ of $R$ we have $I = Ic$ (see [7], Theorem 3.1). Since $I$ is torsion-free, for all elements $x$ of $I$ and regular elements $c$ of $R$ there exists a unique element $x$ of $I$ such that $xc = x$. By defining $xc^{-1} = \bar{x}$ for all $x$ in $I$ and $c$ regular in $R$ we can make $I$ into a right $Q$-module. Since $I \neq 0$ and $Q$ is simple Artinian it follows that $I$ contains a simple right $Q$-module. Since $Q$ is simple Artinian all simple right $Q$-modules are isomorphic. Because $I$ is a finitely generated right $R$-module it follows that $Q$ is a finitely generated right $R$-module. As our remarks above show, in this case $R = Q$ and hence $R$ is left Goldie. Now suppose that $Q \neq R$. Then $I = 0$, $X = P$ and hence $X$ is contained in a free right $R$-module. Thus every 2-generator right $R$-submodule of $Q$ is contained in a free right $R$-module. By Lemma 5.1 $R$ is a left Goldie ring.
**Lemma 5.5.** Let $S$ and $T$ be subrings of a ring $R$ such that $R = S \oplus T$. Let $n$ be a positive integer. Then $R$ is a right FGDPI$_n$-ring if and only if $S$ and $T$ are both right FGDPI$_n$-rings.

**Proof.** Suppose that $R$ is a right FGDPI$_n$-ring. Let $X$ be an $n$-generator right $S$-module. We can make $X$ into an $n$-generator right $R$-module by defining $x(s + t) = xs$ for all $x$ in $X$, $s$ in $S$ and $t$ in $T$. By hypothesis there exists a projective right $R$-module $P$ and an injective right $R$-module $I$ such that $X = P \oplus I$. It can easily be checked that $P$ is a projective right $S$-module and $I$ is an injective right $S$-module. It follows that $S$ is a right FGDPI$_n$-ring. Similarly $T$ is a right FGDPI$_n$-ring.

Conversely, suppose first that $n = 1$; that is, $S$ and $T$ are both right CDPI-rings. Let $E$ be a right ideal of $R = S \oplus T$. Then there exists a right ideal $E_1$ of $S$ and a right ideal $E_2$ of $T$ such that $E = E_1 \oplus E_2$. Since $S$ and $T$ are right CDPI-rings there exist idempotent elements $e_1$ of $S$ and $e_2$ of $T$ such that $E_1 \subseteq e_1S$, $E_2 \subseteq e_2T$, $A = (e_1S)/E_1$ is an injective right $S$-module and $B = (e_2T)/E_2$ is an injective right $T$-module. The Abelian group $C = A \oplus B$ can be made into a right $R$-module by defining $(a, b)(s + t) = (as, bt)$ for all $a$ in $A$, $b$ in $B$, $s$ in $S$ and $t$ in $T$. If $f = e_1 + e_2$ then $f$ is an idempotent element of $R$ and $E \subseteq fR$. Moreover, $(fR)/E$ is isomorphic to the right $R$-module $C$. If $F$ is a right ideal of $R$ then $F = F_1 \oplus F_2$ for some right ideals $F_1$ of $S$ and $F_2$ of $T$, and it can easily be checked that any $R$-homomorphism $\varphi: F \to C$ can be lifted to an $R$-homomorphism $\overline{\varphi}: R \to C$. Thus $C$ is injective. It follows that $R$ is a right CDPI-ring. Now suppose that $n$ is any positive integer and $S$ and $T$ are both right FGDPI$_n$-rings. By Theorem 4.11 the matrix rings $S_n$ and $T_n$ are right CDPI-rings. But clearly $R_n \cong S_n \oplus T_n$ and the above argument shows that $R_n$ is a right CDPI-ring. By Theorem 4.11 $R$ is a right FGDPI$_n$-ring.

It is clear that one consequence of Lemma 5.5 is the following result.

**Corollary 5.6.** Let $S$ and $T$ be subrings of a ring $R$ such that $R = S \oplus T$. Then $R$ is a right FGDPI-ring if and only if both $S$ and $T$ are right FGDPI-rings.

**Theorem 5.7.** Let $R$ be a semiprime right Noetherian ring. Then the following statements are equivalent.

(i) $R$ is a right FGDPI$_2$-ring.
(ii) $R$ is a right FGDPI-ring.
(iii) $R$ is a left Goldie right RIC-ring.
(iv) \( R \) is a finite direct sum \( A \oplus B_1 \oplus B_2 \oplus \cdots \oplus B_n \) where \( A \) is a semiprime Artinian ring and for each \( 1 \leq i \leq n \) the ring \( B_i \) is a simple right and left Noetherian ring Morita equivalent to a Noetherian simple PCI-domain.

**Proof.** (ii) \( \Rightarrow \) (i) is clear. (iii) \( \Rightarrow \) (ii) is a consequence of Corollary 5.3. (iv) \( \Rightarrow \) (iii) is a consequence of [5], Theorem 3.11. It remains to prove (i) \( \Rightarrow \) (iv). Suppose that \( R \) is a right FGDPI\(_2\)-ring. By [5], Theorem 3.11, \( R \) is a finite direct sum \( A \oplus B_1 \oplus B_2 \oplus \cdots \oplus B_n \) where \( A \) is semiprime Artinian and \( B_i \) is a simple right Noetherian ring Morita equivalent to a right Noetherian simple right PCI-domain \( D_i \) for each \( 1 \leq i \leq n \). By Lemmas 5.4 and 5.5 the ring \( B_i \) is a left Goldie ring for each \( 1 \leq i \leq n \). Thus, for each \( 1 \leq i \leq n \), \( D_i \) is left Goldie and hence a Noetherian simple PCI-domain by [3], Theorem 22 and subsequent remarks. It follows that \( B_i \) is left Noetherian (1 \( \leq i \leq n \)). This proves (iv).

**Corollary 5.8.** For any positive integer \( m \) a ring \( R \) is a right Noetherian right FGDPI\(_m\)-ring if and only if \( R \) is a finite direct sum \( A \oplus B_1 \oplus B_2 \oplus \cdots \oplus B_n \) where \( A \) is a right Artinian right FGDPI\(_m\)-ring and the ring \( B_i \) is a simple right and left Noetherian ring Morita equivalent to a Noetherian simple PCI-domain for each \( 1 \leq i \leq n \).

**Proof.** By the theorem and Lemma 5.5.

**Corollary 5.9.** Let \( R \) be a semiprime ring. Then the following statements are equivalent.

(i) \( R \) is a right Noetherian right FGDPI-ring.

(ii) \( R \) is a left Noetherian left FGDPI-ring.

(iii) \( R \) is a right Noetherian right FGDPI-ring.

(iv) \( R \) is a left Noetherian left FGDPI-ring.

**Proof.** By the theorem, Lemma 5.5 and Corollary 5.6.

**Corollary 5.10.** Let \( R \) be a right Noetherian right FGDPI\(_2\)-ring with Jacobson radical \( J \). Then the ring \( R/J \) is a left Noetherian left FGDPI-ring. Moreover \( R \) is a left SI-ring and in particular \( R \) is left hereditary.

**Proof.** By Corollary 5.8 \( R/J \) is a right Noetherian right FGDPI\(_2\)-ring and by Corollary 5.9 \( R/J \) is a left Noetherian left FGDPI\(_2\)-ring. In §1 we noted that right Noetherian right CDPI-rings are right SI-rings. Also by [5], Proposition 3.5, right Artinian right SI-rings
are left SI-rings. The result follows by [5], Theorem 3.11 and Proposition 3.3.

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