Pacific Journal of Mathematics

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Vol. 76, No. 2 December 1978

PACIFIC JOURNAL OF MATHEMATICS
Vol. 76, No 2, 1978

# LOCAL AND GLOBAL CONVEXITY IN COMPLETE RIEMANNIAN MANIFOLDS

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**A connected open set in Euclidean space is convex if it is locally supported at each boundary point; indeed, the same statement holds in any complete Riemannian manifold for which all geodesics are minimal. On the other hand, in an arbitrary complete $n$-dimensional Riemannian manifold $M$ the question, under what circumstances global convexity properties are implied by local ones, involves the notion of cut locus. This question will be considered here.**

Propositions 2 and 3 give sufficient conditions, in terms of the cut loci of boundary points, for a locally supported open subset of $M$ to be weakly convex. Using a theorem of Karcher about hypersurfaces which do not intersect their own cut loci, we then obtain a condition for convexity (Proposition 4), as well as the following (Theorem 3): If $H$ is an imbedded, compact, connected topological hypersurface of $M$ which does not intersect its own cut locus (it follows then that $M \backslash H$ has two components, each with boundary $H$), and if $H$ has a one-sided field of local support elements, then $H$ is homeomorphic to $S^{n-1}$ and the supported component of $M \backslash H$ is convex.

It is hoped that these observations may prove useful in investigating global convexity in certain classes of Riemannian manifolds $M$ for which information on the behavior of cut loci is available.

The paper [6] by Karcher is our reference for facts concerning convexity and weak convexity of subsets of $M$, and cut loci of subsets of $M$. We also use the notion of local convexity defined and investigated by Cheeger and Gromoll [2].

Throughout, $M$ will denote a complete Riemannian manifold of dimension $n$. A subset $B$ of $M$ is *strongly convex* if $M$ contains exactly one minimal geodesic between any two points of $B$ and that geodesic lies in $B$; *convex* if $B$ contains exactly one minimal geodesic between any two points of $B$; and *weakly convex* if $B$ contains at least one minimal geodesic between any two points of $B$. A weakly convex open set contains *every* minimal geodesic in $M$ with endpoints in $B$; thus for open sets, convexity and strong convexity are equivalent. Any $p \in M$ has a strongly convex neighborhood, namely the open metric ball $B(p, \varepsilon)$ for $\varepsilon$ sufficiently small [5]. Finally, a subset $B$ of $M$ is *locally convex* if each point of the closure $\bar{B}$ has a strongly convex neighborhood $U$ such that $B \cap U$ is strongly convex. Clearly, a weakly convex set is locally convex.

If $B$ is an open subset of $M$, then an open halfspace $H_p$ of the tangent space $M_p$ at $p \in \partial B$ is called a *support element for* $B$ if $H_p$ contains the initial tangent vectors of all minimal geodesics from $p$ to points of $B$. $H_p$ is a *local support element for* $B$ if, for some open neighborhood $U$ of $p$, $H_p$ is a support element for $B \cap U$ [6].

The notation $[pq]$ (respectively, $[pq)$) will be used *only* when $p$ and $q$ are not cut points of each other, and will denote the unique (up to oriented reparametrization) minimal geodesic from $p$ to $q$ (resp., that geodesic with endpoint deleted). Whenever $p$ and $q$ lie on a minimal geodesic and one of $p$, $q$ is not an endpoint, we may refer to the geodesic $[pq]$.

**2. The theorem of Karcher.** We shall need the generalized Jordan-Brouwer separation theorem for arbitrary compact topological hypersurfaces of $E^n$. A proof is included because we could not find a reference.

THEOREM 1 (*Generalized Jordan-Brouwer separation theorem*). *Let $H$ be an imbedded, compact, connected topological $(n-1)$-manifold in $E^n$. Then $E^n \backslash H$ consists of two components, each with boundary $H$.*

*Proof.* By Alexander duality, $E^n \backslash H$ has two components $A_1$ and $A_2$ ([4], p. 179). Since these are open, $H \supset \partial A_1 \cup \partial A_2$; by invariance of domain, $H = \partial A_1 \cup \partial A_2$. Furthermore, since $H$ is connected, there exists $q \in H \cap \partial A_1 \cap \partial A_2$. Observe that no closed subset $H'$ of $H \backslash \{q\}$ separates $E^n$. Indeed, if $V$ is an open ball about $q$ in $E^n$ such that $V \cap H' = \phi$, then $V$ contains points of $A_1$ and $A_2$; therefore $A_1 \cup A_2 \cup V$ is a connected subset of $E^n \backslash H'$ whose closure contains $E^n \backslash H'$, and it follows that $E^n \backslash H'$ is connected.

If $p \in H$, then any two neighborhoods in $H$ of $p$ and $q$ respectively contain neighborhoods whose complements in $H$ are homeomorphic, as may be seen by joining $p$ and $q$ by an arc covered by finitely many coordinate neighborhoods. By a theorem of Borsuk ([3], p. 357), if $E^n$ is separated by a compact subset $C$ then $E^n$ is separated by any homeomorph of $C$. Thus it follows from the preceding paragraph that no proper closed subset of $H$ separates $E^n$. Therefore $H$ is the boundary of each component of the complement of $H$ ([3], p. 356). This completes the proof.

For any subset $S$ of $M$, the *cut locus* $C(S)$ of $S$ in $M$ is defined by $C(S) := \bigcup_{p \in S} C(p)$ where $C(p)$ is the cut locus of $p$. The following theorem was proved by Karcher in [6]. (It is stated there for

$H = S^{n-1}$, but the proof holds in the present case also, with the only necessary change being the substitution of Theorem 1 for the original Jordan-Brouwer theorem.)

THEOREM 2 (*Karcher*). *Let $H$ be an imbedded, compact, connected topological $(n-1)$-manifold in $M$ satisfying $H \cap C(H) = \emptyset$. Then $M \backslash H$ consists of two open components $A_1$ and $A_2$, each with boundary $H$, where* (1) *$A_1$ is bounded, and* (2)*$C(\bar{A}_1) \subset A_2$.*

The component $A_1$ is uniquely determined by (1) and (2), and is referred to as the "inside" component of $M \backslash H$.

3. **Local and global convexity.** Concerning the question raised in the introduction, the following information may be found in the paper by Karcher:

PROPOSITION 1 [6]. *A connected open subset $B$ of $M$ is convex if and only if $B$ possesses a local support element at every boundary point and does not intersect its own cut locus.*

If $B$ is a locally convex open set, then as Cheeger and Gromoll have shown [2], $\bar{B}$ is an imbedded topological manifold-with-boundary; furthermore, $B$ possesses a local support element at every boundary point. It is worth noting that an open set may possess a local support element at every boundary point and yet not be locally convex:

EXAMPLE 1. Let $g$ be the standard Riemannian metric on $\mathbf{R}^2$, $B$ be the open subset of $\mathbf{R}^2$ indicated in Figure 1, and $H$ be the indicated arc in $\partial B$. We shall alter the metric $g$ so that the inside loop beginning and ending at $p$ is a geodesic in the new metric. Let $U$ be a connected open set satisfying $H = \bar{U} \cap \partial B$ and carrying Fermi coordinates about $H$. Then there exists a Riemannian metric $h$ on $\mathbf{R}^2$ such that (1) $g$ and $h$ agree on $\mathbf{R}^2 \backslash U$, and (2) $H$ is the image of an $h$-geodesic. Indeed, $h$ may be constructed from $g$ and the flat metric $\tilde{h}$ on $U$ determined by the Fermi coordinates; one uses a smooth Urysohn function vanishing on $\mathbf{R}^2 \backslash U$ and taking value 1 on

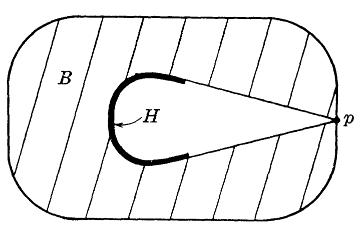


FIGURE 1

a neighborhood of every point of $H$ which does not have a neighborhood on which $\tilde{h}$ agrees with $g$. The metric $h$ is complete since it agrees with $g$ except on a compact set. By (1) and (2), $B$ has a local support element with respect to $h$ at every boundary point, but $B$ is not locally convex at $p$.

Local convexity does follow from being locally supported if $\partial B$ is an imbedded manifold (it is not necessary to assume that $\bar{B}$ is a manifold-with-boundary):

LEMMA 1. *Let $B$ be an open subset of $M$ whose boundary is an imbedded topological $(n-1)$-manifold. If $B$ possesses a local support element at every boundary point, then $B$ is locally convex.*

*Proof.* Fix $p \in \partial B$, and choose $\varepsilon$ sufficiently small that $B(p, \varepsilon')$ is convex for $\varepsilon' \leqq \varepsilon$. It follows from Proposition 1 that every connected component of $B \cap B(p, \varepsilon)$ is convex. Choose $\varepsilon'(0 < \varepsilon' \leqq \varepsilon)$ so that $\partial B \cap B(p, \varepsilon')$ lies in the component through $p$ of $\partial B \cap B(p, \varepsilon)$; this is possible because $\partial B$ is an imbedded manifold. Certainly there is a component $C$ of $B \cap B(p, \varepsilon)$ such that $\partial C \cap B(p, \varepsilon') \neq \emptyset$. Since $C$ is convex, $\partial C$ is an imbedded $(n-1)$-manifold in $M$, and hence in $\partial B$. By invariance of domain, $\partial C \cap B(p, \varepsilon)$ is open in $\partial B \cap B(p, \varepsilon)$; obviously, it is also closed. Thus, by choice of $\varepsilon'$, $p \in \partial C$. Now suppose $C_1$ is another component of $B \cap B(p, \varepsilon)$ whose boundary intersects $B(p, \varepsilon')$. Then $C \cap C_1 = \phi$, and $\bar{C}$ and $\bar{C}_1$ are imbedded manifolds-with-boundary having common boundary in a neighborhood of $p$, in contradiction to the local support hypothesis. Therefore $B \cap B(p, \varepsilon') = C \cap B(p, \varepsilon')$. Since both $C$ and $B(p, \varepsilon')$ are strongly convex, so is $B \cap B(p, \varepsilon')$, as required.

PROPOSITION 2. *A connected open subset $B$ of $M$ is weakly convex if and only if $B$ possesses a local support element at every boundary point and $B \backslash C(p)$ is connected for every $p \in \partial B$.*

*Proof.* Suppose that $B$ is locally supported and $B \backslash C(p)$ is connected for every $p \in \partial B$. Fix $p \in \partial B$, and suppose further that the set $B(p) := \{q \in B \backslash C(p) : [pq] \subset \bar{B}\}$ is nonempty. For a fixed $q \in B(p)$, no point of $(pq]$ falls on $\partial B$, since the existence of a last such point would contradict the local support hypothesis. We may choose $\varepsilon$, by Proposition 1, so that every component of $B \cap B(p, \varepsilon)$ is convex; in particular, the component $C$ containing an initial segment of $(pq]$ is convex. Then $(pq] \subset C \cup U$ where $U$ is a neighborhood in $B$ of $(pq]$. Consider a sequence of points $q_i \in B \backslash C(p)$ converging to $q$. For $i$ sufficiently large, $[pq_i]$ contains a subarc $[r_i q_i] \subset U$ where $r_i \in C \cap U$. Since $C$ is convex, then $[pr_i] \subset \bar{C}$ and hence $[pq_i] \subset \bar{B}$. It follows that $B(p)$ is open in $B \backslash C(p)$. Clearly, $B(p)$ is also closed in $B \backslash C(p)$, and so $B(p) = B \backslash C(p)$.

Now suppose $\gamma$ is a minimal geodesic in $M$ joining any $q, q' \in B$. If $\gamma \not\subset B$ then $\gamma \not\subset \bar{B}$. But then the interior of $\gamma$ contains a point $p \in \partial B$, where $q, q' \in B \backslash C(p)$, $[pq] \subset \bar{B}$, and $[pq'] \not\subset \bar{B}$. Thus $B(p)$ is nonempty and properly contained in $B \backslash C(p)$, and we have just shown that this is impossible. Therefore $\gamma \subset B$ and $B$ is weakly convex.

Conversely, if $B$ is weakly convex, then for any $p \in \partial B$ and $q, r \in B \backslash C(p)$, $B$ contains $[qp)$ and $(pr]$. Since $B$ is locally convex, it is clear that $q$ and $r$ may be joined by a path in $B \backslash C(p)$.

For a locally convex set $B$, Proposition 2 yields a condition for global convexity which involves only the boundary of $B$:

PROPOSITION 3. *Let $B$ be a connected, locally convex, open subset of $M$, and set $H = \partial B$. If $H \backslash C(p)$ is connected for all $p \in H$, then $B$ is weakly convex.*

*Proof.* By Proposition 2, it suffices to observe that $B \backslash C(p)$ is connected for each $p \in H$. Suppose instead that $B \backslash C(p) = S_1 \cup S_2$, where the $S_i$ are nonempty open separated subsets of $M$. Then $H \backslash C(p) = T_1 \cup T_2$ where $T_i = \partial S_i \backslash C(p)$. Assume $p \in T_1$. For any $q \in S_2$, since $S_1$ and $S_2$ are separated and $[pq] \cap C(p) = \emptyset$, $[pq]$ contains a point of $T_2$; thus $T_2$ is nonempty also. By assumption, there is a point $r$ in $T_1 \cap \partial T_2$ or $\partial T_1 \cap T_2$. Since $r$ has a neighborhood $U$ in $M$ not intersecting $C(p)$ and such that $B \cap U$ is connected, by local convexity, it follows that $S_1$ and $S_2$ may be joined by a path in $B \backslash C(p)$, which is impossible.

REMARK 1. The example of a weakly convex open ring on a cylinder illustrates Proposition 2 and shows that the converse of Proposition 3 is false.

PROPOSITION 4. *Let $B$ be a connected, locally convex open subset of $M$, and set $H = \partial B$. If $H$ is connected and compact and does not intersect its own cut locus, then $\bar{B}$ is bounded and strongly convex.*

*Proof.* (Assume $B \neq \emptyset$.) By Proposition 3, $B$ and therefore $\bar{B}$ are weakly convex. Since $H$ is an imbedded topological hypersurface of $M$, then by Theorem 2, $M \backslash H$ consists of two open components $A_1$ and $A_2$ with boundary $H$, where $A_1$ is bounded and $C(\bar{A}_1) \subset A_2$. Since $B$ is open and connected and $\partial B = \partial A_i$, $B$ coincides with $A_1$ or $A_2$.

Suppose $B = A_2$. Let $\gamma$ be a geodesic ray from some $p \in \partial B$

having an initial segment in $A_1$; such a $\gamma$ exists by local convexity of $B$. If there exists a cut point $r$ of $p$ along $\gamma$, then $r \in B$. Therefore there is a subarc $[pq]$ of $\gamma$ such that $p, q \in H$ and $[pq]$ does not lie in $\bar{B}$. This contradicts weak convexity of $\bar{B}$. On the other hand, if $\gamma$ contains no cut point of $p$, then since $A_1$ is bounded, again $\gamma$ must enter $B$, in contradiction to weak convexity of $\bar{B}$. Therefore $B = A_1$. Since $C(\bar{B}) \cap \bar{B} = \emptyset$ and $\bar{B}$ is weakly convex, it is immediate that $\bar{B}$ is strongly convex.

THEOREM 3. *Let $H$ be an imbedded, compact, connected topological $(n-1)$-manifold in $M$ which does not intersect its own cut locus. By Theorem 2, $M \backslash H$ consists of two components, each with boundary $H$. If a component $B$ of $M \backslash H$ has a local support element at every point of $H$, then $H$ is homeomorphic to $S^{n-1}$ and $B$ is bounded and convex.*

*Proof.* By Lemma 1, $B$ is locally convex. Therefore by Proposition 4, $B$ is bounded and convex, and is the inside component of $M \backslash H$. Furthermore, the boundary of a nonempty, bounded, convex, open set is homeomorphic to $S^{n-1}$ [6].

REMARK 2. Theorem 3 was proved by Karcher under the added assumption that $B$ is the *inside* component of $M \backslash H$, in which case the theorem is a direct consequence of Theorem 2 and Proposition 1. We have shown that the outside component of $M \backslash H$ in Theorem 2 can never be locally supported.

COROLLARY 1. *Let $H$ be a compact, connected Riemannian $(n-1)$-manifold, and $i\colon H \to M$ be an isometric imbedding such that $i(H)$ does not intersect its own cut locus. If sectional curvatures satisfy $K_H(\sigma) > K_M(i_*\sigma)$ for every 2-plane $\sigma$ tangent to $H$, then $H$ is homeomorphic to $S^{n-1}$ and $i(H)$ is the boundary of a bounded convex open subset of $M$.*

*Proof.* By assumption, the second fundamental form of $i$ is positive definite with respect to a continuous unit normal field. Therefore if $N$ is a tubular neighborhood of $i(H)$ in $M$, a fixed component of $N \backslash i(H)$ is locally supported at every point of $i(H)$, as Bishop has shown [1]. Thus a component of $M \backslash i(H)$ is locally supported at every point of $i(H)$, and Theorem 3 applies.

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Received December 28, 1976.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

# Pacific Journal of Mathematics

Vol. 76, No. 2 December, 1978