UNIFORM REPRESENTATIONS OF CONGRUENCE SCHEMES

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A congruence scheme Σ is a finite sequence of polynomials. A nontrivial equational class K is a representation of Σ iff the principal congruences in K can be described in a natural fashion by Σ. In this paper it is shown that a necessary and sufficient condition for a congruence scheme Σ whose polynomials do not contain constants to have a representation is that each polynomial in the sequence be at least binary.

1. **Introduction.** For an algebra $\mathfrak{A}$ and $a, b \in A$, let $\Theta(a, b)$ denote the smallest congruence relation under which $a \equiv b$; such relations are called *principal*. For instance, in the class D of distributive lattices (see [1]):

$$c \equiv d(\Theta(a, b)) \text{ iff}$$
$$c = p_0(b, a, b, c, d)$$
$$p_0(a, a, b, c, d) = p_1(a, a, b, c, d)$$
$$p_1(b, a, b, c, d) = p_2(b, a, b, c, d)$$
$$p_2(a, a, b, c, d) = p_3(a, a, b, c, d)$$
$$p_3(b, a, b, c, d) = d ,$$

where

$$p_0(x, y_0, y_1, y_2, y_3) = ((x \wedge y_0) \vee y_2) \wedge (y_2 \vee y_3) ,$$
$$p_1(x, y_0, y_1, y_2, y_3) = (x \vee y_0 \vee y_2) \wedge (y_2 \vee y_3) ,$$
$$p_2(x, y_0, y_1, y_2, y_3) = ((x \vee y_0) \wedge y_2) \vee y_3 ,$$
$$p_3(x, y_0, y_1, y_2, y_3) = (x \wedge y_0 \wedge y_2) \vee y_3 .$$

This is one example of a congruence scheme (for a general definition, see §2). The general definition of a congruence scheme permits an arbitrary sequence $p_0, \cdots, p_{n-1}$ of polynomials and the polynomials may have any number of variables.

As the simplest example of a congruence scheme, let K be an equational class and let us assume that for all $\mathfrak{A} \in K$ the following holds:

$c \equiv d(\Theta(a, b))$ iff $c = a + e$ and $d = b + e$ for some $e \in A$, where $+$ is a binary operation of K.

Congruence schemes have been investigated in [5] under the name 1-good systems and in [1].

An equational class K is a *uniform representation* of a congruence scheme Σ iff the principal congruence relations in all $\mathfrak{A} \in K$ can be described by Σ. The paper [1] starts with an equational class K representing a congruence scheme Σ and considers the consequences of this assumption.

In this paper we start with a congruence scheme Σ and ask under what conditions there is a nontrivial equational class K representing Σ. A trivial necessary condition (see §2) is that each polynomial be at least binary. However, this is not sufficient, as exemplified by the scheme $0 + ((0 + x) + y)$ (see §5). Our main result is that the above condition is necessary and sufficient, provided the polynomials in the scheme contain no nullary operations.

In §2 we give all the basic definitions. In §3 the representation theorem is reduced to finding a suitable simple algebra. The construction of this simple algebra is presented in §4. Some additional results, examples, and problems are given in §5.

2. **Basic definitions.** We assume that the reader is familiar with the basic concepts of universal algebra; see [2].

Given a set of polynomials Ω (of a given type τ) and a class of algebras K (of type μ) we say that K *admits* Ω or that the type of Ω *is contained in* the type of K iff all operation symbols occurring in Ω are operations in K, in other words, if every $p \in \Omega$ can be regarded a polynomial of K.

Now we give the three basic definitions. The reader should be familiar with Mal'cev's lemma (see, e.g., Theorem 10.3 in [2]), which forms the background for these definitions.

DEFINITION 1. For a type τ of algebras, a *congruence scheme* (briefly, scheme) Σ of type τ is a sequence $p_0, \cdots, p_{n-1}$ of polynomials of type τ, where p_i is a polynomial in the variables x and y_j^i, $0 \leq j < n_i$, together with a function $t: \{0, 1, \cdots, n-1\} \to \{0, 1\}$.

We shall say that a class K of algebras *admits* Σ if K admits the polynomials in Σ.

REMARKS. The variables x and y_j^i $(0 \leq j < n_i)$ are assumed to occur explicitly in p_i. Variables with distinct indices (upper or lower) are distinct from each other and from x. The integer n_i can be 0, in which case the only variable of p_i is x. (Some easy changes are necessary in the example given in the introduction to make it satisfy the formal requirements of Definition 1.)

DEFINITION 2. Let Σ be a congruence scheme and $\mathfrak{A}$ an algebra admitting Σ. We define a 4-ary relation on A:

For $a_0, a_1, b_0, b_1 \in A$, $\langle a_0, a_1, b_0, b_1 \rangle$ is in the relation Σ, in symbols, $\Sigma(a_0, a_1, b_0, b_1)$, iff there are elements $c(i, j) \in A$ $(0 \leq i < n, 0 \leq j < n_i)$ satisfying the following equations:

(i) $b_0 = p_0(a_{t(0)}, c(0, 1), \cdots, c(0, n_0 - 1))$,

(ii) $p_i(a_{1-t(i)}, c(i, 0), \cdots, c(i, n_i - 1)) = p_{i+1}(a_{t(i+1)}, c(i + 1, 0), \cdots, c(i + 1, n_{i+1} - 1))$, $i = 0, 1, \cdots, n - 2$,

(iii) $p_{n-1}(a_{1-t(n-1)}, c(n - 1, 0), \cdots, c(n - 1, n_{n-1} - 1)) = b_1$.

REMARK. We call the $c(i, j)$ the elements used to establish $\Sigma(a_0, a_1, b_0, b_1)$. Of course, in general there are many such $c(i, j)$.

Now we can restate Mal'cev's lemma in terms of these definitions:

LEMMA 3. *Let $\mathfrak{A}$ be an algebra. For $a, b, c, d \in A$, $c \equiv d(\Theta(a, b))$ iff there exists a congruence scheme Σ (of the same type as that of $\mathfrak{A}$) such that $\Sigma(a, b, c, d)$.*

DEFINITION 4. Let K be a class of algebras (of the same type) and let Σ be a congruence scheme. K is a *representation* of Σ, in notation, $K \models \Sigma$, iff K admits Σ and for all $\mathfrak{A} \in K$ and $a, b, c, d \in A$,

$$c \equiv d(\Theta(a, b)) \qquad \text{iff} \quad \Sigma(a, b, c, d) \quad \text{holds in} \quad \mathfrak{A}.$$

If $K = \{\mathfrak{A}\}$, then we say that $\mathfrak{A}$ is a representation of Σ or $\mathfrak{A} \models \Sigma$.

REMARK. For $a, b \in A$, then $b \equiv b(\Theta(a, a))$ always holds. Thus if $\mathfrak{A} \models \Sigma$, then we always have $\Sigma(a, a, b, b)$; similarly, we can also conclude $\Sigma(a, b, a, b)$ and $\Sigma(a, b, b, a)$.

DEFINITION 5. A congruence scheme Σ has a *uniform representation K* iff K is a nontrivial equational class representing Σ.

For examples, the reader is referred to [1].
The following observation will be useful:

COROLLARY 6. *Let $\mathfrak{A}$ be an algebra admitting the congruence scheme Σ. For $a, b \in A$, let $c(i, j)$ be the elements used to establish $\Sigma(a, a, b, b)$. Then*

$$p_i(a, c(i, 0), \cdots, c(i, n_i - 1)) = b, \quad \text{for all} \quad 0 \leq i < n.$$

Proof. The polynomials exhibited in Definition 2 with the substitution of a, a, b, b are all congruent to each other and to b under $\Theta(a, a) = \omega$, hence the statement.

Now we come to the necessary condition.

LEMMA 7. *Let Σ be a congruence scheme. If $n_i = 0$ for some $0 \leq i < n$, then Σ has no representation by an algebra with more than one element. In particular, Σ has no uniform representation.*

Proof. Let $\mathfrak{A} \models \Sigma$ and $|A| > 1$. Let $n_i = 0$. Let $a, b \in A$ and $a \neq b$. By Definition 4, $\Sigma(a, a, b, b)$ and so by Corollary 6, $p_i(a) = b$. Similarly, $\Sigma(a, a, a, a)$ and hence $p_i(a) = a$, contradicting that $a \neq b$.

3. The reduction theorem. The proof of the main theorem is given in two stages.

THEOREM 8. *Let Σ be a congruence scheme. Then Σ has a uniform representation iff there exists a simple algebra with more than one element representing Σ.*

Proof. Let us assume that Σ has a uniform representation, that is, $K \models \Sigma$, where K is a nontrivial equational class. By a result of R. Magari [4], K contains a simple algebra $\mathfrak{A}$ with more than one element. Since $\mathfrak{A} \in K$, $\mathfrak{A} \models \Sigma$.

Conversely, let $\mathfrak{A}$ be a simple algebra of type μ with more than one element and let $\mathfrak{A} \models \Sigma$. Let μ' denote the type we obtain by augmenting μ with a ternary operation q and the 4-ary operations $q(i, j)$, $0 \leq i < n$, $0 \leq j < n_i$.

Since $\mathfrak{A}$ is simple and $\mathfrak{A} \models \Sigma$, $\Sigma(a_0, a_1, b_0, b_1)$ holds in $\mathfrak{A}$ whenever $a_0 \neq a_1$ or if $a_0 = a_1$ and $b_0 = b_1$. Let $c(i, j)$ denote the elements used to establish $\Sigma(a_0, a_1, b_0, b_1)$.
Define

$$q(i, j)(a_0, a_1, b_0, b_1) = c(i, j) \, .$$

Define $q(i, j)(a_0, a_0, b_0, b_1)$ for $b_0 \neq b_1$ in an arbitrary way.

Let q be an arbitrary majority function on A, that is, a ternary function satisfying $q(a, b, b) = q(b, b, a) = q(b, a, b) = b$ for all $a, b \in A$.

Let $\mathfrak{A}'$ denote the algebra of type μ' we obtain from $\mathfrak{A}$ by defining q and the $q(i, j)$ as described above.

Let p_i' denote the polynomial defined by

$$p_i' = p_i(x, q(i, 0)(z_0^i, z_1^i, z_2^i, z_3^i), \cdots, q(i, n_i - 1)(z_0^i, z_1^i, z_2^i, z_3^i)) \, .$$

Then $\mathfrak{A}'$ satisfies the following set Φ of first-order axioms:

$$(x)(y)(q(x, y, y) = q(y, x, y) = q(y, y, x) = y)$$
$$(x_0)(x_1)(y_0)(y_1)(x_0 \neq x_1 \longrightarrow (y_0 = p_0'(x_{t(0)}, x_0, x_1, y_0, y_1) \wedge \cdots \wedge$$
$$p_i'(x_{1-t(i)}, x_0, x_1, y_0, y_1) = p_{i+1}'(x_{t(i+1)}, x_0, x_1, y_0, y_1) \wedge \cdots \wedge$$

$$p_{n-1}(x_{1-t(n-1)}, x_0, x_1, y_0, y_1) = y_1))$$
$$(x)(y)(y = p'_0(x, x, x, y, y) = \cdots = p'_{n-1}(x, x, x, y, y)) \ .$$

Let K_1 denote the class of all algebras of type μ' satisfying Φ. Since $\mathfrak{A}' \in K_1$, K_1 contains an algebra of more than one element. K_1 is closed under the formation of ultraproducts (since K_1 is axiomatic), subalgebras (since Φ is universal), and homomorphic images (since all algebras in K_1 are simple, in view of the second sentence in Φ and the fact that K_1 contains the one-element algebra). In view of the first sentence in Φ, all algebras in $K = HSP(K_1)$ are congruence distributive, hence by Jónsson's lemma (see, e.g., [2]) and the observations in the previous sentence, K is the class of subdirect products of algebras in K_1.

Finally, we verify $\mathfrak{B} \models \Sigma$ for all $\mathfrak{B} \in K$. Since $\mathfrak{B} \in K$, $\mathfrak{B}$ is a subdirect product of algebras $\mathfrak{A}_m \in K_1$, $m \in M$. Take $a_0, a_1, b_0, b_1 \in B$ satisfying $b_0 \equiv b_1(\Theta(a_0, a_1))$. For $x \in B$, let x^m denote the projection of x into $\mathfrak{A}_m$. Then $b_0^m \equiv b_1^m(\Theta(a_0^m, a_1^m))$ for all $m \in M$. Thus

$$\Sigma(a_0^m, a_1^m, b_0^m, b_1^m)$$

is established with the elements $q(i, j)(a_0^m, a_1^m, b_0^m, b_1^m)$. Let $c(i, j)$ denote the element of $\Pi(\mathfrak{A}_m \,|\, m \in M)$ for which

$$c(i, j)^m = q(i, j)(a_0^m, a_1^m, b_0^m, b_1^m)$$

for all $m \in M$. Since $c(i, j)$ is a polynomial of a_0, a_1, b_0, b_1, we have $c(i, j) \in B$ and so these elements (the $c(i, j)$, $0 \leqq i < n$, $0 \leqq j < n_i$) establish $\Sigma(a_0, a_1, b_0, b_1)$ in $\mathfrak{B}$. This completes the proof of Theorem 8.

Observe that the equational class K constructed to prove Theorem 8 has a number of interesting properties: K is congruence distributive; K is semisimple, that is, all subdirectly irreducible algebras in K are simple; K has the Congruence Extension Property (this follows from a result in [1]). By adding more operations in the construction of $\mathfrak{A}'$ we could make K satisfy any Mal'cev condition, such as congruence permutability, regularity, and so on.

4. The construction of simple algebras. To construct a simple algebra representing a given congruence scheme Σ, we first prove two theorems about idempotent algebras and a special type of congruence scheme. An algebra $\mathfrak{A}$ is called *idempotent* iff all operations f of $\mathfrak{A}$ have arity > 0 and $f(a, \cdots, a) = a$ for all $a \in A$. A polynomial p is called *idempotent reduced* iff it contains no constants and no subpolynomials of the form $f(u, \cdots, u)$ where f is an operation. (For instance, $(x + y_1) + ((x + y_1) + y_2)$ is idempotent reduced, while

$$(x + (y_1 + y_2)) + (x + (y_1 + y_2)) \quad \text{and} \quad (x + (y_1 + y_1)) + (x + y_2)$$

are not.)

THEOREM 9. *Let $\mathfrak{A}$ be an idempotent algebra of type τ, $\langle a, b, c, d \rangle \in A^4$ and either $a \neq b$ or $a = b$ and $c = d$. Let $p(x, y_1, \cdots, y_n)$ be an idempotent reduced polynomial of type τ, $n \geq 1$. Then there exists an idempotent algebra $\mathfrak{A}'$ containing $\mathfrak{A}$ as a subalgebra and there are elements $e_1, \cdots, e_n \in A'$ such that $c = p(a, e_1, \cdots, e_n)$ and $d = p(b, e_1, \cdots, e_n)$.*

Proof. With a polynomial p we associate a partial algebra $\mathfrak{C}(p)$. First we define $C(p)$:

for $p = x_i$, $C(p) = \{p\}$;

if $p = f(p_0, \cdots, p_{n-1})$,   where f is an operation and $C(p_0), \cdots, C(p_{n-1})$ have already been defined, then

$$C(p) = \{p\} \cup \bigcup (C(p_i) \mid 0 \leq i < n) .$$

For every operation f in the type we define a partial operation f on $C(p)$; the domain of f in $C(p)$ consists of all $\langle p_0, \cdots, p_{n-1} \rangle \in C(p)^n$ (f is n-ary) for which $q = f(p_0, \cdots, p_{n-1}) \in C(p)$ and we define $f(p_0, \cdots, p_{n-1})$ to be q. $\mathfrak{C}(p)$ is the corresponding partial algebra.

Now let $\mathfrak{A}$, $p(x, y_1, \cdots, y_n)$, and a, b, c, d be given. First we consider the case $a \neq b$.

Define $D = A \cup (C(p(x, z_1, \cdots, z_n)) \cup C(p(x', z_1, \cdots, z_n)))$, where $\cup$ stands for disjoint union and $x, x', z_1, \cdots, z_n$ are distinct variables. Observe that $C(p(x, z_1, \cdots, z_n)) \cap C(p(x', z_1, \cdots, z_n))$ consists of all subpolynomials of $p(x, z_1, \cdots, z_n)$ that do not contain x while

$$C(p(x, z_1, \cdots, z_n)) \cap [x] = \{x\} \quad \text{and} \quad C(p(x', z_1, \cdots, z_n)) \cap [x] = \varnothing .$$

Define on D the operations f of $\mathfrak{A}$ as partial operations. If f is m-ary, the domain of f in D, Dom $(f, \mathfrak{D})$, is

$$A^m \cup \text{Dom} (f, \mathfrak{C}(p(x, z_1, \cdots, z_n))) \cup \text{Dom} (f, \mathfrak{C}(p(x', z_1, \cdots, z_n)))$$

and f is defined on its domain to make $\mathfrak{A}$, $\mathfrak{C}(p(x, z_1, \cdots, z_n))$ and $\mathfrak{C}(p(x', z_1, \cdots, z_n))$ subalgebras of $\mathfrak{D}$. It is easily seen that this definition is not ambiguous.

Now let Θ be the smallest equivalence relation on D under which $a \equiv x$, $b \equiv x'$, $c \equiv p(x, z_1, \cdots, z_n)$, and $d \equiv p(x', z_1, \cdots, z_n)$. We claim that Θ is a congruence relation of $\mathfrak{D}$. Indeed, let f be an m-ary partial operation, let $u_i \equiv v_i(\Theta)$, $i = 1, 2, \cdots, m$, and let $f(u_1, \cdots, u_m)$, $f(v_1, \cdots, v_m)$ be both defined. Since Θ restricted to A is ω, we can assume without loss of generality that

$$\langle u_1, \cdots, u_n \rangle \in \mathrm{Dom}\ (f,\ \mathfrak{C}(p(x, z_1, \cdots, z_n)))\ .$$

Since p is not a proper subpolynomial of itself, no u_i is $p(x, z_1, \cdots, z_n)$. Hence either $\langle v_1, \cdots, v_n \rangle \in A^n$ and

$$\langle u_1, \cdots, u_n \rangle = \langle x, \cdots, x \rangle,\ \langle v_1, \cdots, v_n \rangle = \langle a, \cdots, a \rangle\ ,$$

contradicting that p is idempotent reduced or

$$\langle v_1, \cdots, v_n \rangle \in \mathrm{Dom}\ (f,\ \mathfrak{C}(p(x, z_1, \cdots, z_n)))$$

in which case we must have $u_i = v_i$ for all $i = 1, \cdots, m$ since Θ restricted to $C(p(x, z_1, \cdots, z_n))$ is ω. This proves that Θ is a congruence relation.

Now let $\mathfrak{A}'$ be any completion of $\mathfrak{D}/\Theta$ into an idempotent algebra. If $a = b$, then we start with $D = A \cup C(p(x, z_1, \cdots, z_n))$ and proceed as above. This completes the proof of Theorem 9.

Let us call a congruence scheme Σ *idempotent reduced* iff each polynomial occurring in Σ is idempotent reduced.

THEOREM 10. *Let Σ be an idempotent reduced congruence scheme satisfying $n_i \geqq 1$ for $0 \leqq i < n$. Then there exists a countable nontrivial simple idempotent algebra $\mathfrak{A}$ representing Σ.*

Proof. We proceed by repeated applications of Theorem 9. Let $\mathfrak{A}_0$ be an idempotent algebra of a type admitting Σ and $1 < |A_0| \leqq \aleph_0$ (such an $\mathfrak{A}_0$ obviously exists). Now let us assume that we already have $\mathfrak{A}_0, \cdots, \mathfrak{A}_m$ with the following properties:

(i) $\mathfrak{A}_{i-1}$ is a subalgebra of $\mathfrak{A}_i$ for $i < m$; (ii) if $\langle a_0, a_1, b_0, b_1 \rangle \in A_{i-1}$ and $a_0 \neq a_1$ or $a_0 = a_1$ and $b_0 = b_1$, then $\Sigma(a_0, a_1, b_0, b_1)$ in $\mathfrak{A}_i$ for $i \leqq m$; (iii) $|A_i| \leqq \aleph_0$ for $i \leqq m$. To construct $\mathfrak{A}_{m+1}$ let

$$\{\langle a_0^i, a_1^i, b_0^i, b_1^i \rangle \,|\, 0 < i < \omega\}$$

be all elements $\langle a_0, a_1, b_0, b_1 \rangle$ of A_m^4 such that $a_0 \neq a_1$ or $a_0 = a_1$ and $b_0 = b_1$ and $\Sigma(a_0, a_1, b_0, b_1)$ does not hold in $\mathfrak{A}_m$. We define a sequence of idempotent algebras $\mathfrak{B}_0, \mathfrak{B}_1, \cdots, \mathfrak{B}_k \cdots k < \omega$ such that $\mathfrak{B}_0 = \mathfrak{A}_m$, $\mathfrak{B}_i$ is a subalgebra of $\mathfrak{B}_{i+1}$, $|B_i| \leqq \aleph_0$, and $\Sigma(a_0^i, a_1^i, b_0^i, b_1^i)$ in $\mathfrak{B}_i$. Indeed, if $\mathfrak{B}_0, \cdots, \mathfrak{B}_i$ are already constructed, then we construct $\mathfrak{B}_{i+1}$ as follows: if $\Sigma(a_0^{i+1}, a_1^{i+1}, b_0^{i+1}, b_1^{i+1})$ in $\mathfrak{B}_i$, then $\mathfrak{B}_i = \mathfrak{B}_{i+1}$. If not, we adjoin to B_i $n-1$ new elements $w_0, \cdots, w_{n-2}$ and take an idempotent algebra $\mathfrak{B}_i'$ containing $\mathfrak{B}_i$ as a subalgebra, $|B_i'| \leqq \aleph_0$, such that $w_0, \cdots, w_{n-2} \in B_i'$. Now successively apply (the construction of) Theorem 9 to $\langle a_{t(0)}^i, a_{1-t(0)}^i, b_0^i, w_0 \rangle, \langle a_{t(1)}^i, a_{1-t(1)}^i, w_0, w_1 \rangle, \cdots, \langle a_{t(n-1)}^i, a_{1-t(n-1)}^i, w_{n-2}, b_1^i \rangle$ and call the resulting algebra $\mathfrak{B}_{i+1}$. Then $|B_{i+1} - B_i| < \aleph_0$, hence $|B_{i+1}| \leqq \aleph_0$.

Let $\mathfrak{A}_m$ be the direct limit of the $\mathfrak{B}_i$, $i < \omega$, and let $\mathfrak{A}$ be the

direct limit of $\mathfrak{A}_m$, $m < \omega$. $\mathfrak{A}$ is obviously idempotent and nontrivial, and $|A| \leqq \aleph_0$. To show that $\mathfrak{A}$ is simple we verify that for a, b, c, $d \in A$ and $a \neq b$, we have $c \equiv d(\Theta(a, b))$. We prove more: if $\langle a_0, a_1, b_0, b_1 \rangle \in A^4$ and $a_0 \neq a_1$ or $a_0 = a_1$ and $b_0 = b_1$, then $\Sigma(a_0, a_1, b_0, b_1)$. This is obvious, since $\langle a_0, a_1, b_0, b_1 \rangle \in \mathfrak{A}_m$ for some $m < \omega$ and hence $\Sigma(a_0, a_1, b_0, b_1)$ in $\mathfrak{A}_{m+1}$ and therefore in $\mathfrak{A}$. This shows that $\mathfrak{A}$ is simple and that $\mathfrak{A}$ represents Σ. This completes the proof of Theorem 10.

To obtain the main result we have to get rid of the condition "idempotent reduced" in the previous results.

DEFINITION 11. Define a map $p \to \hat{p}$ from the set of polynomials containing no constants of a given type into itself by the rules:

(i) $\hat{x} = x$ for any variable x.

(ii) Let $p = f(q_1, \cdots, q_m)$, where f is an operation. Then $\hat{p} = \hat{q}_1$, if $\hat{q}_1 = \cdots = \hat{q}_m$ and $\hat{p} = f(\hat{q}_1, \cdots, \hat{q}_m)$ otherwise.

For a congruence scheme Σ, $\hat{\Sigma}$ is the scheme obtained from Σ by replacing each polynomial p by $\hat{p}$ (and retaining t).

COROLLARY 12. (i) *Let p be a polynomial. If the variable x occurs in p, then it occurs in $\hat{p}$.*

(ii) *Let $\mathfrak{A}$ be an idempotent algebra admitting the polynomial p. Then $p(a_1, \cdots, a_n) = \hat{p}(a_1, \cdots, a_n)$ for all $a_1, \cdots, a_n \in A$.*

(iii) *Let $\mathfrak{A}$ be an idempotent algebra and let Σ be a congruence scheme. Then $\mathfrak{A}$ represents Σ iff $\mathfrak{A}$ represents $\hat{\Sigma}$.*

Now our main result is a simple combination of Lemma 7, Theorems 8, 10, and Corollary 12(iii):

THEOREM 13. *A congruence scheme Σ containing no constant has a uniform representation iff all polynomials in Σ are at least binary.*

This construction can be easily modified to give a single uniform representation of any set of congruence schemes satisfying the hypothesis of Theorem 13.

5. **Concluding remarks.** We start out with a scheme containing a constant which does not have a uniform representation even though it satisfies the necessary condition of Lemma 7. In fact, we prove more.

THEOREM 14. *Let Σ be the congruence scheme given by $p = 0 + ((0 + x) + y)$ and $t(0) = 0$. Then there is no algebra $\mathfrak{A}$ with more than one element representing Σ.*

Proof. Let $\mathfrak{A}$ represent Σ, $|A| > 1$. We claim that $x \to 0 + x$ is a permutation of A. Indeed, let $a, b \in A$, $0 + a = 0 + b$, and $c \equiv d(\Theta(a, b))$ in $\mathfrak{A}$. Then $\Sigma(a, b, c, d)$, that is, $c = 0 + ((0 + a) + e)$ and $d = 0 + ((0 + b) + e)$, hence $c = d$. Thus $\Theta(a, b) = \omega$ and so $a = b$. Since $a \equiv a(\Theta(0, 0))$, there is an $e \in A$, such that $a = 0 + ((0 + a) + e)$, hence $a = 0 + a'$ with $a' = (0 + a) + e$. This shows that to every $a \in A$, there is a unique $a' \in A$ satisfying $a = 0 + a'$.

Next fix an $a \in A$, $a \neq 0'$ and consider the congruence $a \equiv b(\Theta(0', a))$. We claim that b is uniquely determined. Indeed, $\Sigma(0', a, a, b)$, hence $a = 0 + ((0 + 0') + e)$ and $b = 0 + ((0 + a) + e)$. In the first equation, $0 + 0' = 0$, hence $a = 0 + (0 + e)$. Thus $0 + e = a'$ and so e is uniquely determined (in fact, $e = a''$), yielding $b = 0 + ((0 + a) + a'')$. Thus b is uniquely determined. But this is clearly a contradiction since both $b = a$ and $b = 0'$ satisfy the congruence.

Binary polynomials play a special role in connection with representations of congruence schemes as witnessed by the following results:

(i) Let the scheme Σ have a uniform representation. Then there exists a binary scheme Σ' (that is, all polynomials in Σ' are binary) such that Σ' has a uniform representation and every equational class K representing Σ' also represents Σ.

(ii) An equational class representing a binary congruence scheme contains no finite algebra of more than one element.

(iii) Any equational class K representing a congruence scheme satisfies an identity of the form $x = q(x, y)$, where q is a binary polynomial in which x and y occur. In particular, K cannot be regular.

(ii) is the most complicated of these three statements so we shall prove it. Let Σ be a congruence scheme with the polynomials $p_i(x, y_i)$, $0 \leq i < n$. Let $\mathfrak{A}$ be an algebra, $\mathfrak{A} \models \Sigma$, and let A be finite, $|A| > 1$. Fix $a_0, a_1 \in A$, $a_0 \neq a_1$. For all $b \in B$ we have $\Sigma(a_0, a_0, b, b)$ and $\Sigma(a_1, a_1, b, b)$; let $f_i^0(b)$, $0 \leq i < n$, and $f_i^1(b)$, $0 \leq i < n$, be the elements establishing these. By Corollary 6,

$$p_i(a_j, f_i^j(b)) = b, \ j = 0, 1; \ i = 0, 1, \cdots, n - 1 \, .$$

Thus each f_i^j is one-to-one. Since A is finite, each f_i^j is a permutation of A. In particular, each f_i^j is onto, hence

$$p_i(x, u) = p_i(x, v) \quad \text{implies that} \quad u = v \, .$$

Consider now any $b \in A$ satisfying $\Sigma(a_0, a_1, a_0, b)$, and let $c_0, \cdots, c_{n-1}$ be the elements used to establish it. Thus

$$a_0 = p_0(a_{t(0)}, c_0), \ \cdots, \ p_i(a_{1-t(i)}, c_i)$$
$$= p_{i+1}(a_{t(i+1)}, c_{i+1}), \ \cdots, \ p_{n-1}(a_{1-t(n-1)}, c_{n-1}) = b \, .$$

Thus c_0 is uniquely determined, in fact, $c_0 = f_0^{t(0)}(a_0)$. Similarly, $c_1 = f_1^{t(1)}(p_0(a_{1-t(0)}c_0))$. Continuing thus, we find that each c_i, $1 \leqq i \leqq n$, is uniquely determined and so b is uniquely determined. But this is clearly a contradiction since both $b = a_0$ and $b = a_1$ satisfy $\Sigma(a_0, a_1, a_0, b)$ and $a_0 \neq a_1$. This completes the proof of (ii).

(ii) is particularly interesting because it provides a large variety of equational classes K of algebras with no nontrivial finite algebras. In fact our method finds such K that are idempotent, congruence distributive, etc.

Another result of some interest is

(iv) Let Σ be a congruence scheme with $n = 1$ (that is, with a single polynomial). Then every equational class representing K is congruence permutable.

This follows from H. Lakser [3]; see also H. Werner [6].

Some of the open problems can be stated in terms of three quasi-orderings defined on congruence schemes:

$$\Sigma_0 \subseteqq_i \Sigma_1 \Longleftrightarrow (\Sigma_0(a, b, c, d) \Longrightarrow \Sigma_1(a, b, c, d))$$
$$\Sigma_0 \subseteqq_a \Sigma_1 \Longleftrightarrow (\mathfrak{A} \models \Sigma_0 \Longrightarrow \mathfrak{A} \models \Sigma_1)$$
$$\Sigma_0 \subseteqq_e \Sigma_1 \Longleftrightarrow (K \models \Sigma_0 \Longrightarrow K \models \Sigma_1) ,$$

where K is an equational class.

Natural problems are:

I. Describe the relations $\subseteqq_i, \subseteqq_a, \subseteqq_e$.

II. Define $\Sigma_0 \equiv_i \Sigma_1$ iff $\Sigma_0 \subseteqq_i \Sigma_1$ and $\Sigma_1 \subseteqq_i \Sigma_0$, similarly define $\equiv_a$ and $\equiv_e$. Given a scheme Σ, find a scheme Σ' of smallest "rank" such that $\Sigma' \equiv_i \Sigma$; similarly for $\equiv_e$ and $\equiv_a$.

III. It would be interesting to investigate equational classes of semigroups and some other algebraic systems form the point of view of congruence schemes.

The main result of this paper is no longer true if in Definition 1 a p_i and a p_j ($i \neq j$) can have more than one variable in common.

IV. Find the corresponding result.

Rather than using a scheme to describe principal congruences, one can devise schemes to describe $\bigvee (\Theta(a_i, b_i) \mid i = 1, \cdots, n)$.

V. Generalize the main result to such schemes.

Given an algebra $\mathfrak{A}$, define Sch $\mathfrak{A}$ as the set of all congruence schemes Σ such that $\mathfrak{A} \models \Sigma$. For a class K of algebras, set

$$\text{Sch } \boldsymbol{K} = \bigcap (\text{Sch } \mathfrak{A} \,|\, \mathfrak{A} \in \boldsymbol{K}) \,.$$

Observe that $\subseteqq_a$ and $\subseteqq_e$ can be defined in terms of Sch $\mathfrak{A}$ and Sch $\boldsymbol{K}$, respectively.

This notation justifies the use of the term "representation" since $\boldsymbol{K}$ represents Σ iff $\Sigma \in \text{Sch } \boldsymbol{K}$, that is, Σ actually occurs in Sch $\boldsymbol{K}$.

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