CENTERS OF REGULAR SELF-INJECTIVE RINGS

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This paper is concerned with calculating centers of regular self-injective rings, particularly those obtained by completions with respect to rank functions, and those obtained as factor rings of other regular self-injective rings. Sufficient conditions are developed under which the completion of a regular ring $R$ has the same center as $R$. For any regular self-injective ring $R$ of Type $I_f$, it is shown that the center of any factor ring of $R$ is a factor ring of the center of $R$. These results are used to distinguish among the simple regular self-injective rings of Type $II_f$ by their possible centers.

All rings in this paper are associative with unit, and all ring maps are assumed to preserve the unit.

1. Introduction. The class of regular, right self-injective rings may be divided into subclasses using the theory of types as in [6, Chapters 5-7]. In particular, any indecomposable, regular, right self-injective ring must be one of Types $I_f$, $I_\infty$, $II_f$, $II_\infty$, or $III$ [6, Corollary 7.6]. The indecomposable, regular, right self-injective rings of Types $I_f$ and $I_\infty$ are easy enough to describe, since those of Type $I_f$ are the simple artinian rings, while those of Type $I_\infty$ are the endomorphism rings of infinite-dimensional right vector spaces [6, Theorem 5.4].

In the remaining cases, however, very little is known. The center suggests itself as a reasonable invariant with which to distinguish among different rings of the same type, particularly in the indecomposable case, where the center is a field. In this paper, we develop some techniques for calculating centers, and we apply these techniques to the standard Type $II$ examples (described below). In particular, we show that any field can be the center of a simple, regular, right and left self-injective ring of Type $II$, and that the standard Type $II$ examples can be distinguished by means of their centers.

Both of the standard Type $II$ examples are built up from certain sequences of simple artinian rings, one by the completion of a direct limit, the other by a factor ring of the direct product. The second of these is the easiest to describe, as follows. Let $R_1, R_2, \cdots$ be simple artinian rings whose composition series lengths are unbounded, and set $R = \Pi R_n$. If $M$ is any maximal two-sided ideal of $R$ which contains $\bigoplus R_n$, then it follows from [6, Corollary
that $R/M$ is a simple, regular, right and left self-injective ring of Type II.

To describe the other example, we outline the completion process as developed in [8, 4, 5]. A pseudo-rank function on a regular ring $R$ [4, p. 269] is a map $N: R \to [0, 1]$ such that

(a) $N(1) = 1$.
(b) $N(xy) \leq N(x), N(y)$ for all $x, y \in R$.
(c) $N(e + f) = N(e) + N(f)$ for all orthogonal idempotents $e, f \in R$.

A rank function on $R$ [13, p. 231] is a pseudo-rank function $N$ such that

(d) $N(x) > 0$ for all nonzero $x \in R$.

It follows from (b) and (c) that $N(x + y) \leq N(x) + N(y)$ for all $x, y \in R$ [13, Corollary, p. 231].

Any pseudo-rank function $N$ on $R$ induces a pseudo-metric $\delta$ on $R$ according to the rule $\delta(x, y) = N(x - y)$, and the ring operations on $R$ are uniformly continuous with respect to $\delta$ [13, pp. 231, 232]. Thus the (Hausdorff) completion of $R$ with respect to $\delta$ is a ring $\overline{R}$, which we refer to as the $N$-completion of $R$. Note that the kernel of the natural map $R \to \overline{R}$ equals the kernel of $N$. According to [8, Theorem 3.7], $\overline{R}$ is a regular ring, $N$ extends continuously to a rank function $\overline{N}$ on $\overline{R}$, and $\overline{R}$ is complete with respect to $\overline{N}$. Also, $\overline{R}$ is right and left self-injective by [4, Corollary 15].

Given a simple regular ring $R$ with a rank function $N$, the $N$-completion $\overline{R}$ of $R$ need not be a simple ring or even indecomposable, as [4, Examples B, C] show. Necessary and sufficient conditions for $\overline{R}$ to be simple are given in [4, Corollary 20]. In particular, [4, Corollary 21] says that if $N$ is unique then $\overline{R}$ is simple.

The remaining Type II example is now constructed as in the following proposition.

**Proposition 1.1.** Let $R_1 \to R_2 \to \cdots$ be a sequence of simple artinian rings and ring maps such that the composition series lengths of the $R_n$ are unbounded, and set $R = \lim R_n$. Then there exists a unique rank function $N$ on $R$, and the $N$-completion of $R$ is a simple, regular, right and left self-injective ring of Type II.

**Proof.** According to [4, Proposition 2], there is a unique rank function on each $R_n$. Inasmuch as rank functions on $R$ are induced by compatible rank functions on the $R_n$, it follows that there is a unique rank function $N$ on $R$.

Let $\overline{R}$ denote the $N$-completion of $R$. Then $\overline{R}$ is regular by
[8, Theorem 3.7] and [4, Corollary 15] says that $R$ is right and left self-injective. Since $N$ is unique, [4, Corollary 21] says that $R$ is simple.

Inasmuch as the composition series lengths of the $R_n$ are unbounded, we infer that there exists an infinite sequence of nonzero orthogonal idempotents in $R$. Since $R$ is simple, the natural map $R \to \bar{R}$ is injective, hence $\bar{R}$ contains an infinite sequence of nonzero orthogonal idempotents. Thus $\bar{R}$ is not artinian. Consequently, we conclude as in [6, pp. 33, 34] that $\bar{R}$ is Type II.

We conclude this section by showing that any field can be the center of a regular, right self-injective ring of Type III. This result was proved by Handelman after seeing a preliminary version of this paper.

**Proposition 1.2 (Handelman).** Let $F$ be any field. Then there exists a simple, regular, right self-injective ring $R$ of Type III such that $\text{center}(R) \cong F$.

**Proof.** Set $S_n = M_{2n}(F)$ for all $n = 1, 2, \ldots$. Map each $S_n \to S_{n+1}$ along the diagonal, i.e., map $x \mapsto \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$, and set $S = \lim S_n$. Note that $S$ is a simple regular ring, and that $\text{center}(S) \cong F$.

Now let $R$ be the maximal right quotient ring of $S$. Since $S$ is a simple regular ring, we see that $R$ is a simple, regular, right self-injective ring. Given any $x \in \text{center}(R)$, we see that $J = \{ s \in S \mid xs \in S \}$ is a nonzero two-sided ideal of $S$, whence $J = S$ and so $x \in S$. Consequently, $\text{center}(R) = \text{center}(S) \cong F$.

In [7, Example (e), pp. 831, 832], it is shown that $R$ is directly infinite. Since $R$ is simple, it follows that $R$ contains no nonzero directly finite idempotents. Therefore $R$ is Type III.

2. Completions. This section is concerned with calculating the center of the completion $\bar{R}$ of a regular ring $R$ with respect to a pseudo-rank function, and in particular with conditions which ensure that the center of $\bar{R}$ coincides with the center of $R$. As our methods deal with direct limits of semisimple (artinian) rings, we begin with the case where $R$ equals the ring $M_n(D)$ of all $n \times n$ matrices over a division ring $D$. Our method, which is an extension of [9, Theorem 5], involves comparing the ranks of additive commutators $xy - yx$ with the ranks of differences $x - z$, where $z \in \text{center}(R)$. (By the rank of a matrix $x \in M_n(D)$, we mean the number of linearly independent rows (or columns) of $x$.)

In order to construct additive commutators in $M_n(D)$ with
suitably large ranks, we use the rational canonical form for matrices in $M_n(D)$, which works as well over division rings as over fields [10, p. 50]. Specifically, any matrix in $M_n(D)$ is similar to a block diagonal matrix where each block is a companion matrix, i.e., a matrix of the form

$$
\begin{bmatrix}
0 & 1 & & \\
0 & 0 & 1 & \\
& & \ddots & \ddots \\
& & & 0 & 1 \\
* & \cdots & & & *
\end{bmatrix}
$$

**Lemma 2.1.** Let $D$ be a division ring, let $n \geq 2$ be an integer, and let $x \in M_n(D)$ be a companion matrix. Then there exists $y \in M_n(D)$ such that $xy - yx$ is invertible.

**Proof.** If $n$ is even, set

$$
y = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 1 & 0 & \ddots \\
& 1 & 0 & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & 1 & 0 \\
* & \cdots & & & 1
\end{bmatrix}
$$

In this case, we compute that $xy - yx$ has the form

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & 0 & \ddots \\
& 0 & 0 & 0 & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & 1 \\
* & \cdots & \cdots & \cdots & \cdots & 0
\end{bmatrix} - \begin{bmatrix}
1 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & 0 & \ddots \\
& 0 & 0 & 0 & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & 1 \\
* & \cdots & \cdots & \cdots & \cdots & 0
\end{bmatrix} = \begin{bmatrix}
1 & -1 & 1 & -1 & \cdots & 0 \\
0 & 0 & 0 & 1 & 0 & \ddots \\
& 0 & 0 & 0 & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & 1 \\
* & \cdots & \cdots & \cdots & \cdots & -1
\end{bmatrix}
$$

which is clearly invertible. If $n$ is odd, set
\[
V = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 1 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & & & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0
\end{pmatrix}
\]
In this case, we compute that $xy - yx$ has the form
\[
\begin{pmatrix}
0 & 0 & 1 & 0 & & \\
0 & 1 & 0 & 1 & \cdots & \\
\vdots & & & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0
\end{pmatrix}
\]
which we see is invertible (rearrange the columns in the order $2, 3, \cdots, n - 1, 1, n$ to obtain a triangular matrix).

**Lemma 2.2.** Let $D$ be a division ring, and let $n$ be a positive integer. Let $x \in M_n(D)$ be a diagonal matrix with diagonal entries $a_1, b_1, a_2, b_2, \cdots, a_t, b_t, a_{t+1}, a_{t+1}, \cdots, a_{t+1}$ such that $a_i \neq b_i$ for all $i = 1, \cdots, t$.

(a) If $a_{t+1} \not\in \text{center}(D)$, then there exists $y \in M_n(D)$ such that $xy - yx$ is invertible.

(b) If $a_{t+1} \in \text{center}(D)$, then there exists $y \in M_n(D)$ such that $\text{rank}(xy - yx) = 2t$.

**Proof.** It clearly suffices to show that if $z = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M_2(D)$ with $a \neq b$, then there exists $w \in M_2(D)$ such that $zw - wz$ is invertible. For this we need only take $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, since then $zw - wz =$
PROPOSITION 2.3. Let $D$ be a division ring, let $n$ be a positive integer, and let $x \in M_n(D)$. Then there exist matrices $y \in M_n(D)$ and $z \in \text{center}(M_n(D))$ such that $\text{rank}(x - z) \leq \text{rank}(xy - yx)$.

Proof. By a change of basis, we may assume that $x$ is in rational canonical form. Then, by a permutation of the basis, we may put $x$ in the form $\begin{bmatrix} x' & 0 \\ 0 & x'' \end{bmatrix}$, where $x'$ is an $n' \times n'$ diagonal matrix and $x''$ is an $n'' \times n''$ block diagonal matrix whose diagonal blocks are companion matrices of degree at least 2. In view of Lemma 2.1, there exists an $n'' \times n''$ matrix $y''$ over $D$ such that $x''y'' - y''x''$ is invertible. We may arrange the diagonal entries of $x'$ in the order $a_1, b_1, a_2, b_2, \ldots, a_t, b_t, a_{t+1}, a_{t+1}, \ldots, a_{t+1}$ with $a_i \neq b_i$ for all $i = 1, \ldots, t$. If $a_{t+1} \in \text{center}(D)$, then by Lemma 2.2 there exists an $n' \times n'$ matrix $y'$ over $D$ such that $x'y' - y'x'$ is invertible. In this case, set $y = \begin{bmatrix} y' & 0 \\ 0 & y'' \end{bmatrix}$ in $M_n(D)$, so that $xy - yx$ is invertible. Setting $z = 0$, we obtain $\text{rank}(x - z) \leq n = \text{rank}(xy - yx)$, as desired.

Now assume that $a_{t+1} \in \text{center}(D)$, and define $z \in \text{center}(M_n(D))$ to be the diagonal matrix with all diagonal entries equal to $a_{t+1}$. Note that $\text{rank}(x - z) \leq 2t + n''$. According to Lemma 2.2, there exists an $n' \times n'$ matrix $y'$ over $D$ such that $\text{rank}(x'y' - y'x') = 2t$. Setting $y = \begin{bmatrix} y' & 0 \\ 0 & y'' \end{bmatrix}$ in $M_n(D)$, we conclude that $\text{rank}(xy - yx) = 2t + n'' \geq \text{rank}(x - z)$, as desired.

DEFINITION. Given modules $A$ and $B$, we write $A \leq B$ to mean that $A$ is isomorphic to a submodule of $B$. Given elements $x$ and $y$ in a regular ring $R$, we note that $xR \leq yR$ if and only if $x = ayb$ for some $a, b \in R$.

COROLLARY 2.4. Let $R$ be a semisimple ring, and let $x \in R$. Then there exist elements $y \in R$ and $z \in \text{center}(R)$ such that $(x - z)R \leq (xy - yx)R$.

Proof. It suffices to consider the case when $R$ is simple artinian, hence we may assume that $R = M_n(D)$ for some positive integer $n$ and some division ring $D$. According to Proposition 2.3, there exist $y \in R$ and $z \in \text{center}(R)$ such that $\text{rank}(x - z) \leq \text{rank}(xy - yx)$. As a result, we see that the composition series length of $(x - z)R$ is less than or equal to the composition series length of $(xy - yx)R$. Consequently, we conclude that $(x - z)R \leq (xy - yx)R$. 

\[
\begin{bmatrix}
0 & a - b \\
b - a & 0
\end{bmatrix}.
\]
COROLLARY 2.5. Let $R$ be a directed union of semisimple sub-rings $R_i$, and assume that $\text{center}(R_i) \subseteq \text{center}(R_j)$ whenever $R_i \subseteq R_j$. Given any $x \in R$, there exist elements $y \in R$ and $z \in \text{center}(R)$ such that $(x - z)R \leq (xy - yx)R$.

Proof. We have $x \in R_k$ for some $k$, hence by Corollary 2.4 there exist $y \in R_k$ and $z \in \text{center}(R_k)$ such that $(x - z)R_k \leq (xy - yx)R_k$. Since $\text{center}(R_i) \subseteq \text{center}(R_j)$ whenever $R_i \subseteq R_j$, we see that $z \in \text{center}(R)$. Inasmuch as $R$ is a flat left $R_k$-module, we conclude that $(x - z)R \leq (xy - yx)R$.

THEOREM 2.6. Let $R$ be a directed union of semisimple sub-rings $R_i$, and assume that $\text{center}(R_i) \subseteq \text{center}(R_j)$ whenever $R_i \subseteq R_j$. Let $N$ be a pseudo-rank function on $R$, let $\bar{R}$ denote the $N$-completion of $R$, and let $\phi: R \rightarrow \bar{R}$ be the natural map. Then $\phi(\text{center}(R))$ is dense in $\text{center}(\bar{R})$.

Proof. Let $\bar{N}$ be the natural extension of $N$ to $\bar{R}$. Let $x \in \text{center}(\bar{R})$, and let $\varepsilon > 0$ be a real number.

Choose $w \in R$ such that $\bar{N}(\phi(w) - x) < \varepsilon/3$. According to Corollary 2.5, there exist $y \in R$ and $z \in \text{center}(R)$ such that $(w - z)R \leq (wy - yw)R$. Then $w - z = a(wy - yw)b$ for some $a, b \in R$, from which we see that $N(w - z) \leq N(wy - yw)$. Since $x$ commutes with $\phi(y)$, we obtain

\[
N(w - z) \leq N(wy - yw) = \bar{N}(\phi(w)\phi(y) - \phi(y)\phi(w)) = \bar{N}((\phi(w) - x)\phi(y) - \phi(y)(\phi(w) - x)) \leq 2\bar{N}(\phi(w) - x) < 2\varepsilon/3,
\]

and consequently

\[
\bar{N}(\phi(z) - x) \leq \bar{N}(\phi(z) - \phi(w)) + \bar{N}(\phi(w) - x) < N(w - z) + (\varepsilon/3) < \varepsilon.
\]

Therefore $\phi(\text{center}(R))$ is dense in $\text{center}(\bar{R})$.

COROLLARY 2.7. Let $R$ be a directed union of semisimple sub-rings $R_i$, and assume that $\text{center}(R_i) \subseteq \text{center}(R_j)$ whenever $R_i \subseteq R_j$. Let $N$ be a rank function on $R$, let $\bar{R}$ denote the $N$-completion of $R$, and assume that $\bar{R}$ is indecomposable (as a ring). Then the natural map $\text{center}(R) \rightarrow \text{center}(\bar{R})$ is an isomorphism.

Proof. Since $N$ is a rank function, we see that the natural
map \( \phi: \mathbb{R} \to \bar{\mathbb{R}} \) must be injective. Let \( \bar{N} \) be the natural extension of \( N \) to \( \bar{\mathbb{R}} \). According to Theorem 2.6, \( \phi(\text{center}(\mathbb{R})) \) is dense in \( \text{center}(\bar{\mathbb{R}}) \) in the \( \bar{N} \)-metric. Since \( \bar{\mathbb{R}} \) is an indecomposable regular ring, \( \text{center}(\bar{\mathbb{R}}) \) is a field. Then for any \( x \in \text{center}(\bar{\mathbb{R}}) \), either \( x = 0 \) or \( x \) is invertible, hence either \( \bar{N}(x) = 0 \) or \( \bar{N}(x) = 1 \). Thus the \( \bar{N} \)-metric on \( \text{center}(\bar{\mathbb{R}}) \) is discrete, hence we conclude that \( \phi(\text{center}(\mathbb{R})) = \text{center}(\bar{\mathbb{R}}) \). Therefore \( \tilde{\phi} \) restricts to an isomorphism of \( \text{center}(\mathbb{R}) \) onto \( \text{center}(\bar{\mathbb{R}}) \).

Corollary 2.7 does not remain valid without the compatibility condition on the centers of the \( \mathbb{R}_i \), as Example 2.10 shows.

The calculation of the center of the completion of a direct limit of semisimple rings with respect to a rank function was first performed by Alexander in the following special case [1, Theorem 12.8], using a fairly involved procedure. Specializing this case to the situation where \( D \) is a field of characteristic zero, Handelman developed a relatively short proof in [9, Proposition 6].

**Theorem 2.8.** Let \( D \) be a division ring, let \( n(1) < n(2) < \cdots \) be positive integers such that \( n(k)|n(k+1) \) for all \( k \), and set \( R_k = M_{n(k)}(D) \) for all \( k \). Map each \( R_k \to R_{k+1} \) by the obvious block diagonal map, and set \( R = \lim \to R_k \). Then

(a) There exists a unique rank function \( N \) on \( R \).

(b) The \( N \)-completion \( \bar{R} \) of \( R \) is a simple, regular, right and left self-injective ring of Type II.

(c) The natural map \( \text{center}(D) \to \text{center}(\bar{R}) \) is an isomorphism.

**Proof.** (a) and (b) are given by Proposition 1.1.

(c) It is clear that the natural map \( \text{center}(D) \to \text{center}(R) \) is an isomorphism. Since the maps \( \text{center}(R_k) \to \text{center}(R_{k+1}) \) are isomorphisms for all \( k \), we conclude from Corollary 2.7 that the natural map \( \text{center}(R) \to \text{center}(\bar{R}) \) is an isomorphism. Thus the natural map \( \text{center}(D) \to \text{center}(\bar{R}) \) must be an isomorphism.

**Corollary 2.9.** Let \( F \) be any field. Then there exists a simple, regular, right and left self-injective ring \( \bar{R} \) of Type II such that \( \text{center}(\bar{R}) \cong F \).

**Proof.** Set \( D = F \) and \( n(k) = 2^k \) for all \( k \), and construct \( \bar{R} \) as in Theorem 2.8.

We close this section with an example which shows that Theorem 2.6 and Corollary 2.7 may fail if we do not assume that \( \text{center}(R_i) \subseteq \text{center}(R_j) \) whenever \( R_i \subseteq R_j \).
EXAMPLE 2.10. There exists a simple regular ring $R$ such that
(a) $R$ is the direct limit of a sequence $R_1 \to R_2 \to \cdots$ of simple artinian rings and ring maps.
(b) There exists a unique rank function $N$ on $R$.
(c) The $N$-completion $\bar{R}$ of $R$ is a simple, regular, right and left self-injective ring of Type II.
(d) $\text{center}(\bar{R}) \cong R$ but $\text{center}(\bar{R}) \cong C$.

Proof. (a) Set $s(n) = 2^{n(n+1)/2}$ and $R_n = M_{s(n)}(C)$ for all $n = 1, 2, \cdots$. Define $R$-algebra maps $\varphi_n: R_n \to R_{n+1}$ according to the rule
\[
\phi_n(x) = \begin{pmatrix}
x \\
x \\
\vdots \\
\vdots \\
x \\
\bar{x}
\end{pmatrix},
\]
where $\bar{x}$ denotes the conjugate (not transposed) of the matrix $x$. Set $R = \lim \rightarrow R_n$, and for all $n$ let $\psi_n: R_n \to \bar{R}$ be the natural map.

(b) For each $n$, there is a unique rank function $P_n$ on $R_n$, given by the rule $P_n(x) = \text{rank}(x)/s(n)$. Observing that $P_{n+1} \varphi_n = P_n$ for all $n$, we see that there is a rank function $N$ on $R$ such that $N\psi_n = P_n$ for all $n$. Since the $P_n$ are unique, $N$ must be unique.

(c) is given by Proposition 1.1.

(d) It is clear that $\text{center}(R) \cong R$. Since $N$ is a rank function, we may identify $R$ with its image in $\bar{R}$. Let $\bar{N}$ be the natural extension of $N$ to $\bar{R}$.

For all $n$, let $\theta_n: C \to R_n$ be the natural isomorphism of $C$ onto the center of $R_n$. Given any $x \in C$, we claim that the sequence $\{\psi_n \theta_n(x)\} \subseteq R$ is Cauchy with respect to $N$. For each $n$, we see that $\theta_{n+1}(x) - \phi_n \theta_n(x)$ is a diagonal matrix with 0 for the first $s(n+1) - s(n)$ diagonal entries and $x - \bar{x}$ for the remaining $s(n)$ diagonal entries, whence $\text{rank}(\theta_{n+1}(x) - \phi_n \theta_n(x)) \leq s(n)$. Consequently,
\[
N(\psi_{n+1} \theta_{n+1}(x) - \psi_n \theta_n(x)) = N\psi_{n+1}(\theta_{n+1}(x) - \phi_n \theta_n(x))
= P_{n+1}(\theta_{n+1}(x) - \phi_n \theta_n(x)) \leq s(n)/s(n+1) = 1/2^{n+1}.
\]
As a result, we see for all $k > n$ that
\[
N(\psi_k \theta_k(x) - \psi_n \theta_n(x)) \leq \sum_{j=n}^{k-1} N(\psi_{j+1} \theta_{j+1}(x) - \psi_j \theta_j(x)) \leq \sum_{j=n}^{k-1} 1/2^{j+1} < 1/2^n.
\]
Therefore \( \{\psi_n \theta_n(x)\} \) is indeed Cauchy with respect to \( N \), hence there is a unique element \( \gamma(x) \in \bar{R} \) such that \( \psi_n \theta_n(x) \to \gamma(x) \) in the \( \bar{N} \)-metric. Inasmuch as \( \bar{N}(\psi_n \theta_n(x) - \psi_k \theta_k(x)) < 1/2^n \) for all \( k > n \), we see that \( \bar{N}(\psi_n \theta_n(x) - \gamma(x)) \leq 1/2^n \) for all \( n \).

We now have a map \( \gamma : C \to \bar{R} \), and it is clear that \( \gamma \) is an injective ring map. We claim that \( \gamma(C) = \text{center}(\bar{R}) \).

In order to prove that \( \gamma(C) \subseteq \text{center}(\bar{R}) \), it suffices to show that for any \( x \in C \), \( \gamma(x) \) commutes with any \( y \in R \). Choose \( k \) such that \( y \in \psi_k(R_k) \). Given any \( n \geq k \), we have \( y = \psi_n(z_n) \) for some \( z_n \in R_n \). Since \( \theta_n(x) \) commutes with \( z_n \), we see that \( \psi_n \theta_n(x) \) commutes with \( y \). Taking limits, we find that \( \gamma(x) \) commutes with \( y \), as desired.

Finally, consider any \( w \in \text{center}(\bar{R}) \). There exists \( x \in R_n \) for some \( n \) such that \( \bar{N}(\psi_n(x) - w) < 1/8 \). According to Corollary 2.4, there exist elements \( y \in R_n \) and \( z \in \text{center}(R_n) \) such that \( (x-z)R_n \subseteq (xy - yx)R_n \), whence \( P_n(x-z) \leq P_n(xy - yx) \). Since \( w \) commutes with \( \psi_n(y) \), we obtain

\[
P_n(x-z) \leq P_n(xy - yx) = N(\psi_n(y) \psi_n(x) - \psi_n(y) \psi_n(x))
= \bar{N}((\psi_n(x) - w) \psi_n(y) - \psi_n(y)(\psi_n(x) - w))
\leq 2 \bar{N}(\psi_n(x) - w) < 1/4.
\]

In addition, we have \( z = \theta_n(t) \) for some \( t \in C \), hence

\[
\bar{N}(\psi_n(x) - \psi_n \theta_n(t)) = N(\psi_n(x) - z) = P_n(x-z) < 1/4.
\]

Recalling that \( \bar{N}(\psi_n \theta_n(t) - \gamma(t)) \leq 1/2^n \), we find that

\[
\bar{N}(w - \gamma(t)) \leq \bar{N}(w - \psi_n(x)) + \bar{N}(\psi_n(x) - \psi_n \theta_n(t)) + \bar{N}(\psi_n \theta_n(t) - \gamma(t))
\leq (1/8) + (1/4) + (1/2^n) < 1.
\]

Consequently, \( w - \gamma(t) \) is a noninvertible element of the field \( \text{center}(\bar{R}) \), hence \( w - \gamma(t) = 0 \). Thus \( w \in \gamma(C) \).

Therefore \( \gamma(C) = \text{center}(\bar{R}) \) as claimed, so that \( C \cong \text{center}(\bar{R}) \).

3. Factor rings. This section is concerned with calculating the centers of factor rings of regular, right self-injective rings.

**Definition.** For any ring \( R \), we use \( B(R) \) to stand for the Boolean algebra of all central idempotents of \( R \). If \( R \) is regular and right self-injective, then \( B(R) \) is complete by [6, Proposition 4.1].

**Lemma 3.1.** Let \( R \) be a regular, right self-injective ring, and let \( X \subseteq R \). Let \( Y \subseteq B(R) \), and set \( f = \vee Y \). If \( xe = 0 \) for all \( x \in X \) and all \( e \in Y \), then \( xf = 0 \) for all \( x \in X \).
Proof. Let $J$ be the right annihilator of $X$, and note that $(R/J)_R$ is nonsingular. According to [6, Proposition 4.1], $\sum_{e \in \mathcal{F}} eR$ is an essential right $R$-submodule of $fR$. Inasmuch as $\sum_{e \in \mathcal{F}} eR \leq J$, we conclude that $fR \leq J$, so that $xf = 0$ for all $x \in X$.

Lemma 3.2. Let $R$ be a regular, right self-injective ring. Given any $x \in R$, there exist elements $y \in R$ and $f \in B(R)$ such that $fx \in \text{center}(R)$ and $(xy - yx)e \neq 0$ for all nonzero $e \leq 1 - f$ in $B(R)$.

Proof. Set $Y = \{e \in B(R) | (xt - tx)e = 0 \text{ for all } t \in R\}$ and $f = \bigvee Y$. According to Lemma 3.1, $(fx)t - t(fx) = (xt - tx)f = 0 \text{ for all } t \in R$, hence $fx \in \text{center}(R)$.

Let $W$ be the set of those $g \in B(R)$ for which there exists an element $t \in R$ such that $(xt - tx)e \neq 0$ for all nonzero $e \leq g$ in $B(R)$. We claim that every nonzero $h \leq 1 - f$ in $B(R)$ must lie above some nonzero member of $W$.

Now $h \not\leq f$ and so $h \in Y$, hence $(xt - tx)h \neq 0$ for some $t \in R$. Setting $k = \bigvee \{e \in B(R) | (xt - tx)e = 0\}$, we see by Lemma 3.1 that $(xt - tx)k = 0$. Then $h \not\leq k$, hence $g = h - hk$ is a nonzero member of $B(R)$. It is clear that $g \leq h$, and that $(xt - tx)e \neq 0 \text{ for all nonzero } e \leq g$ in $B(R)$, whence $g \in W$.

Thus every nonzero $h \leq 1 - f$ in $B(R)$ lies above some nonzero member of $W$, as claimed. Consequently, we infer that there exist orthogonal idempotents $g_i \in W$ such that $\bigvee g_i = 1 - f$. For each $i$, there exists an element $y_i \in R$ such that $(xy_i - y_ix)e \neq 0$ for all nonzero $e \leq g_i$ in $B(R)$. Since $R$ is right self-injective, there exists $y \in R$ such that $yg_i = y,g_i$ for all $i$. Given any nonzero $e \leq 1 - f$ in $B(R)$, we have $eg_i \neq 0$ for some $i$, whence $(xy - yx)eg_i = (xy_i - y_ix)eg_i \neq 0$. Therefore $(xy - yx)e \neq 0$ for all nonzero $e \leq 1 - f$ in $B(R)$.

Theorem 3.3. Let $R$ be a regular, right self-injective ring, and let $P$ be a minimal prime ideal of $R$. Then the natural map $\text{center}(R) \to \text{center}(R/P)$ is surjective.

Proof. Given any $x \in R$ such that $\bar{x} \in \text{center}(R/P)$, we must show that $\bar{x} = \bar{z}$ in $R/P$ for some $z \in \text{center}(R)$. According to Lemma 3.2, there exist $y \in R$ and $f \in B(R)$ such that $fx \in \text{center}(R)$ and $(xy - yx)e \neq 0$ for all nonzero $e \leq 1 - f$ in $B(R)$.

Since $\bar{x} \in \text{center}(R/P)$, we obtain $xy - yx \in P$. According to [3, Theorem 2.3], $P = [P \cap B(R)]_R$, from which it follows that $xy - yx = g(xy - yx)$ for some $g \in P \cap B(R)$. Consequently, $(xy - yx)(1 - g) = 0$, which implies that $(1 - g)(1 - f) = 0$. Thus $1 - f = g(1 - f) \in P$, and so $x - fx \in P$.

Therefore we have $fx \in \text{center}(R)$ such that $\bar{x} = \bar{fx}$ in $R/P$. 
DEFINITION. For any ring $R$, we use $BS(R)$ to stand for the set of all maximal ideals of the Boolean algebra $B(R)$.

**Lemma 3.4.** Let $R$ be a regular, right self-injective ring which is Type I$_n$ for some $n$. For any $M \in BS(R)$, $R/MR$ is a simple artinian ring, and the natural map $\text{center}(R) \rightarrow \text{center}(R/MR)$ is surjective.

*Proof.* According to [3, Theorem 2.3], $MR$ is a minimal prime ideal of $R$, hence Theorem 3.3 says that the natural map $\text{center}(R) \rightarrow \text{center}(R/MR)$ is surjective. Now $R \cong M_n(S)$ for some abelian, regular, right self-injective ring $S$, hence $R/MR \cong M_n(S/Q)$ for some prime ideal $Q$ of $S$. Inasmuch as $S$ is strongly regular, $S/Q$ is a division ring, whence $R/MR$ is simple artinian.

**Proposition 3.5.** Let $R$ be a regular, right self-injective ring of Type I$_r$. Given any $x \in R$, there exist elements $y \in R$ and $z \in \text{center}(R)$ such that $(x - z)R \preceq (xy - yx)R$.

*Proof.* According to [6, Corollary 6.5], there exist regular, right self-injective rings $R_1, R_2, \ldots$ such that $R \cong \Pi R_n$ and each $R_n$ is Type I$_n$. Consequently, there exist orthogonal central idempotents $e_1, e_2, \ldots \in B(R)$ such that $\vee e_n = 1$ and each $e_nR$ is Type I$_n$.

Let $X$ be the set of those $f \in B(R)$ for which there exist elements $y, a, b \in R$ and $z \in \text{center}(R)$ such that $f(x - z) = fa(xy - yx)b$. We claim that any nonzero $g \in B(R)$ must lie above some nonzero member of $X$.

Now $g e_n \neq 0$ for some $n$, hence there exists $M \in BS(R)$ such that $ge_n \in M$. Since $e_n \notin M$, we see that $M \cap e_nR \in BS(e_nR)$ and that $e_nR/(M \cap e_nR)e_nR \cong R/MR$. Consequently, it follows from Lemma 3.4 that $R/MR$ is a simple artinian ring, and that the natural map $\text{center}(R) \rightarrow \text{center}(R/MR)$ is surjective. Applying Corollary 2.4 to the element $\bar{x} \in R/MR$, we obtain elements $y' \in R/MR$ and $z' \in \text{center}(R/MR)$ such that $(\bar{x} - z')(R/MR) \preceq (\bar{x}y' - y'\bar{x})(R/MR)$. Thus $\bar{x} - z' = a'(\bar{x}y' - y'\bar{x})b'$ for some $a', b' \in R/MR$. Now there exist elements $y, a, b \in R$ such that $\bar{y} = y'$, $\bar{a} = a'$, and $\bar{b} = b'$, and there exists $z \in \text{center}(R)$ such that $\bar{z} = z'$.

Thus $x - z - a(xy - yx)b \in MR$, from which it follows that

$$x - z - a(xy - yx)b = h[x - z - a(xy - yx)b]$$

for some $h \in M$. Since $g \in M$ and $h \in M$, we see that $f = g(1 - h)$ is not in $M$, whence $f$ is a nonzero member of $B(R)$ such that $f \preceq g$.

In addition, $f[x - z - a(xy - yx)b] = 0$, so that $f \in X$. Thus the
Inasmuch as every nonzero member of $B(R)$ lies above a nonzero member of $X$, we infer that there exist orthogonal idempotents $f_i \in X$ such that $\vee f_i = 1$. For each $i$, there exist elements $y_i, a_i, b_i \in R$ and $z_i \in \text{center}(R)$ such that $f_i(x - z_i) = f_i a_i (xy_i - yx)b_i$. Since $R$ is right self-injective, there exist elements $y, a, b, z \in R$ such that $y f_i = y_i f_i$, $a f_i = a_i f_i$, $b f_i = b_i f_i$, and $z f_i = z_i f_i$ for all $i$. Then $f_i(x - z) = f_i a (xy - yx)b$ for all $i$, hence we see by Lemma 3.1 that $x - z = a(xy - yx)b$. Thus $(x - z)R \leq (xy - yx)R$. Since $z f_i = z_i f_i \in \text{center}(R)$ for all $i$, we conclude from Lemma 3.1 that $z \in \text{center}(R)$.

**Theorem 3.6.** Let $R$ be a regular, right self-injective ring of Type I. If $K$ is any two-sided ideal of $R$, then the natural map $\text{center}(R) \to \text{center}(R/K)$ is surjective.

**Proof.** Consider any $x \in R$ such that $\bar{x} \in \text{center}(R/K)$. According to Proposition 3.5, there exist $y \in R$ and $z \in \text{center}(R)$ such that $(x - z)R \leq (xy - yx)R$. Since $R/K$ is a flat left $R$-module, it follows that $(\bar{x} - \bar{z})(R/K) \leq (\bar{xy} - \bar{yx})(R/K)$. Since $\bar{x} \in \text{center}(R/K)$, we obtain $\bar{xy} = \bar{yx}$, and consequently $\bar{x} = \bar{z}$.

The centers of the other standard Type II examples may now be calculated using Theorem 3.6, as follows.

**Theorem 3.7.** Let $D_1, D_2, \ldots$ be division rings, let $n(1) < n(2) < \cdots$ be positive integers, and set $R_k = M_{n(k)}(D_k)$ and $F_k = \text{center}(R_k)$ for all $k$. Set $R = \Pi R_k$, and let $M$ be any maximal two-sided ideal of $R$ which contains $\oplus R_k$. Then $R/M$ is a simple, regular, right and left self-injective ring of Type II, and $\text{center}(R/M) \cong (II F_k)/[M \cap (II F_k)]$.

**Proof.** Since each $R_k$ is a regular, right self-injective ring of Type I$_{n(k)}$, [6, Corollary 11.10] says that $R/M$ is a simple, regular, right and left self-injective ring of Type II. According to [6, Corollary 6.5], $R$ is Type I$_r$, hence Theorem 3.6 says that the natural map $\text{center}(R) \to \text{center}(R/M)$ is surjective. Observing that $\text{center}(R) = II F_k$, we conclude that $\text{center}(R/M) \cong (II F_k)/[M \cap (II F_k)]$.

Unlike the Type II examples obtained from completions, the rings $R/M$ in Theorem 3.7 do not have completely arbitrary centers, as the following corollary (of Theorem 3.6) shows.

**Corollary 3.8.** Let $R_1, R_2, \ldots$ be simple artinian rings, set $R = \Pi R_k$, and let $M$ be any maximal two-sided ideal of $R$ which
contains \( \bigoplus R_k \). Then \( \text{center}(R/M) \) is either finite or uncountable.

Proof. Set \( F_k = \text{center}(R_k) \) for all \( k \), and note that \( \text{center}(R) = IF_k \). Note also that \( P = M \cap (IF_k) \) is a prime ideal of \( IF_k \) which contains \( \bigoplus F_k \). By [6, Corollary 6.5], \( R \) is Type I, hence Theorem 3.6 says that the natural map \( \text{center}(R) \to \text{center}(R/M) \) is surjective. Thus \( \text{center}(R/M) \cong (IF_k)/P \).

Since \( IF_k \) is a commutative regular ring, all of its prime ideals are maximal, hence it follows that \( P \) is a minimal prime ideal of \( IF_k \). Consequently, [3, Proposition 3.3] says that there exists an ultrafilter \( \mathcal{F} \) on the index set \( N = \{1, 2, \ldots \} \) such that

\[
P = \{ x \in IF_k \mid \{ k \in N \mid x_k = 0 \} \in \mathcal{F} \}.
\]

Thus \( (IF_k)/P \) is an ultraproduct of the \( F_k \). Since \( \bigoplus F_k \subseteq P \), we see that \( \mathcal{F} \) is a nonprincipal ultrafilter on \( N \).

Now if the set \( \{ k \in N \mid F_k \text{ is finite} \} \) belongs to \( \mathcal{F} \), then we see by [2, Lemmas 3.7, 3.11] that \( (IF_k)/P \) is either finite or uncountable. On the other hand, if the set \( \{ k \in N \mid F_k \text{ is infinite} \} \) belongs to \( \mathcal{F} \), then we see by [2, Corollaries 1.10, 3.14] that \( (IF_k)/P \) is uncountable. Therefore in all cases \( \text{center}(R/M) \cong (IF_k)/P \) is either finite or uncountable.

The following example, and the basic idea for its proof, was suggested by Handelman in correspondence.

Example 3.9. There exists a simple regular ring \( R \) such that

(a) \( R \) is the direct limit of a sequence \( R_1 \to R_2 \to \cdots \) of simple artinian rings and ring maps.

(b) There exists a unique rank function \( N \) on \( R \).

(c) The \( N \)-completion \( \bar{R} \) of \( R \) is a simple, regular, right and left self-injective ring.

(d) There do not exist simple artinian rings \( S_1, S_2, \cdots \) such that \( \bar{R} \) is isomorphic to a factor ring of \( \Pi S_n \).

Proof. Set \( D = \mathbb{Q} \) and \( n(k) = 2^k \) for all \( k = 1, 2, \cdots \), and construct \( R \) as in Theorem 2.8. Then (a), (b), (c) hold, and center \( (\bar{R}) \cong \mathbb{Q} \).

(d) Suppose that there do exist simple artinian rings \( S_1, S_2, \cdots \) such that \( \bar{R} \cong (\Pi S_n)/M \) for some maximal two-sided ideal \( M \) of \( \Pi S_n \). Since \( \bar{R} \) is Type II, we see that \( \bar{R} \not\cong S_n \) for all \( n \), whence \( \bigoplus S_n \subseteq M \). Then Corollary 3.8 says that the center of \( (\Pi S_n)/M \) is either finite or uncountable. Since center \( (\bar{R}) \) is countably infinite, this is impossible.

In view of Example 3.9, we ask whether the simple, regular,
right and left self-injective rings \( R/M \) of Theorem 3.7 can be obtained as in Theorem 2.8. More generally, can every simple, regular, right and left self-injective ring of Type II be obtained as in Theorem 2.8? The following example of von Neumann [12] indicates that the answer to this second question is probably negative.

Set \( D = \mathbb{C} \), choose positive integers \( n(1) < n(2) < \cdots \) such that \( n(k) \mid n(k + 1) \) for all \( k \), and construct \( \bar{R} \) as in Theorem 2.8. Let \( S \) be the "regular ring" of a complex \( W^* \)-factor of Type II (denoted \( U(M) \) in [12, §5]). As indicated in [12, §5], \( S \) is a continuous regular ring. It is not hard to check that \( S \) is simple, and [11, Theorem 7.9] says that \( S \) is right and left self-injective. By [11, Theorem 5.1], \( S \) is directly finite, from which we infer that \( S \) is Type II. However, [12, Theorem E] says that \( \bar{R} \not\cong S \).

This example shows that there exists a simple, regular, right and left self-injective ring \( S \) of Type II such that \( S \) cannot be obtained as in Theorem 2.8 in case \( D \) is a field (since then \( D \cong \mathbb{C} \)). It still might be possible to obtain \( S \) as in Theorem 2.8 with some noncommutative \( D \), but this seems unlikely.

**References**


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