Pacific Journal of Mathematics

SOME RADICAL PROPERTIES OF RINGS WITH $(a, b, c) = (c, a, b)$

DAVID POKRASS

Vol. 76, No. 2 December 1978

PACIFIC JOURNAL OF MATHEMATICS
Vol. 76, No. 2, 1978

SOME RADICAL PROPERTIES OF RINGS WITH $(a, b, c) = (c, a, b)$

DAVID POKRASS

A ring is an s-ring if (for fixed s) A^s is an ideal whenever A is. We show that at least two different definitions for the prime radical are equivalent in s-rings. If R satisfies $(a, b, c) = (c, a, b)$ then R is a 2-ring. In this note we investigate various properties of the prime and nil radicals of R. In addition, if R is a finite dimensional algebra over a field of characteristic $\neq 2$ of 3 we show that the concepts of nil and nilpotent are equivalent.

In [1] Brown and McCoy studied a collection of prime radicals and nil radicals in an arbitrary nonassociative ring. In light of their treatment we will consider these radicals in rings which satisfy the identity

$$(a, b, c) = (c, a, b) . \tag{1}$$

While these rings may be viewed as an extension of alternative rings, they are in general not even power associative. Examples of (not power associative) rings satisfying (1) appear in [2] and [4].

1. s-rings and the prime radical. Prime radicals for an arbitrary ring R were treated in [1] in the following way. Let $\mathscr{A}$ be the set of all finite nonassociative products of at least two elements from some countable set of indeterminates $x_1, x_2, x_3, \cdots$. Then if $u \in \mathscr{A}$ we call an ideal P u-prime if $u(A_1, A_2, \cdots, A_n) \subseteqq P$ implies some $A_i \subseteqq P$ for ideals $A_1, A_2, \cdots, A_n$. For example if $u = (x_1x_2)x_3$ then P is u-prime if whenever $(A_1A_2)A_3 \subseteqq P$ we have one of the A_i's in P. The u-prime radical R^u is then the intersection of all u-prime ideals in R. It was shown that if u^* is the word having the same association as u, but in only one variable, then $R^u = R^{u^*}$. For example if $u = (x_1x_2)x_3$ then $u^* = (xx)x$, and R^{u^*} is the intersection of ideals P with the property that if $(AA)A \subseteqq P$ for an ideal A, then $A \subseteqq P$.

Another theory of the prime radical was given in [9]. Call a ring R an s-ring if for some fixed positive integer s, A^s is an ideal whenever A is. Call an ideal P prime if $A_1A_2 \cdots A_s \subseteqq P$ implies some $A_i \subseteqq P$ for ideals $A_1, \cdots, A_s$. The prime radical $P(R)$ of an s-ring R is then the intersection of all prime ideals.

In the case of s-rings we see that these approaches are essentially the same:

THEOREM 1. *Let R be an s-ring. Then for each $u \in \mathscr{A}$ having degree $\geqq s$, R^u coincides with $P(R)$.*

Proof. If A is an ideal of R, consider the two descending chains: $A^{(0)} = A_0 = A$, $A^{(n+1)} = A^{(n)}A^{(n)}$, and $A_{n+1} = (A_n)^s$. It is easily seen that $\langle A_n \rangle$ is a chain of ideals in R and for each n, $A_n \subseteqq A^{(n)}$. Next choose $u \in \mathscr{A}$. We first show that there is an integer r such that $A^{(r)} \subseteqq u^*(A, A, \cdots, A)$. We induct on $\deg u^*$. When $u^* = x^2$, take $r = 1$. Assuming $\deg u^* > 2$, write $u^* = v_1 v_2$ where each v_i has degree less than that of u^*. Then there exists r_1, r_2 such that $A^{(r_i)} \subseteqq v_i(A, A, \cdots, A)$. Letting $r = \max\{r_1, r_2\}$, $A^{(r+1)} = A^{(r)}A^{(r)} \subseteqq A^{(r_1)}A^{(r_2)} \subseteqq v_1(A)v_2(A) \subseteqq u^*(A)$, which completes the induction. Now assume P is prime (in the sense of [9]). Then P is also u^*-prime. For if A is any ideal with $u^*(A, A, \cdots, A) \subseteqq P$ we may choose r such that $A_r \subseteqq A^{(r)} \subseteqq u^*(A) \subseteqq P$. Using repeatedly the fact that P is prime we see that $A \subseteqq P$. We have shown $R^u = R^{u^*} \subseteqq P(R)$.

To see the other inclusion, assume $\deg u \geqq s$. Let P be u^*-prime. Then P is also prime. For if A is an ideal with $A^s \subseteqq P$ it follows that $u^*(A) \subseteqq A^{\deg u^*} \subseteqq A^s \subseteqq P$, and so $A \subseteqq P$. This shows $P(R) \subseteqq R^{u^*} = R^u$, which completes the proof.

COROLLARY. *If R is a 2-ring, the u-prime radicals all coincide.*

Rich has shown that in an s-ring the prime radical $P(R)$ is the intersection of all ideals Q such that R/Q has no nonzero nilpotent ideals [5]. However, if R/Q has no nonzero nilpotent ideals it also has no nonzero solvable ideals: For if $A^{(n)} \subseteqq Q$ for some ideal A, then $A_n \subseteqq A^{(n)} \subseteqq Q$ using the same notation as above. It follows that $A \subseteqq Q$. This shows that the word "nilpotent" may be replaced by "solvable" in Rich's characterization of $P(R)$.

2. **Nilalgebras.** In this section we let R denote a ring satisfying equation (1) and having characteristic not equal to 2 or 3. Outcalt showed that if R is simple then it is alternative (and hence a Caycley-Dickson algebra or associative) [3]. Sterling extended this result by showing that if R has no nonzero ideals whose square is zero then R is alternative [8].

We see that rings R which satisfy (1) are 2-rings. For if A is an ideal with $a_1, a_2 \in A$, then $(a_1a_2)x = (a_1, a_2, x) + a_1(a_2x) = (a_2, x, a_1) + a_1(a_2x) \in A^2$. In fact, it is easily shown that A^n is an ideal for each $n \geqq 2$.

Next recall that an element a is nilpotent if there is some association u^* such that $u^*(a) = 0$. An ideal A is a nil ideal if each element in A is nilpotent. We call A solvable if $A^{(n)}$ (defined above) is zero for some n. Finally, A is right nilpotent if the sequence $A, A^2, A^2A, (A^2A)A, \cdots$ reaches zero in a finite number of steps.

LEMMA. *Let R be a ring satisfying* (1). *Then R is nilpotent if and only if R is right nilpotent.*

Proof. The proof of this lemma, which appears in [4], only required identity (1) and is therefore valid.

We will need the following identity [8, eq. 4] which holds in R

$$(2) \qquad 9(((a, x, x), x, x), x, x) = (a, (x, x, x), (x, x, x)) .$$

LEMMA. *Let R be a finite dimensional algebra, satisfying* (1), *over a field F of characteristic* $\neq 2, 3$. *If R is solvable then R is nilpotent.*

Proof. We induct on $\dim R$. When $\dim R = 1$ the result is obvious, so assume $\dim R > 1$. By the previous lemma it is sufficient to show that R is right nilpotent. Let S_a denote the right multiplication operator $x \to xa$. Let $\hat{R}$ be the subalgebra of the multiplication algebra R^* which is generated by $\{S_a \mid a \in R\}$. Note that R is right nilpotent if and only if $\hat{R}$ is nilpotent. Now by the solvability of R we may write $R = B + Fx$ where B is an ideal containing R^2 and $B \subsetneqq R$. Since $\dim B < \dim R$, B is nilpotent by the induction assumption. Suppose $B^k = 0$. We claim $(\hat{R})^{6k^2} = 0$.

Treating a as the independent variable and expanding (2) it becomes apparent that $(S_x)^6$ may be written as the sum of 15 terms each containing S_{x^2}, S_{x^2x}, or S_{xx^2}. These factors are in $(R^2)^* \subseteqq B^*$. This implies that $(S_x)^{6k}$ can be expressed as a sum of terms each containing at least k factors from B^*. Since B^n is as ideal for each n, it follows that $(S_x)^{6k} = 0$. Now choose $T \in (\hat{R})^{6k2}$. Then T is a sum of terms each containing a factor of the form

$$(S_{y_1}S_{y_2} \cdots S_{y_{6k}})(S_{z_1}S_{z_2} \cdots S_{z_{6k}}) \cdots (S_{w_1}S_{w_2} \cdots S_{w_{6k}}) ,$$

where each subscript is either equal to x or is a member of B. Note there are k "blocks" each having length $6k$. If k of the S's have elements from B attached to them then the above expression is 0 since B^n is always an ideal. On the other hand if there are not k such S's, then one of the blocks must be of the form $S_xS_x \cdots S_x$, or $(S_x)^{6k} = 0$. In any case $T = 0$, so R is nilpotent completing the proof.

THEOREM 2. *If R is a finite dimensional nilalgebra, satisfying* (1), *over a field of characteristic* $\neq 2, 3$, *then R is nilpotent.*

Proof. We induct on $\dim R$. Assume $\dim R > 1$. If R is alternative we are done. If not, by Sterling's result [8], there exists an ideal $J \neq 0$ such that $J^2 = 0$. Then R/J is solvable by the induction assumption. Since J is solvable it follows that R must be. By the previous lemma R is nilpotent.

3. Radicals. If v is a word in one variable, then a is called v-nilpotent if the sequence $a, v(a), v(v(a)), \cdots$ ends in 0. An ideal is v-nil if each of its elements is v-nilpotent. Every ring has a maximal v-nil ideal N_v and a maximal nil ideal N[1]. We shall call N_v the v-nil radical and N the nil radical. The Jacobson radical J is the set of all elements which generate quasi-regular ideals. It is shown in [1] that for each word $u^* = v$ we have

$$R^u \subseteqq N_v \subseteqq N \subseteqq J .$$

THEOREM 3. *Let R be a ring of characteristic $\neq 2, 3$ and satisfying* (1). *Then all of the u-prime radicals coincide and each of the v-nil radicals coincides with N.*

Proof. The first statement follows from the corollary to Theorem 1 and the fact that R is a 2-ring. The second statement follows from Sterling's theorem: The ring R/R^u contains no nonzero ideals whose square is zero (since $A^2 \subseteqq R^u$ implies $A \subseteqq R^u$). Hence R/R^u is alternative, and so R/N_v is alternative. Since R/N_v is power associative, N/N_v is a v-nil ideal in R/N_v, and so N must be a v-nil ideal in R. This means $N_v = N$.

THEOREM 4. *If R is a finite dimensional algebra, satisfying* (1) *over a field of characteristic $\neq 2, 3$, then the Jacobson radical R is nilpotent.*

Proof. By the reasoning in the proof of Theorem 3 we may conclude that R/N is alternative. A result of Slater's says that in an alternative ring with d.c.c. on right ideals, the nil radical equals the Jacobson radical]7]. Hence $0 = N(R/N) = J(R/N)$. It follows that $J \subseteqq N$ so J is nilpotent.

We will add one final note. If R is a ring the attached ring R^+ is the ring where multiplication is redefined by $a \cdot b = ab + ba$. Rich has shown that if R is alternative and having characteristic $\neq 2, 3$, then the (Jordan) ring R^+ has the same prime radical as R [6]. That is, $P(R) = P(R^+)$ using the notation of §1. This result may be generalized slightly: If R satisfies (1) and has characteristic

$\neq 2, 3$, then the prime radical of R coincides with each of the u-prime radicals $(R^+)^u$ in R^+. This is interesting because while Jordan rings are 3-rings, it does not seem likely that in general R^+ will be an s-ring. The proof (which we omit) is similar to the one found in [6].

REFERENCES

1. B. Brown and N. McCoy, *Prime ideals in nonassociative rings*, Trans. Amer. Math. Soc., **89** (1958), 245-255.
2. E. Kleinfeld, *Assosymmetric rings*, Proc. Amer. Math. Soc., **8** (1957), 983-986.
3. D. L. Outcalt, *An extension of the class of alternative rings*, Canad. J. Math., **17** (1965), 130-141.
4. D. Pokrass and D. Rodabaugh, *Solvable assosymmetric rings are nilpotent*, Proc. Amer. Math. Soc., **64** (1977), 30-34.
5. M. Rich, *Some radical properties of s-rings*, Proc. Amer. Math. Soc., **30** (1971), 40-42.
6. ———, *The prime radical in alternative rings*, Proc. Amer. Math. Soc., **56** (1976), 11-15.
7. M. Slater, *Alternative rings with d.c.c., I*, J. of Algebra, **11** (1969), 102-110.
8. N. Sterling, *Rings satisfying* $(x, y, z) = (y, z, x)$, Canad. J. Math., **20** (1968), 913-918.
9. P. Zwier, *Prime ideals in a large class of nonassociative rings*, Trans. Amer. Math. Soc., **158** (1971), 257-271.

Received December 6, 1976.

EMORY UNIVERSITY
ATLANTA, GA 30322.

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708. Older back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 76, No. 2 December, 1978