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**SOME RADICAL PROPERTIES OF RINGS WITH  
 $(a, b, c) = (c, a, b)$**

DAVID POKRASS

## SOME RADICAL PROPERTIES OF RINGS WITH $(a, b, c) = (c, a, b)$

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**A ring is an  $s$ -ring if (for fixed  $s$ )  $A^s$  is an ideal whenever  $A$  is. We show that at least two different definitions for the prime radical are equivalent in  $s$ -rings. If  $R$  satisfies  $(a, b, c) = (c, a, b)$  then  $R$  is a 2-ring. In this note we investigate various properties of the prime and nil radicals of  $R$ . In addition, if  $R$  is a finite dimensional algebra over a field of characteristic  $\neq 2$  or  $3$  we show that the concepts of nil and nilpotent are equivalent.**

In [1] Brown and McCoy studied a collection of prime radicals and nil radicals in an arbitrary nonassociative ring. In light of their treatment we will consider these radicals in rings which satisfy the identity

$$(1) \quad (a, b, c) = (c, a, b).$$

While these rings may be viewed as an extension of alternative rings, they are in general not even power associative. Examples of (not power associative) rings satisfying (1) appear in [2] and [4].

1.  $s$ -rings and the prime radical. Prime radicals for an arbitrary ring  $R$  were treated in [1] in the following way. Let  $\mathcal{A}$  be the set of all finite nonassociative products of at least two elements from some countable set of indeterminates  $x_1, x_2, x_3, \dots$ . Then if  $u \in \mathcal{A}$  we call an ideal  $P$   $u$ -prime if  $u(A_1, A_2, \dots, A_n) \subseteq P$  implies some  $A_i \subseteq P$  for ideals  $A_1, A_2, \dots, A_n$ . For example if  $u = (x_1 x_2) x_3$  then  $P$  is  $u$ -prime if whenever  $(A_1 A_2) A_3 \subseteq P$  we have one of the  $A_i$ 's in  $P$ . The  $u$ -prime radical  $R^u$  is then the intersection of all  $u$ -prime ideals in  $R$ . It was shown that if  $u^*$  is the word having the same association as  $u$ , but in only one variable, then  $R^u = R^{u^*}$ . For example if  $u = (x_1 x_2) x_3$  then  $u^* = (xx)x$ , and  $R^{u^*}$  is the intersection of ideals  $P$  with the property that if  $(AA)A \subseteq P$  for an ideal  $A$ , then  $A \subseteq P$ .

Another theory of the prime radical was given in [9]. Call a ring  $R$  an  $s$ -ring if for some fixed positive integer  $s$ ,  $A^s$  is an ideal whenever  $A$  is. Call an ideal  $P$  prime if  $A_1 A_2 \dots A_s \subseteq P$  implies some  $A_i \subseteq P$  for ideals  $A_1, \dots, A_s$ . The prime radical  $P(R)$  of an  $s$ -ring  $R$  is then the intersection of all prime ideals.

In the case of  $s$ -rings we see that these approaches are essentially the same:

**THEOREM 1.** *Let  $R$  be an  $s$ -ring. Then for each  $u \in \mathcal{A}$  having degree  $\geq s$ ,  $R^u$  coincides with  $P(R)$ .*

*Proof.* If  $A$  is an ideal of  $R$ , consider the two descending chains:  $A^{(0)} = A_0 = A$ ,  $A^{(n+1)} = A^{(n)}A^{(n)}$ , and  $A_{n+1} = (A_n)^s$ . It is easily seen that  $\langle A_n \rangle$  is a chain of ideals in  $R$  and for each  $n$ ,  $A_n \subseteq A^{(n)}$ . Next choose  $u \in \mathcal{A}$ . We first show that there is an integer  $r$  such that  $A^{(r)} \subseteq u^*(A, A, \dots, A)$ . We induct on  $\deg u^*$ . When  $u^* = x^2$ , take  $r = 1$ . Assuming  $\deg u^* > 2$ , write  $u^* = v_1 v_2$  where each  $v_i$  has degree less than that of  $u^*$ . Then there exists  $r_1, r_2$  such that  $A^{(r_i)} \subseteq v_i(A, A, \dots, A)$ . Letting  $r = \max\{r_1, r_2\}$ ,  $A^{(r+1)} = A^{(r)}A^{(r)} \subseteq A^{(r_1)}A^{(r_2)} \subseteq v_1(A)v_2(A) \subseteq u^*(A)$ , which completes the induction. Now assume  $P$  is prime (in the sense of [9]). Then  $P$  is also  $u^*$ -prime. For if  $A$  is any ideal with  $u^*(A, A, \dots, A) \subseteq P$  we may choose  $r$  such that  $A_r \subseteq A^{(r)} \subseteq u^*(A) \subseteq P$ . Using repeatedly the fact that  $P$  is prime we see that  $A \subseteq P$ . We have shown  $R^u = R^{u^*} \subseteq P(R)$ .

To see the other inclusion, assume  $\deg u \geq s$ . Let  $P$  be  $u^*$ -prime. Then  $P$  is also prime. For if  $A$  is an ideal with  $A^s \subseteq P$  it follows that  $u^*(A) \subseteq A^{\deg u^*} \subseteq A^s \subseteq P$ , and so  $A \subseteq P$ . This shows  $P(R) \subseteq R^{u^*} = R^u$ , which completes the proof.

**COROLLARY.** *If  $R$  is a 2-ring, the  $u$ -prime radicals all coincide.*

Rich has shown that in an  $s$ -ring the prime radical  $P(R)$  is the intersection of all ideals  $Q$  such that  $R/Q$  has no nonzero nilpotent ideals [5]. However, if  $R/Q$  has no nonzero nilpotent ideals it also has no nonzero solvable ideals: For if  $A^{(n)} \subseteq Q$  for some ideal  $A$ , then  $A_n \subseteq A^{(n)} \subseteq Q$  using the same notation as above. It follows that  $A \subseteq Q$ . This shows that the word "nilpotent" may be replaced by "solvable" in Rich's characterization of  $P(R)$ .

**2. Nilalgebras.** In this section we let  $R$  denote a ring satisfying equation (1) and having characteristic not equal to 2 or 3. Outcalt showed that if  $R$  is simple then it is alternative (and hence a Cayley-Dickson algebra or associative) [3]. Sterling extended this result by showing that if  $R$  has no nonzero ideals whose square is zero then  $R$  is alternative [8].

We see that rings  $R$  which satisfy (1) are 2-rings. For if  $A$  is an ideal with  $a_1, a_2 \in A$ , then  $(a_1 a_2)x = (a_1, a_2, x) + a_1(a_2 x) = (a_2, x, a_1) + a_1(a_2 x) \in A^2$ . In fact, it is easily shown that  $A^n$  is an ideal for each  $n \geq 2$ .

Next recall that an element  $a$  is nilpotent if there is some association  $u^*$  such that  $u^*(a) = 0$ . An ideal  $A$  is a nil ideal if each element in  $A$  is nilpotent. We call  $A$  solvable if  $A^{(n)}$  (defined above)

is zero for some  $n$ . Finally,  $A$  is right nilpotent if the sequence  $A, A^2, A^2A, (A^2A)A, \dots$  reaches zero in a finite number of steps.

**LEMMA.** *Let  $R$  be a ring satisfying (1). Then  $R$  is nilpotent if and only if  $R$  is right nilpotent.*

*Proof.* The proof of this lemma, which appears in [4], only required identity (1) and is therefore valid.

We will need the following identity [8, eq. 4] which holds in  $R$

$$(2) \quad 9(((a, x, x), x, x), x, x) = (a, (x, x, x), (x, x, x)).$$

**LEMMA.** *Let  $R$  be a finite dimensional algebra, satisfying (1), over a field  $F$  of characteristic  $\neq 2, 3$ . If  $R$  is solvable then  $R$  is nilpotent.*

*Proof.* We induct on  $\dim R$ . When  $\dim R = 1$  the result is obvious, so assume  $\dim R > 1$ . By the previous lemma it is sufficient to show that  $R$  is right nilpotent. Let  $S_a$  denote the right multiplication operator  $x \rightarrow xa$ . Let  $\hat{R}$  be the subalgebra of the multiplication algebra  $R^*$  which is generated by  $\{S_a | a \in R\}$ . Note that  $R$  is right nilpotent if and only if  $\hat{R}$  is nilpotent. Now by the solvability of  $R$  we may write  $R = B + Fx$  where  $B$  is an ideal containing  $R^2$  and  $B \cong R$ . Since  $\dim B < \dim R$ ,  $B$  is nilpotent by the induction assumption. Suppose  $B^k = 0$ . We claim  $(\hat{R})^{6k^2} = 0$ .

Treating  $a$  as the independent variable and expanding (2) it becomes apparent that  $(S_x)^6$  may be written as the sum of 15 terms each containing  $S_{x^2}, S_{x^2x}$ , or  $S_{xx^2}$ . These factors are in  $(R^2)^* \subseteq B^*$ . This implies that  $(S_x)^{6k}$  can be expressed as a sum of terms each containing at least  $k$  factors from  $B^*$ . Since  $B^n$  is an ideal for each  $n$ , it follows that  $(S_x)^{6k} = 0$ . Now choose  $T \in (\hat{R})^{6k^2}$ . Then  $T$  is a sum of terms each containing a factor of the form

$$(S_{y_1}S_{y_2} \cdots S_{y_{6k}})(S_{z_1}S_{z_2} \cdots S_{z_{6k}}) \cdots (S_{w_1}S_{w_2} \cdots S_{w_{6k}}),$$

where each subscript is either equal to  $x$  or is a member of  $B$ . Note there are  $k$  "blocks" each having length  $6k$ . If  $k$  of the  $S$ 's have elements from  $B$  attached to them then the above expression is 0 since  $B^n$  is always an ideal. On the other hand if there are not  $k$  such  $S$ 's, then one of the blocks must be of the form  $S_x S_x \cdots S_x$ , or  $(S_x)^{6k} = 0$ . In any case  $T = 0$ , so  $R$  is nilpotent completing the proof.

**THEOREM 2.** *If  $R$  is a finite dimensional nilalgebra, satisfying (1), over a field of characteristic  $\neq 2, 3$ , then  $R$  is nilpotent.*

*Proof.* We induct on  $\dim R$ . Assume  $\dim R > 1$ . If  $R$  is alternative we are done. If not, by Sterling's result [8], there exists an ideal  $J \neq 0$  such that  $J^2 = 0$ . Then  $R/J$  is solvable by the induction assumption. Since  $J$  is solvable it follows that  $R$  must be. By the previous lemma  $R$  is nilpotent.

3. Radicals. If  $v$  is a word in one variable, then  $a$  is called  $v$ -nilpotent if the sequence  $a, v(a), v(v(a)), \dots$  ends in 0. An ideal is  $v$ -nil if each of its elements is  $v$ -nilpotent. Every ring has a maximal  $v$ -nil ideal  $N_v$  and a maximal nil ideal  $N$ [1]. We shall call  $N_v$  the  $v$ -nil radical and  $N$  the nil radical. The Jacobson radical  $J$  is the set of all elements which generate quasi-regular ideals. It is shown in [1] that for each word  $u^* = v$  we have

$$R^u \subseteq N_v \subseteq N \subseteq J.$$

**THEOREM 3.** *Let  $R$  be a ring of characteristic  $\neq 2, 3$  and satisfying (1). Then all of the  $u$ -prime radicals coincide and each of the  $v$ -nil radicals coincides with  $N$ .*

*Proof.* The first statement follows from the corollary to Theorem 1 and the fact that  $R$  is a 2-ring. The second statement follows from Sterling's theorem: The ring  $R/R^u$  contains no nonzero ideals whose square is zero (since  $A^2 \subseteq R^u$  implies  $A \subseteq R^u$ ). Hence  $R/R^u$  is alternative, and so  $R/N_v$  is alternative. Since  $R/N_v$  is power associative,  $N/N_v$  is a  $v$ -nil ideal in  $R/N_v$ , and so  $N$  must be a  $v$ -nil ideal in  $R$ . This means  $N_v = N$ .

**THEOREM 4.** *If  $R$  is a finite dimensional algebra, satisfying (1) over a field of characteristic  $\neq 2, 3$ , then the Jacobson radical  $R$  is nilpotent.*

*Proof.* By the reasoning in the proof of Theorem 3 we may conclude that  $R/N$  is alternative. A result of Slater's says that in an alternative ring with d.c.c. on right ideals, the nil radical equals the Jacobson radical [7]. Hence  $0 = N(R/N) = J(R/N)$ . It follows that  $J \subseteq N$  so  $J$  is nilpotent.

We will add one final note. If  $R$  is a ring the attached ring  $R^+$  is the ring where multiplication is redefined by  $a \cdot b = ab + ba$ . Rich has shown that if  $R$  is alternative and having characteristic  $\neq 2, 3$ , then the (Jordan) ring  $R^+$  has the same prime radical as  $R$  [6]. That is,  $P(R) = P(R^+)$  using the notation of §1. This result may be generalized slightly: If  $R$  satisfies (1) and has characteristic

$\neq 2, 3$ , then the prime radical of  $R$  coincides with each of the  $u$ -prime radicals  $(R^+)^u$  in  $R^+$ . This is interesting because while Jordan rings are 3-rings, it does not seem likely that in general  $R^+$  will be an  $s$ -ring. The proof (which we omit) is similar to the one found in [6].

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