

# Pacific Journal of Mathematics

***(hnp)*-RINGS OVER WHICH EVERY MODULE ADMITS A  
BASIC SUBMODULE**

SURJEET SINGH

## $(hnp)$ -RINGS OVER WHICH EVERY MODULE ADMITS A BASIC SUBMODULE

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**The structure of those bounded  $(hnp)$ -rings over which every module admits a basic submodule, is determined. It is shown that such rings are precisely the block lower triangular matrix rings over  $D \setminus M$  where  $D$  is a discrete valuation ring with  $M$  as its maximal ideal.**

In [12], the author generalized some well known results on decomposability of torsion abelian groups to torsion modules over bounded  $(hnp)$ -rings. Let  $R$  be a bounded  $(hnp)$ -ring and  $M$  be a (right)  $R$ -module. A submodule  $N$  of  $M$  is called a *basic* submodule of  $M$  if it satisfies the following conditions:

- (i)  $N$  is decomposable in the sense that it is a direct sum of uniserial modules and finitely generated uniform torsion free modules.
- (ii)  $N$  is a pure submodule of  $M$ .
- (iii)  $M/N$  is a divisible module.

The following result has been proved by the author (see [9] for details):

**THEOREM 1.** *Any torsion module  $M$  over a bounded  $(hnp)$ -ring has a basic submodule and any two basic submodules of  $M$  are isomorphic.*

In general an  $R$ -module need not have a basic submodule. However Marubayashi [8, Theorem (3.6)] showed that every module over a  $g$ -discrete valuation ring has a basic submodule. In this paper we determine the structure of those bounded  $(hnp)$ -rings, over which every (right) module admits a basic submodule (Theorems 3 and 4).

As defined by Marubayashi [8, p. 432], a prime, right as well as left principal ideal ring  $R$ , such that its Jacobson radical  $J(R)$  is the only maximal ideal, and idempotents modules  $J(R)$  can be lifted, is called a  $g$ -discrete valuation ring; further if  $R/J(R)$  is a division ring, then  $R$  is called a discrete valuation ring. In view of [8, Lemma (3.1)] and [7, Lemma (2.1)],  $g$ -discrete valuation rings are precisely the matrix rings over discrete valuation rings. Modules considered will be unital right modules and the notations and terminology of [12, 13] will be used without comment.

Henceforth in all lemmas,  $R$  is a bounded  $(hnp)$ -ring over which every module admits a basic submodule. Further  $Q$  stands for the classical quotient ring of  $R$ .

LEMMA 1. *A submodule  $N$  of a torsion free module  $M$  over an  $(hnp)$ -ring  $S$  is pure if and only if  $M/N$  is torsion free.*

*Proof. Necessity.* Let for some  $x \in M$  and a regular element  $b$  in  $S$ ,  $xb = y \in N$ . As  $N$  is pure, for some  $z \in N$ ,  $xb = zb$ . This in turn gives  $x = z \in N$ . This proves that  $M/N$  is torsion free.

SUFFICIENCY. Let  $M/N$  be torsion-free. Consider a finite system of equations  $\sum_i x_i r_{ij} = s_j$ ,  $s_j \in N$ , having a solution  $\{x_i\}$  in  $M$ . If  $K = \sum x_i S + N$ , then  $K/N$  being finitely generated and torsion free, is projective. Hence  $K = K_1 \oplus N$ . This gives that the above system of equations have a solution in  $N$ . Hence  $N$  is pure in  $M$ .

LEMMA 2. *If  $U$  is a uniform torsion free right  $R$ -module, then either  $U$  is finitely generated, or divisible.*

*Proof.* Since by Lemma 1,  $0$  and  $U$  are only pure submodules of  $U$ , so  $0$  or  $U$  is the basic submodule of  $U$ . Hence  $U$  is divisible or finitely generated.

LEMMA 3. *Every over-ring of  $R$  different from  $Q$  is finitely generated as an  $R$ -module.*

*Proof.* Consider an over ring  $S$  of  $R$  such that  $S \neq Q$ . Now  $S = \bigoplus \sum U_i$ ,  $U_i$  are uniform as right  $S$ -modules, since  $S$  is an  $(hnp)$ -ring [6]. If any  $U_i$  is divisible as a right  $R$ -module, then  $S = Q$ , otherwise by Lemma 2,  $S_x$  is finitely generated.

Let  $L$  be any ring and  $J$  be an ideal of  $L$ . Let  $n$  be a positive integer and  $(k_1, k_2, \dots, k_r)$  be an ordered  $r$ -tuple of positive integers such that  $k_1 + k_2 + \dots + k_r = n$ . In the notations of Reiner [10, Chapter 8], we can form a block matrix ring of the type:

$$\begin{bmatrix} (L) & (J) & \dots & (J) \\ (L) & (L) & \dots & (J) \\ (L) & (L) & \dots & (L) \end{bmatrix} (k_1, k_2, \dots, k_r).$$

In the terminology of Robson [11], any such matrix ring is said to be a block lower triangular matrix ring over  $L \setminus J$ .

THEOREM 2. *Let  $R$  be a bounded  $(hnp)$ -ring over which every module admits a basic submodule. Then there exists a discrete valuation ring  $D$  with maximal ideal  $M$  such that  $R$  is a block lower triangular matrix ring over  $D \setminus M$ .*

*Proof.* First of all we show that  $R$  has only one maximal invertible ideal. Let  $A$  be a maximal invertible ideal of  $R$ . If  $A$  is not the only maximal invertible ideal, then in the notations of [13]  $R < R_A < Q$ . There exists a non unit regular element  $a$  in  $R$  such that  $a$  is a unit modulo  $A$ . Then  $U_n a^{-n} R \subset R_A$  and  $U_n a^{-n} R$  is not finitely generated as a right  $R$ -module. This contradicts Lemma 3. Hence  $A$  is the only maximal invertible ideal of  $R$  and  $R = R_A$ . Then  $J(R) = A$ . This then gives  $R$  has only finitely many idempotent ideals. Let  $B$  be a minimal nonzero idempotent ideal of  $R$ . Then  $O_i(B) = \{x \in Q: xB \subset B\}$  is a Dedekind prime ring [3, Proposition (1.8)]. As for  $R$ , every torsion free uniform  $O_i(B)$ -module is either finitely generated or divisible. As a consequence  $O_i(B)$  has only one maximal ideal  $P$  and  $O_i(B) = O_i(B)_P$ . So by [7, Lemma (2.1)]  $O_i(B) = D_n$  for some discrete valuation ring  $D$ . By Jacobson [5, p. 120],  $R$  is equivalent to  $O_i(B)$ . Hence by Robson [11, Theorem (6.3) and Corollary (2.8)],  $R$  is a block lower triangular matrix ring over  $D \setminus M$ , where  $D$  is a discrete valuation ring with  $M$  as its maximal ideal.

It is clear that any non block lower triangular matrix ring over  $D \setminus M$  where  $D$  is a discrete valuation ring with  $M$  is its maximal ideal, is equivalent to  $D_n$ . So to prove the converse of the above theorem it is enough to prove the following:

**THEOREM 3.** *Let  $R$  be a bounded (hnp)-ring such that  $R$  is equivalent to  $S$ , for some  $g$ -discrete valuation ring  $S$ , which is an overring of  $R$ , then every  $R$ -module admits a basic submodule.*

*Proof.* First of all we show that any uniform torsion free  $R$ -module  $U$  is either divisible or finitely generated. Suppose  $U$  is not divisible. Now  $S = D_n$ . There exist regular elements  $a$  and  $b$  in  $R$  such that  $aSb \subset R$ . Since  $S$  is bounded there exists a nonzero ideal  $\mathcal{S}$  of  $S$  such that  $\mathcal{S} \subset Sb$ . Then  $a\mathcal{S} \subset R$  and the fact that  $S_S$  is embeddable in  $a\mathcal{S}$  gives that  $S_R$  is finitely generated. Similarly  ${}_R S$  is finitely generated. So using [3, Theorem (1.6)], we get  $S = O_i(A) = A^* = AA^*$  for some idempotent ideal  $A$  of  $R$ . We can suppose that  $U \subset Q$ , the classical quotient ring of  $R$ . If  $US = eQ$ , then  $UAA^*A = eQA = eQ$ . However  $UAA^*A \subset U$ . Thus in this case  $U$  is divisible. Hence  $US$  is finitely generated as  $S$ -module [8, Lemma (3.2)]. This gives  $U_R$  is finitely generated, since as proved above  $S_R$  is finitely generated.

Thus every uniform torsion free right  $R$ -module is injective or projective. Consider any right  $R$ -module  $M$  and let  $T$  be its torsion submodule.  $T$  admits a basic submodule  $B$  by Theorem 1. Then  $\bar{B}$  is a pure submodule of  $M$  and  $T/B$  is divisible; further  $T/B$  is the torsion submodule of  $M/B$ . So we can write

$$M/B = L/B \oplus T/B \oplus K/B$$

where  $L/B$  is torsion free, divisible  $R$ -module and  $K/B$  is a torsion free reduced  $R$ -module. If  $K/B = 0$ , we get  $B$  itself is a basic submodule. So let  $K/B \neq 0$ . We can find a maximal uniform submodule  $U/B$  of  $K/B$ . By what has been proved above  $U/B$  is finitely generated and hence projective. So by Lemma 1,  $U$  is a pure submodule of  $K$ , and  $U = U_1 \oplus B$ , where  $U_1$  is a finitely generated uniform torsion submodule. By Zorn's lemma, we can find a maximal direct sum  $E = B \oplus \sum \oplus U_i$ , in  $K$  such that  $E$  is a pure submodule of  $K$ ,  $U_i$  are finitely generated uniform, torsion free  $R$ -modules. By Lemma 1,  $K/E$  is torsion free. If  $K/E$  is not divisible, then as before we get a nonzero finitely generated uniform submodule  $V/E$  of  $K/E$  such that  $V/E$  is pure in  $K/E$ . Then  $V = V_1 \oplus E$  and  $V$  is a pure submodule of  $K$ . This contradicts the maximality of  $E$ . Hence  $K/E$  is divisible.  $E$  is clearly decomposable and is a basic submodule of  $M$ . This completes the proof.

We remark that any two basic submodules of a module over the ring of the above theorem, can be shown to be isomorphic.

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#### REFERENCES

1. D. Eisenbud and P. Griffith, *Serial rings*, J. Algebra, **17** (1971), 389-400.
2. D. Eisenbud and J. C. Robson, *Modules over Dedekind prime rings*, J. Algebra, **16** (1970), 67-85.
3. ———, *Hereditary noetherian prime rings*, J. Algebra, **16** (1970), 86-104.
4. L. Fuchs, *Abelian Groups*, Pergamon Press, 1960.
5. N. Jacobson, *The Theory of Rings*, Mathematical Survey No. 2, Amer. Math. Soc., 1943.
6. J. Kuzmanovitch, *Localizations of Dedekind prime rings*, J. Algebra, **21** (1972), 371-395.
7. H. Marubayashi, *Modules over bounded Dedekind prime rings*, Osaka J. Math., **9** (1972), 95-110.
8. ———, *Modules over bounded Dedekind prime rings II*, **9** (1972), 427-445.
9. Musharafuddin Khan, Ph. D. Thesis, Aligarh Muslim University, Aligarh, 1976.
10. I. Reiner, *Maximal Orders*, Academic Press, 1975.
11. J. C. Robson, *Idealizers and hereditary noetherian prime rings*, J. Algebra, **22** (1972), 45-81.
12. S. Singh, *Modules over hereditary noetherian prime rings*, Canad J. Math., **27** (1975), 867-883.
13. ———, *Modules over hereditary noetherian prime rings II*, Canad. J. Math., **28** (1976), 73-82.

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|                                                                                                                                                                                         |     |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----|
| Stephanie Brewster Brewer Taylor Alexander, <i>Local and global convexity in complete Riemannian manifolds</i> .....                                                                    | 283 |
| Claudi Alsina i Català, <i>On countable products and algebraic convexifications of probabilistic metric spaces</i> .....                                                                | 291 |
| Joel David Berman and George Grätzer, <i>Uniform representations of congruence schemes</i> .....                                                                                        | 301 |
| Ajit Kaur Chilana and Kenneth Allen Ross, <i>Spectral synthesis in hypergroups</i> .....                                                                                                | 313 |
| David Mordecai Cohen and Howard Leonard Resnikoff, <i>Hermitian quadratic forms and Hermitian modular forms</i> .....                                                                   | 329 |
| Frank Rimi DeMeyer, <i>Metabelian groups with an irreducible projective representation of large degree</i> .....                                                                        | 339 |
| Robert Ellis, <i>The Furstenberg structure theorem</i> .....                                                                                                                            | 345 |
| Heinz W. Engl, <i>Random fixed point theorems for multivalued mappings</i> .....                                                                                                        | 351 |
| William Andrew Ettl, <i>On arc length sharpenings</i> .....                                                                                                                             | 361 |
| Kent Ralph Fuller and Joel K. Haack, <i>Rings with quivers that are trees</i> .....                                                                                                     | 371 |
| Kenneth R. Goodearl, <i>Centers of regular self-injective rings</i> .....                                                                                                               | 381 |
| John Gregory, <i>Numerical algorithms for oscillation vectors of second order differential equations including the Euler-Lagrange equation for symmetric tridiagonal matrices</i> ..... | 397 |
| Branko Grünbaum and Geoffrey Shephard, <i>Isotoxal tilings</i> .....                                                                                                                    | 407 |
| Myron Stanley Henry and Kenneth Leroy Wiggins, <i>Applications of approximation theory to differential equations with deviating arguments</i> .....                                     | 431 |
| Mark Jungerman, <i>The non-orientable genus of the <math>n</math>-cube</i> .....                                                                                                        | 443 |
| Robert Richard Kallman, <i>Only trivial Borel measures on <math>S_\infty</math> are quasi-invariant under automorphisms</i> .....                                                       | 453 |
| Joyce Longman and Michael Rich, <i>Scalar dependent algebras in the alternative sense</i> .....                                                                                         | 463 |
| Richard A. Mollin, <i>The Schur group of a field of characteristic zero</i> .....                                                                                                       | 471 |
| David Pokrass, <i>Some radical properties of rings with <math>(a, b, c) = (c, a, b)</math></i> .....                                                                                    | 479 |
| Margaret Shay and Paul Ruel Young, <i>Characterizing the orders changed by program translators</i> .....                                                                                | 485 |
| Jerrold Norman Siegel, <i>On the structure of <math>B_\infty(F)</math>, <math>F</math> a stable space</i> .....                                                                         | 491 |
| Surjeet Singh, <i>(hnp)-rings over which every module admits a basic submodule</i> .....                                                                                                | 509 |
| A. K. Snyder, <i>Universal interpolating sets and the Nevanlinna-Pick property in Banach spaces of functions</i> .....                                                                  | 513 |
| Jeffrey D. Vaaler, <i>On the metric theory of Diophantine approximation</i> .....                                                                                                       | 527 |
| Roger P. Ware, <i>When are Witt rings group rings? II</i> .....                                                                                                                         | 541 |