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CHEBYSHEV CENTERS AND UNIFORM CONVEXITY

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If E is a uniformly convex Banach space and T is any topological space, then in the space $X = C(T, E)$ of E -valued bounded continuous functions on E , every bounded set has a Chebyshev center. Moreover, the set function $A \rightarrow Z(A)$, corresponding to A the set of its Chebyshev centers, is uniformly continuous on bounded subsets of the space $\mathcal{B}(X)$ of bounded subsets of X with the Hausdorff metric. This is contrasted with the fact that a normed space X in which $Z(A)$ is a singleton for every bounded A is uniformly convex iff $A \rightarrow Z(A)$ is uniformly continuous on bounded subsets of $\mathcal{B}(X)$.

Let (X, d) be a metric space. Denote by $\mathcal{B}(X)$ the space of nonempty bounded subsets of X and let h be the Hausdorff semi-metric on $\mathcal{B}(X)$:

$$h(A, B) = \max \left(\sup_{u \in A} \inf_{v \in B} d(u, v), \sup_{v \in B} \inf_{u \in A} d(u, v) \right).$$

For $x \in X$, $r \geq 0$, let $B(x, r) = \{y \in X; d(x, y) \leq r\}$ be the closed r -ball around x . For $A \in \mathcal{B}(X)$ and $x \in X$ denote $r(x, A) = \inf \{r \geq 0; B(x, r) \supset A\}$, $r(A) = \inf_{x \in X} r(x, A)$ is the *Chebyshev radius of A* , and $Z(A) = \{x \in X; r(x, A) = r(A)\}$ is the set of *Chebyshev centers of A* . For $Y \subset X$ we can consider also the *relative Chebyshev radius of A in Y* , $r_Y(A) = \inf_{y \in Y} r(y, A)$, and the set of *relative Chebyshev centers of A in Y* , $Z_Y(A) = \{y \in Y; r(y, A) = r_Y(A)\}$. In the case that $A = \{x\}$ is a singleton, then $Z_Y(A)$ is just the set of best approximations in Y to x , $P_Y x$.

We say that X admits centers if every bounded set in X has Chebyshev centers. The classical Banach spaces, i.e., the spaces $L_p(\mu)$, $1 \leq p \leq \infty$, over any measure space and the spaces $C(T)$ of continuous real-valued functions on compact Hausdorff T , admit centers ([1], [3]). However, Garkavi ([1]) gave an example of a 3-point set in a maximal subspace H of $C[0, 1]$ which has no Chebyshev center in H . The problem of characterizing all Banach space which admit centers is still open.

Ward ([5]) proved that the space $C(T, E)$ of E -valued bounded continuous functions on the topological space T , with the norm $\|x\| = \sup_{t \in T} \|x(t)\|$, admits centers in each of the following two cases: (a) E is a finite-dimensional strictly convex (hence uniformly convex) normed space and T is paracompact. (b) E is a Hilbert

space and T is normal. Ward asked whether both results can be strengthened by taking in (b) any uniformly convex Banach space E . Our first result shows that the answer is in the affirmative.

We use the following characterization of uniform convexity, due to Ruston ([4]). We include a proof for completeness sake.

1. LEMMA. *A normed space E is uniformly convex iff for every $\varepsilon > 0$ there is $\delta'(\varepsilon) > 0$ such that if $x, y \in E$ and $\phi \in E^*$ are such that $\|x\| = \|y\| = 1 = \|\phi\| = \phi(y)$ and $\phi(x) > 1 - \delta'(\varepsilon)$, then $\|x - y\| < \varepsilon$. We can take, of course, $\delta'(\varepsilon) \leq 1/2\varepsilon$.*

Proof. If E is uniformly convex, then $\delta(\varepsilon) \equiv \inf \{1 - \|(u - v)/2\|; \|u\| = \|v\| = 1, \|u - v\| \geq \varepsilon\}$ is positive. Take $\delta'(\varepsilon) < 2\delta(\varepsilon)$. If $\|x - y\| \geq \varepsilon$ then $\|(x + y)/2\| \leq 1 - \delta(\varepsilon)$ hence $(\phi(x) + 1)/2 = \phi((x + y)/2) \leq 1 - \delta(\varepsilon)$ and $\phi(x) \leq 1 - 2\delta(\varepsilon) < 1 - \delta'(\varepsilon)$.

Conversely, we claim that $\delta(\varepsilon) \geq \delta'(\varepsilon/4)$. Indeed, if $\|x\| = \|y\| = 1$ and $\|x - y\| > \varepsilon$, take $\phi \in E^*$ with $\|\phi\| = 1$, $\phi(x + y) = \|x + y\|$. Then either $\|(x + y)/2\| < 1 - \varepsilon/4 \leq 1 - \delta'(\varepsilon/4)$, or $\|(x + y)/2\| \geq 1 - \varepsilon/4$, hence $\|(x + y)/2 - (x + y)/\|x + y\|\| \leq \varepsilon/4$ and $\|x - (x + y)/\|x + y\|\| \geq \varepsilon/4$, $\|y - (x + y)/\|x + y\|\| \geq \varepsilon/4$, hence $\phi(x) \leq 1 - \delta'(\varepsilon/4)$, $\phi(y) \leq 1 - \delta'(\varepsilon/4)$ and $\|x + y\| = \phi(x + y) \leq 2(1 - \delta'(\varepsilon/4))$.

2. THEOREM. *If E is a uniformly convex Banach space and T is any topological space, then $C(T, E)$ admits centers.*

Proof. Given any bounded $A \subset C(T, E)$ we may assume, without loss of generality, that $r(A) = 1$. Given $\varepsilon > 0$, choose any $f_0 \in C(T, E)$ with $r(f_0, A) \leq 1 + \delta'(\varepsilon)$. We claim that there is $f_1 \in C(T, E)$ with $r(f_1, A) \leq 1 + \delta'(\varepsilon/2)$ and $\|f_1 - f_0\| \leq 2\varepsilon$. Indeed, take any $g \in C(T, E)$ with $r(g, A) \leq 1 + \delta'(\varepsilon/2)$ and define:

$$\beta(t) = \begin{cases} 1 & \text{if } \|g(t) - f_0(t)\| \leq 2\varepsilon \\ \frac{2\varepsilon}{\|g(t) - f_0(t)\|} & \text{if } \|g(t) - f_0(t)\| > 2\varepsilon \end{cases}$$

$$f_1(t) = f_0(t) + \beta(t)(g(t) - f_0(t)).$$

Clearly, $f_1 \in C(T, E)$ and $\|f_1 - f_0\| \leq 2\varepsilon$. Take any $a \in A$. We have to show that $\|f_1(t) - a(t)\| \leq 1 + \delta'(\varepsilon/2)$. This is clear if $\beta(t) = 1$, since then $f_1(t) = g(t)$, or if $\beta(t) < 1$ but $\|g(t) - a(t)\| \geq \|f_0(t) - a(t)\|$, since $f_1(t)$ lies on the segment $[f_0(t), g(t)]$. Therefore we may assume $1 + \delta'(\varepsilon) \geq \|f_0(t) - a(t)\| > \|g(t) - a(t)\|$. Denote $u = f_0(t) - a(t)$, $v = g(t) - a(t)$, so that $\|v\| \leq 1 + \delta'(\varepsilon/2)$ and $1 + \delta'(\varepsilon) \geq \|u\| > \|v\|$ and we want to show that going a distance of 2ε from u towards v , we enter the $(1 + \delta'(\varepsilon/2))$ -ball around 0. Since this is true if $\|v\| = 0$,

it suffices to show it when $\|v\| = 1 + \delta'(\varepsilon/2)$.

In the 2-dimensional space spanned by u and v let z be on the $\|v\|$ -sphere, on the same side of the line through 0 and u as v is, such that the line \overline{uz} supports the sphere. Extend this line to a hyperplane $H = \psi^{-1}$ supporting the $\|v\|$ -ball in E . Let $\phi = \|v\|\psi$, $x = u/\|u\|$, $y = z/\|z\|$. Then $\|\phi\| = \phi(y) = 1 = \|y\| = \|x\|$ and $\phi(x) = 1/\|u\| \geq 1/(1 + \delta'(\varepsilon)) > 1 - \delta'(\varepsilon)$, hence $\|x - y\| < \varepsilon$ and $\|u - z\| < \varepsilon + \|u - x\| + \|z - y\| \leq \varepsilon + \delta'(\varepsilon) + \delta'(\varepsilon/2) < 2\varepsilon$. This proves our claim, since the distance from u to the $\|v\|$ -ball in the direction of v is less than the maximal of the distances in the directions of x (which is $\leq \delta'(\varepsilon)$) and z (which is $< 2\varepsilon$).

Inductively, we find f_{n+1} with $\|f_{n+1} - f_n\| \leq 2\varepsilon/2^n$ and $r(f_{n+1}, A) \leq 1 + \delta'(\varepsilon/2^{n+1})$. The Cauchy sequence (f_n) converges to some f with $\|f - f_0\| \leq 2\varepsilon$. $r(f, A) \leq \lim r(f_n, A) \leq 1$, hence $r(f, A) = 1$ and f is a Chebyshev center for A . (See Remark 6.)

3. COROLLARY. *If $X = C(T, E)$, E a uniformly convex Banach space, then the mapping $A \rightarrow Z(A)$ in $\mathcal{B}(X)$ is uniformly continuous on bounded subsets of $\mathcal{B}(X)$.*

Proof. By the proof of Theorem 1, if $r(f_0, A) \leq (1 + \delta'(\varepsilon))r(A)$, then there is $f \in Z(A)$ with $\|f - f_0\| \leq 4\varepsilon r(A)$. Given R and $\varepsilon > 0$, let $0 < \delta \leq R\delta'(\varepsilon)/2$. If $r(A) \leq R$, $r(B) \leq R$ and $h(A, B) < \delta$, then for every x we have $|r(x, A) - r(x, B)| < \delta$ (given $u \in A$, find $v \in B$ with $d(u, v) < \delta$ and then $d(x, u) < d(x, v) + \delta$ etc.), hence $|r(A) - r(B)| < \delta$, and for every $z \in Z(A)$ we have $r(z, B) < r(A) + \delta < r(B) + 2\delta \leq (1 + \delta'(\varepsilon))r(B)$. Therefore we can find $w \in Z(B)$ with $\|w - z\| \leq 4\varepsilon R$. Similarly $\sup_{w \in Z(B)} d(w, Z(A)) \leq 4\varepsilon R$, i.e., $h(Z(A), Z(B)) \leq 4\varepsilon R$.

4. COROLLARY. *If $X = C(T)$ and Y is a closed linear sublattice of X , then for every bounded $A \subset X$ there is a relative Chebyshev center in Y , and $A \rightarrow Z_Y(A)$ is uniformly continuous on bounded subsets of $\mathcal{B}(X)$.*

Proof. In the proof of the theorem, if f_0 and g are chosen in Y , then by the lattice property also $f_1 \in Y$.

The continuity property of the operation $A \rightarrow Z(A)$ in the "most square" space $X = C(T)$, obtained in Corollary 3, is somewhat surprising in view of the next theorem.

5. THEOREM. *A Banach space X is uniformly convex iff for every $A \in \mathcal{B}(X)$ $Z(A)$ is a singleton, and $A \rightarrow Z(A)$ is uniformly continuous on bounded subsets of $\mathcal{B}(X)$.*

Proof. Assume first that X is uniformly convex. Since X is reflexive $Z(A) \neq \emptyset$ for every $A \in \mathcal{B}(X)$, while uniform convexity guarantees that $Z(A)$ is a singleton ([1]). It is known (and easily proved) that if $\|x\|, \|y\| \leq M$ and $\|x - y\| \geq \varepsilon$, then $\|(x + y)/2\| \leq (1 - \delta(\varepsilon/M))M$. Suppose now $z = Z(A)$, $w = Z(B)$, $r(A) < R$ and $h(A, B) < \eta < 1$. Then $r(B) \leq r(z, B) < r(z, A) + \eta = r(A) + \eta$, and $r(A) \leq r(w, A) < r(B) + \eta < r(A) + 2\eta$. Therefore for $u \in A$ we have $\|u - z\| \leq r(A)$, $\|u - w\| < r(A) + 2\eta$ and

$$\left\| u - \frac{z + w}{2} \right\| = \left\| \frac{(u - z) + (u - w)}{2} \right\| \leq \left(1 - \delta\left(\frac{\varepsilon}{R + 2}\right) \right) (r(A) + 2\eta).$$

But $\|u - (z + w)/2\| \geq r(A)$, for some $u \in A$, hence if $\|z - w\| \geq \varepsilon$ then

$$\eta \geq \frac{r(A)\delta\left(\frac{\varepsilon}{R + 2}\right)}{2\left(1 - \delta\left(\frac{\varepsilon}{R + 2}\right)\right)} \geq \frac{r(A)}{2}\varepsilon\left(\frac{\varepsilon}{R + 2}\right).$$

Thus if $\eta < r(A)\delta(\varepsilon/(R + 2))/2$ then $\|z - w\| < \varepsilon$. So that fixing $\eta = \varepsilon\delta(\varepsilon/(R + 2))/4$ we have either $r(A) \geq \varepsilon/2$ and then $\|z - w\| < \varepsilon$, or $r(A) < \varepsilon/2$ and then taking any $u \in A$ we have $\|z - w\| \leq \|z - u\| + \|u - w\| < r(A) + r(A) + 2\eta < \varepsilon$.

Conversely, if E is not uniformly convex, there are $x_n, y_n \in E$ with $\|x_n\| = \|y_n\| = 1$, $\|x_n - y_n\| = \varepsilon$ and $\|x_n + y_n\| \rightarrow 2$. Let

$$z_n = \frac{x_n + y_n}{\|x_n + y_n\|}, \quad A_n = \text{conv} \left\{ x_n, \frac{x_n + y_n}{2}, -\frac{x_n + y_n}{2}, -y_n \right\},$$

$$B_n = \text{conv} \{ x_n, z_n, -z_n, -y_n \}.$$

Then $h(A_n, B_n) \rightarrow 0$, but $(x_n - y_n)/4 \in Z(A_n)$ while $0 \in Z(B_n)$. Thus if $Z(A_n)$ and $Z(B_n)$ are singletons, we have $h(Z(A_n), Z(B_n)) = \|(x_n - y_n)/4\| = \varepsilon/4$.

REMARKS. (1) By the proof it is clear that it is enough to check the uniform continuity of $A \rightarrow Z(A)$ on the 2-dimensional subsets A of the unit ball of X .

(2) There are nonuniformly convex spaces in which $Z(A)$ is a singleton for every bounded nonempty A . It is known ([1]) that if X is reflexive then $Z(A) \neq \emptyset$ for every $A \in \mathcal{B}(X)$ while the condition that $Z(A)$ is at most a singleton for every $A \in \mathcal{B}(X)$ is equivalent to X being u.c.e.d (uniformly convex in every direction, which means that $\delta_z(\varepsilon) \equiv \inf \{ 1 - \|(x + y)/2\|; \|x\| = \|y\| = 1, x - y = \lambda z, \|x - y\| \geq \varepsilon \} > 0$ for every $z \neq 0$). Since every separable space can be equivalently renormed to be u.c.e.d ([6]) while only super-

reflexive spaces can be renormed to be uniformly convex ([2]), every reflexive but nonsuperreflexive separable X can be renormed so that $Z(A)$ is a singleton for every $A \in \mathcal{B}(X)$, while $A \rightarrow Z(A)$ is not uniformly continuous.

(3) If we wish to drop the condition that $Z(A)$ is a singleton, the same proof yields.

“A normed space X is uniformly convex iff every selection for the set-valued map $Z: A \rightarrow Z(A)$ is uniformly continuous on bounded subsets of the domain of definition of Z in $\mathcal{B}(X)$.”

Indeed, A_n and B_n in the proof above have Chebyshev centers, while continuity of every selection of Z implies that Z is single-valued. Again, we may restrict ourselves to 2-dimensional sets of the type A_n, B_n .

(4) Say that X is “uniformly convex with respect to Y ”, where Y is a closed subspace of X , if $\delta_Y(\varepsilon) \equiv \inf \{1 - \|(x + y)/2\|; \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon, x - y \in Y\} > 0$ for every $\varepsilon > 0$. The same proof will yield: The Banach space X is uniformly convex with respect to its subspace Y iff $A \rightarrow Z_Y(A)$ is a locally uniformly continuous function from $\mathcal{B}(X)$ (or even the 2-dimensional sets in $\mathcal{B}(X)$) to Y .

(5) A related result is the following theorem of P. Smith (cf. [7], p. 188): If E is an “ E -space” (i.e., a Banach space with a Fréchet differentiable dual or, equivalently, a reflexive strictly convex space in which $x_n \xrightarrow{w} x, \|x_n\| \rightarrow \|x\| \Rightarrow x_n \rightarrow x$) then $A \rightarrow Z(A)$ is a continuous single-valued map from the space of compact subsets of E (in the Hausdorff metric) into E .

(6) Theorem 2 has been obtained, independently, by Ka-Sing Lau, who proved in a similar way the following more general result: For any bounded set-valued map ϕ from a topological space X into a uniformly convex Banach space E and for every closed $C(X)$ -submodule M in $C(X, E)$ there is a best approximation from M to ϕ .

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