Pacific Journal of Mathematics

EQUIVALENCE OF DECOMPOSABLE AND 2-DECOMPOSABLE OPERATORS

Mehdi Radjabalipour

Vol. 77, No. 1 January 1978

PACIFIC JOURNAL OF MATHEMATICS
Vol. 77, No. 1, 1978

EQUIVALENCE OF DECOMPOSABLE AND 2-DECOMPOSABLE OPERATORS

MEHDI RADJABALIPOUR

It is shown that every 2-decomposable operator (in the sense of S. Plafker) is decomposable (in the sense of C. Foias); this answers a question raised by Plafker.

Throughout the paper, T is a bounded linear operator defined on a Banach space Y. An invariant subspace X of T is called a *spectral maximal subspace* of T if $M \subseteqq X$ for all invariant subspaces M of T for which $\sigma(T|M) \subseteqq \sigma(T|X)$. The operator T is called *n-decomposable* [10] (n a fixed integer greater than 1) if for every open covering $\{G_1, G_2, \cdots, G_n\}$ of $\sigma(T)$ there exist spectral maximal subspaces $X_1, X_2, \cdots, X_n$ of T such that $\sigma(T|X_i) \subseteqq \bar{G}_i$ $(i = 1, 2, \cdots, n)$ and $Y = X_1 + X_2 + \cdots + X_n$; T is called *decomposable* [6] if it is n-decomposable for all $n \geqq 2$.

Plafker [10] asked whether every n-decomposable operator is decomposable; the question is answered by E. Albrecht and F.-H. Vasilescu [1] for $n \geqq 3$ in a general Banach space and by S. Frunză [7] for $n \geqq 2$ in a reflexive Banach space. Here we extend Frunză's result to a general Banach space by a shorter and simpler proof, and thus we completely solve Plafker's problem.

For a closed subset F of C, we let $X_T(F) = \{x \in Y$: there exists an analytic function $f_x: C\backslash F \to Y$ such that $(z - T)f_x(z) \equiv x\}$. If T is 2-decomposable, then $X_T(F)$ is a spectral subspace for all F, and every spectral maximal subspace of T is of this form [10] (see also [4] for the case of decomposable operators). I. Colojoară and C. Foiaș [4, page 217] ask whether the restriction of a decomposable operator to every spectral maximal subspace is again decomposable; an operator whose restriction to every spectral maximal subspace is decomposable is called *strongly decomposable* by C. Apostol [2]. Obviously every strongly decomposable operator is decomposable and whether the converse is true is what Colojoară and Foiaș ask. I. Bacalu [3] shows that decomposable operators with nowhere dense spactra are strongly decomposable. The problem in the general case seems to be difficult and has been attacked by many authors; the following question is simpler and arises in a natural way.

Question 1. Does every decomposable operator T satisfy the following condition (1):

$$X_T(F) \subseteqq X_T(\bar{G}_1) + X_T(\bar{G}_2) + \cdots + X_T(\bar{G}_n) \tag{1}$$

for any closed set F and any open covering $\{G_1, G_2, \cdots, G_n\}$ of F.

Decomposable operators satisfying (an equivalent modification of) condition (1) are said to have *almost localized spectra* by Vasilescu [13], who also shows that the duals of such operators are again decomposable operators of the same type. Frunză [7], [8] shows that the dual of any 2-decomposable operator is decomposable and satisfies (1); hence in a reflexive Banach space all decomposable operators satisfy (1). We improve this result by showing that every decomposable operator (on any Banach space) satisfies (1).

We will make use of the following proposition due to Frunză [8, Proposition 1].

PROPOSITION 1. *Let T be a 2-decomposable operator. Assume $X_T(F) \subseteq X_T(\bar{G}_1) + X_T(\bar{G}_2)$ for any closed set F and any open covering $\{G_1, G_2\}$ of F. Then T is a decomposable operator satisfying* (1).

The following lemma is probably known to the experts; we include a proof for an easy reference. Recall that the operator T is said to have the single-valued extension property if there exists no nonzero, Y-valued, analytic function f such that $(z - T)f(z) \equiv 0$; it is known that every 2-decomposable operator T has the single-valued extension property and $\sigma(T | X_T(F)) \subseteq F \cap \sigma(T)$ for all closed sets F [10] (see also [4]).

LEMMA 1. *Let T be a 2-decomposable operator and let E be a closed subset of C. Let $\hat{Y} = Y/X_T(E)$, $\hat{T}$ be the operator on $\hat{Y}$ induced by T, and let $\hat{x}$ in $\hat{Y}$ denote the image under the canonical mapping of $x \in Y$. Assume $\hat{x} \in X_{\hat{T}}(F)$ for some closed set F. Then $x \in X_T(E \cup F)$.*

Proof. Let G_1 be an open neighborhood of $E \cup F$ and let G_2 be another open set such that $\sigma(T) \subseteq G_1 \cup G_2$ and $\bar{G}_2 \cap (E \cup F) = \varnothing$. Let $x = x_1 + x_2$, where $x_i \in X_T(\bar{G}_i)$ $(i = 1, 2)$. Since $\hat{x}_2 = \hat{x} - \hat{x}_1$, there exists and analytic function $g: C\backslash\bar{G}_1 \rightarrow \hat{Y}$ such that $(z - \hat{T})g(z) \equiv \hat{x}_2$. Let $M = X_T(\bar{G}_2 \cup E)/X_T(E)$. Since M is a spectral maximal subspace of $\hat{T}$ containing $\hat{x}_2$ [2, Proposition 1.3.2(3)], it follows that $g(z) \in M$ for all $z \notin \bar{G}_1$ [4, page 19]. (Note that, in view of [12, Theorem 2.1], T has the single-valued extension property.) Now let $S: X_T(\bar{G}_2) \rightarrow M$ be the operator defined by $Sy = \hat{y} (y \in X_T(\bar{G}_2))$. In the light of the Riesz decomposition theorem (applied to $T | X_T(\bar{G}_2 \cup E)$), S is bijective and $S(T | X_T(\bar{G}_2)) = (\hat{T} | M)S$. Thus $(z - T)(S^{-1}g(z)) \equiv x$ for $z \notin \bar{G}_1$ and hence $x_2 \in X_T(\bar{G}_1)$. Therefore $x \in X_T(\bar{G})$ for all neighborhoods G of $E \cup F$ which implies that $x \in X_T(E \cup F)$.

Now we prove the main result of the paper.

THEOREM 1. *Every 2-decomposable operator T is decomposable and satisfies condition* (1), i.e., *it has almost localized spectrum.*

Proof. Let F be a closed subset of the plane and let $\{G_1, G_2\}$ be an open covering of F. Let $E = \bar{G}_1 \cap \bar{G}_2$, $\hat{Y} = Y/X_T(E)$, and let $\hat{T}$ be the operator on $\hat{Y}$ induced by T. Let $x \in X_T(F)$ and let $f: C\backslash F \to Y$ be an analytic function such that $(z - T)f(z) \equiv x$. Obviously $\hat{f}: C\backslash F \to \hat{Y}$ is analytic and $(z - \hat{T})\hat{f}(z) \equiv \hat{x}$. (Here again $\hat{u}$ denotes the image of u under the canoninal mapping from Y onto $\hat{Y}$.) Since $\sigma(\hat{T}) \subseteqq C\backslash(G_1 \cap G_2)$ [9, Lemma 1], it follows that $\hat{f}$ has an analytic extension (denoted by the same symbol $\hat{f}$) to $(C\backslash F) \cup (G_1 \cap G_2)$, which also preserves the identity $(z - \hat{T})\hat{f}(z) \equiv \hat{x}$.

Let D_1 be a Cauchy domain containing $F\backslash G_2$ and let D_2 be another one containing $F\backslash G_1$. Since $F\backslash G_1$ and $F\backslash G_2$ are disjoint, D_1 and D_2 can be chosen such that $\bar{D}_1 \cap \bar{D}_2 = \varnothing$. Now for $j = 1, 2$ define

$$\xi_j = (2\pi i)^{-1} \int_{+\partial D_j} \hat{f}(\lambda) d\lambda ,$$

and

$$g_j(z) = (2\pi i)^{-1} \int_{+\partial D_j} (z - \lambda)^{-1} \hat{f}(\lambda) d\lambda \ (z \notin D_j) .$$

Obviously $\hat{x} = \xi_1 + \xi_2$. Also,

$$\begin{aligned}(z - \hat{T})g_j(z) &= (2\pi i)^{-1} \int_{+\partial D_j} ((z - \lambda) + (\lambda - \hat{T}))(z - \lambda)^{-1} \hat{f}(\lambda) d\lambda \\ &= (2\pi i)^{-1} \int_{+\partial D_j} \hat{f}(\lambda) d\lambda + (2\pi i)^{-1} \int_{+\partial D_j} (z - \lambda)^{-1} \hat{x} d\lambda = \xi_j ,\end{aligned}$$

for $z \notin D_j$ and $j = 1, 2$. This shows that $\xi_1 \in X_{\hat{T}}(\bar{D}_1)$, and since D_1 is an arbitrary Cauchy domain containing $F\backslash G_2$, it follows that $\xi_1 \in X_{\hat{T}}(F\backslash G_2)$. Similarly $\xi_2 \in X_{\hat{T}}(F\backslash G_1)$. Let x_j be a vector in Y such that $\xi_j = \hat{x}_j (j = 1, 2)$. By Lemma 1, $x_j \in X_T(\bar{G}_j)(j = 1, 2)$. Since $\hat{x} = \hat{x}_1 + \hat{x}_2$, $x = (x_1 + u) + x_2$, where $u \in X_T(E)$; thus $x \in X_T(\bar{G}_1) + X_T(\bar{G}_2)$. Now the rest of the proof follows from Proposition 1.

Part (c) of the following corollary gives a new characterization of decomposable operators.

COROLLARY 1. *The following assertions are equivalent.*

(a) *T is decomposable.*

(b) *T is 2-decomposable.*

(c) *For every closed set F, $X_T(F)$ is closed and $\sigma(T^F) \subseteqq \sigma(T)\backslash F^\circ$, where T^F denotes the operator induced by T on $Y/X_T(F)$ and F° denotes the interior of F.*

Proof. First note that the operator T in each case has the single-valued extension property (for case (c) see [11, Remark 2]). Now the equivalence of (a) and (b) follows from Theorem 1 and the equivalence of (b) and (c) follows from [9].

REMARK. J. Daughtry [5] defines a superinvariant subspace of T to be a subspace M invariant under all operators A such that $(AT - TA)M \subseteqq M$; he shows that an invariant subspace of a normal operator is superinvariant if and only if it is the range of a spectral projection. Note that if T is a normal operator with the resolution of the identity E_T, then $X_T(F) = E_T(F)Y$ for all closed sets F. The following proposition shows that for any decomposable operator T the subspaces $X_T(F)$ are superinvariant.

PROPOSITION 2. *Let T be a decomposable operator and let F be a closed subset of C. Then $X_T(F)$ is a superinvariant subspace of T.*

Proof. Let $T_F = T|X_T(F)$. Let $x \in X_T(F)$ and let $f: C\backslash F \rightarrow Y$ be an analytic function such that $(T - \lambda)f(\lambda) \equiv x$. Let A be an operator such that $(TA - AT)X_T(F) \subseteqq X_T(F)$ and let

$$g(\lambda) = Af(\lambda) - (T_F - \lambda)^{-1}(TA - AT)f(\lambda) \ (\lambda \notin F) .$$

Since $\sigma(T_F) \subseteqq F$ and $f(\lambda) \in X_T(F)$ for $\lambda \notin F$ [4, page 19], it follows that g is a well-defined analytic function and $(T - \lambda)g(\lambda) \equiv Ax$. Thus $Ax \in X_T(F)$ and the proof is complete.

REFERENCES

1. E. Albrecht and F.-H. Vasilescu, *On spectral capacities*, Rev. Roum. Math. Pures Appl., **19** (1974), 701-705.

2. C. Apostol, *Spectral decomposition and functional calculus*, Rev. Roum. Math. Pures Appl., **13** (1968), 1481-1528.

3. I. Bacalu, *On restrictions and quotients of decomposable operators*, Rev. Roum. Math. Pures Appl., **18** (1973), 809-813.

4. I. Colojoară and C. Foiaş, *TheTheory of Generalized Spectral Operators*, Gordon Breach, Science Publ., New York (1968).

5. J. Daughty, *Superinvariant subspaces*, (to appear).

6. C. Foiaş, *Spectral maximal spaces and decomposable operators in Banach spaces*, Arch. Math. (Basel), **14** (1963), 341-349.

7. S. Frunză, *Spectral decomposition and duality*, Illinois J. Math., (to appear).

8. ———, *A new result of duality for spectral decompositions*, (to appear).

9. A. A. Jafarian and F.-H. Vasilescu, *A characterization of 2-decomposable operators*, Rev. Roum. Math. Pures Appl., **19** (1974), 769-771.

10. S. Plafker, *On decomposable operators*, Proc. Amer. Math. Soc., **24** (1970), 215-216.

11. M. Radjabalipour, *On subnormal operators*, Trans. Amer. Math. Soc., **211** (1975), 377-389.

12. F.-H. Vasilescu, *Residual properties for closed operators on Frechet spaces*, Illinois J. Math., **15** (1971), 377-386.
13. ———, *On the residual decomposability in dual spaces*, Rev. Roum. Pures. Appl., **16** (1971), 1573-1578.

Received August 2, 1976 and in revised form November 1, 1977. The research was done in an operator theory institute at the University of New Hampsphire.

REZA SHAH KABIR UNIVERSITY
P. O. Box 444
BABOLSAR, IRAN

PACIFIC JOURNAL OF MATHEMATICS

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 77, No. 1 January, 1978