

Pacific Journal of Mathematics

**THE NUMBER OF NONFREE COMPONENTS IN THE
DECOMPOSITION OF SYMMETRIC POWERS IN
CHARACTERISTIC p**

GERT EINAR TORSTEN ALMKVIST

THE NUMBER OF NONFREE COMPONENTS IN THE DECOMPOSITION OF SYMMETRIC POWERS IN CHARACTERISTIC p

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If G is the group with p ($=$ prime) elements and k a field of characteristic p let V_1, V_2, \dots, V_p denote the indecomposable $k[G]$ -modules of k -dimension $1, 2, \dots, p$ respectively. Let $e_{n,\nu}$ denote the number of nonfree components of the decomposition of the symmetric power $S^\nu V_{n+1}$. Then the following symmetry relation is proved

$$e_{n,p-n-\nu-1} = e_{n,\nu}.$$

As a corollary we find that $S^r V_{n+1}$ has exactly one nonfree component when $n + r = p - 2$ thus solving a problem in a previous paper by R. Fossum and the author. An explicit formula for $e_{n,\nu}$ expressed in numbers of restricted partitions is obtained.

Let G be the group with p elements where p is a prime number. Let k be a field of characteristic p . Then there are p indecomposable $k[G]$ -modules V_1, V_2, \dots, V_p where

$$V_n \cong k[x]/(x - 1)^n.$$

Note that $V_p = k[G]$ is free and $\dim_k V_n = n$.

The symmetric power $S^\nu V_{n+1}$ taken over k is again a $k[G]$ -module and can be decomposed into a direct sum of the V_i : s

$$S^\nu V_{n+1} = \bigoplus_{j=1}^p c_{\nu,j}(n) V_j$$

where the integer $c_{\nu,j}(n)$ is the number of times V_j is repeated. Let

$$e_{n,\nu} = \sum_{j=1}^{p-1} c_{\nu,j}(n)$$

be the number of nonfree components in $S^\nu V_{n+1}$.

If we write down these numbers in triangular form we get the following pictures where the number in the $(\nu + 1)$ th place in the $(n + 1)$ th row from below is $e_{n,\nu}$.

P = 11

1 ↘ ν

1 1

1 1 1

1 1 1 1

1 1 2 1 1

1 1 2 2 1 1

1 1 3 3 3 1 1

1 1 2 3 3 2 1 1

1 1 2 2 3 2 2 1 1

1 1 1 1 1 1 1 1 1 1

$n \uparrow$

1 1 1 1 1 1 1 1 1 1 1 1

P = 13

1

1 1

1 1 1

1 1 1 1

1 1 2 1 1

1 1 2 2 1 1

1 1 3 3 3 1 1

1 1 3 4 4 3 1 1

1 1 3 4 5 4 3 1 1

1 1 2 3 4 4 3 2 1 1

1 1 2 2 3 3 3 2 2 1 1

1 1 1 1 1 1 1 1 1 1 1 1

1 1 1 1 1 1 1 1 1 1 1 1 1 1

P = 17

1

1 1

1 1 1

1 1 1 1

1 1 2 1 1

1 1 2 2 1 1

1 1 3 3 3 1 1

1 1 3 5 5 3 1 1

1 1 4 6 7 6 4 1 1

1 1 4 6 9 9 6 4 1 1

1 1 4 6 10 10 10 6 4 1 1

1 1 3 6 9 10 10 9 6 3 1 1

1 1 3 5 7 9 10 9 7 5 3 1 1

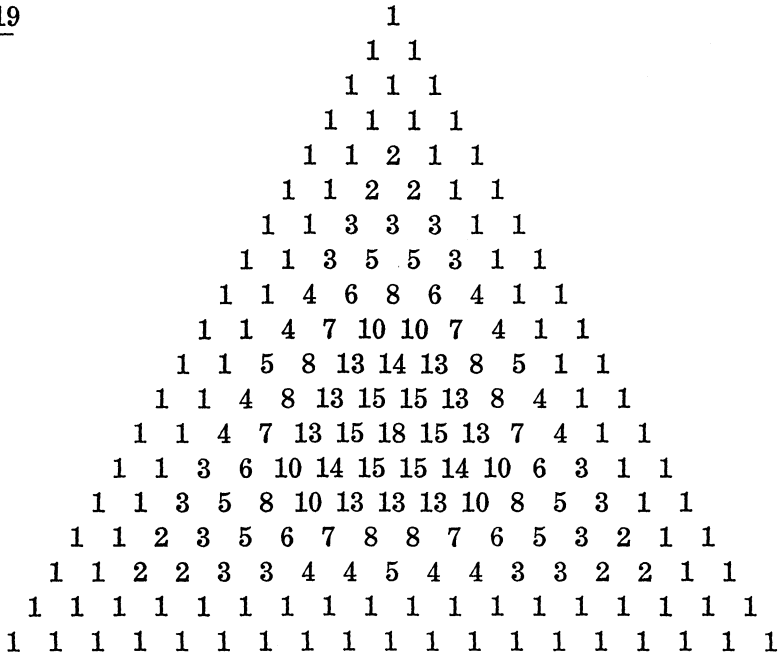
1 1 2 3 5 6 6 6 6 5 3 2 1 1

1 1 2 2 3 3 4 4 4 3 3 2 2 1 1

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

P = 19



The first triangle is in [1] III. 4 (compare also Problem VI. 3.10) and the other ones are computed by using methods explained there. The symmetry of the triangles suggests the following result.

- THEOREM 1.** (1) $e_{n,p-n-\nu-1} = e_{n,\nu}$
 (2) $e_{p-n-\nu-1,\nu} = e_{n,\nu}$
 (3) $e_{n,\nu} = e_{\nu,n}$
 (4) $e_{n,\nu+p} = e_{n,\nu}$.

Proof. The third relation is a consequence of

$$S^\nu V_{n+1} \cong S^n V_{\nu+1}$$

(see [1] III. 2.7b).

The fourth relation follows from (see [1] III. 2.5)

$$S^{\nu+p} V_{n+1} \cong \text{free} \oplus S^\nu V_{n+1} .$$

To prove (1) and (2) we are going to find a formula for $e_{n,\nu}$ or rather for the generating function

$$\eta_n(t) = \sum_{\nu=0}^{\infty} e_{n,\nu} t^\nu .$$

The proof is rather technical and will use the method of Fourier series. For the notation see [1] Ch. V. 4.

The number $a_{n,r}$ of *all* components of $S^r V_{n+1}$ is up to $r = p - 1$ given by the p first coefficients of

$$\tilde{\varphi}_n = \psi_n + \sum_{j=1}^{\infty} (u_{n,2jp} + u_{n,2jp+1})$$

where

$$\begin{aligned} \psi_n(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(\varphi)(1 + \cos \varphi)d\varphi \\ u_{n,j}(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(\varphi) \cot \frac{\varphi}{2} (\sin j\varphi - \sin (j - 1)\varphi)d\varphi \end{aligned}$$

with

$$g_n(\varphi) = \prod_{\nu=0}^n (1 - te^{i(n-2\nu)\varphi})^{-1}.$$

By considering the decomposition of $S^r V_{n+1}$ into the virtual indecomposable $k[G]$ -modules W_i for all $i \geq 0$ (see [1] I. 1.9) we find that the number of *free* components of $S^r V_{n+1}$ (for $r < p$) will be given by the p first coefficients of

$$u_{n,p} + u_{n,p+1} + u_{n,3p} + u_{n,3p+1} + u_{n,5p} + u_{n,5p+1} + \dots$$

Hence

$$\eta_n = \psi_n + \sum_{j+1}^{\infty} (u_{n,2jp} + u_{n,2jp+1}) - \sum_{j=0}^{\infty} (u_{n,(2j+1)p} + u_{n,(2j+1)p+1})$$

will give the number $e_{n,r}$ of *nonfree* components for $r = 0, 1, 2, \dots, p - 1$.

The first part

$$\tilde{\varphi}_n = \psi_n + \sum_{j=1}^{\infty} (u_{n,2jp} + u_{n,2jp+1})$$

is computed in [1] V. 4.7.

The second sum becomes

$$\begin{aligned} &\sum_{j=0}^{\infty} (u_{n,(2j+1)p} + u_{n,(2j+1)p+1}) \\ &= \lim_{m \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(\varphi) \cot \frac{\varphi}{2} \sum_{j=0}^{m-1} [\sin ((2j + 1)p + 1)\varphi \\ &\quad - \sin ((2j + 1)p - 1)\varphi]d\varphi \\ &= \lim_{m \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(\varphi)(1 + \cos \varphi) \frac{\sin 2mp\varphi}{\sin p\varphi}d\varphi. \end{aligned}$$

Using that

$$\tilde{\Phi}_n = \lim_{m \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(\varphi)(1 + \cos \varphi) \frac{\sin(2m + 1)p\varphi}{\sin p\varphi} d\varphi$$

we get

$$\begin{aligned} \tilde{\gamma}_n &= \lim_{m \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(\varphi)(1 + \cos \varphi) \frac{\sin(2m + 1)p\varphi - \sin(2mp\varphi)}{\sin p\varphi} d\varphi \\ &= \lim_{m \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(\varphi)(1 + \cos \varphi) \frac{\cos\left(2m + \frac{1}{2}\right)p\varphi}{\cos \frac{p\varphi}{2}} d\varphi . \end{aligned}$$

We want to rewrite this limit as a sum containing the p th roots of unity. Making a linear substitution we get the Dirichlet kernel in the integrand and then we can use Lemma V. 4.8 in [1].

Put $\varphi = \pi + 2\theta$. Then we get

$$\begin{aligned} \tilde{\gamma}_n &= \lim_{m \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(\pi + 2\theta)(1 - \cos 2\theta) \frac{\sin(4m + 1)p\theta}{\sin p\theta} d\theta \\ &= \frac{1}{p} \sum_{\mu=0}^{p-1} g_n\left(\pi + \frac{2\mu\pi}{p}\right) \left(1 - \cos \frac{2\mu\pi}{p}\right) \\ &= \frac{1}{2p} \sum_{\mu=0}^{p-1} \frac{2 - \frac{e^{i2\mu\pi}}{p} - \frac{e^{-i2\mu\pi}}{p}}{\prod_{\nu=0}^n (1 - te^{i(n-2\nu)(\pi + (2\mu\pi/p))})} . \end{aligned}$$

To get any further we have to treat the cases n even or odd separately.

Case 1. n is even.

Then $e^{i(n-2\nu)\pi} = 1$ and we get with $\alpha = e^{i2\pi/p}$

$$\begin{aligned} \tilde{\gamma}_n &= \frac{1}{2p} \sum_{\mu=0}^{p-1} \frac{2 - \alpha^\mu - \alpha^{-\mu}}{\prod_{\nu=0}^n (1 - t\alpha^{(n-2\nu)\mu})} \\ (*) \quad &= \frac{1}{2p} \sum_{\mu=0}^{p-1} (2 - \alpha^\mu - \alpha^{-\mu}) \sum_{\nu=0}^{\infty} G_{n+\nu, n}(\alpha^\mu, \alpha^{-\mu}) t^\nu \\ &= \frac{1}{p} \sum_{\gamma \in H} (1 - \gamma) \sum_{\nu=0}^{\infty} G_{n+\nu, n}(\gamma, \gamma^{-1}) t^\nu \end{aligned}$$

where H is the group of p th roots of unity. $G_{n,r}$ is the *homogeneous Gaussian polynomial* defined in [1] Ch. II. 4

$$G_{n,r}(X, Y) = \frac{(X^n - Y^n)(X^{n-1} - Y^{n-1}) \dots (X^{n-r+1} - Y^{n-r+1})}{(X^r - Y^r)(X^{r-1} - Y^{r-1}) \dots (X - Y)} .$$

We also used the formula II. 4.3 in [1]

$$\prod_{j=0}^r (1 - X^{n-j} Y^j t)^{-1} = \sum_{\nu=0}^{\infty} G_{r+\nu, r}(X, Y) t^\nu .$$

From the definition of the Gaussian polynomials we get

$$G_{n+\nu+p, n}(\gamma, \gamma^{-1}) = G_{n+\nu, n}(\gamma, \gamma^{-1})$$

and

$$G_{n+\nu, n}(\gamma, \gamma^{-1}) = 0 \quad \text{if} \quad p - n \leq \nu \leq p - 1 .$$

Hence

$$\begin{aligned} \tilde{\eta}_n(t) &= \frac{1}{p(1 - t^p)} \sum_{\tau \in H} (1 - \gamma) \sum_{\nu=0}^{p-n-1} G_{n+\nu, n}(\gamma, \gamma^{-1}) t^\nu \\ &= \frac{1}{1 - t^p} \sum_{\nu=0}^{p-n-1} \left(\frac{1}{p} \sum_{\tau \in H} (1 - \gamma) G_{n+\nu, n}(\gamma, \gamma^{-1}) \right) t^\nu . \end{aligned}$$

It follows that

$$(1 - t^p) \tilde{\eta}_n(t) = \sum_{\nu=0}^{p-n-1} \tilde{e}_{n, \nu} t^\nu$$

where

$$\tilde{e}_{n, \nu} = \frac{1}{p} \sum_{\tau \in H} (1 - \gamma) G_{n+\nu, n}(\gamma, \gamma^{-1}) .$$

Then $\tilde{e}_{n, \nu} = e_{n, \nu}$ for $\nu = 0, 1, \dots, p - 1$. But

$$\tilde{e}_{n, \nu+p} - \tilde{e}_{n, \nu} = 0$$

and hence $\tilde{e}_{n, \nu} = e_{n, \nu}$ for all $\nu \geq 0$ and

$$\tilde{\eta}_n(t) = \eta_n(t) .$$

From (*) we infer that

$$\eta_n(t^{-1}) = -t^{n+1} \eta_n(t)$$

and $\eta_n(t)$ is *symmetric* in the sense of Stanley (see [1] V. 5.1). Using V. 5.6 in [1] we get

$$e_{n, -\nu} = e_{n, \nu-n-1} \quad \text{for} \quad \nu > n .$$

But $e_{n, p-\nu} = e_{n, -\nu} = e_{n, \nu-n-1}$ and replacing ν by $\nu + n + 1$ we get

$$e_{n, p-n-\nu+1} = e_{n, \nu}$$

which proves (1).

From (3) $e_{n, \nu} = e_{\nu, n}$ we get (2) from (1)

$$e_{p-n-\nu-1, \nu} = e_{\nu, p-n-\nu-1} = e_{\nu, n} = e_{n, \nu}$$

and we are done in case n is even.

Case 2. n is odd.

Then $e^{i(n-2\nu)\pi} = -1$ and

$$\tilde{\eta}_n = \frac{1}{2p} \sum_{\mu=0}^{p-1} \frac{2 - \alpha^\mu - \alpha^{-\mu}}{\prod_{\nu=0}^n (1 + t\alpha^{(n-2\nu)\mu})} = (1 + t^p)^{-1} \sum_{\nu=0}^{p-n-1} \tilde{e}_{n,\nu} t^\nu.$$

We get

$$\eta_n(t) = \frac{1 + t^p}{1 - t^p} \tilde{\eta}_n = \frac{1 + t^p}{1 - t^p} \cdot \frac{1}{2p} \sum_{\mu=0}^{p-1} \frac{2 - \alpha^\mu - \alpha^{-\mu}}{\prod_{\nu=0}^n (1 + t\alpha^{(n-2\nu)\mu})}$$

and it follows

$$\eta_n(t^{-1}) = -t^{n+1} \eta_n(t).$$

The proof is then finished as in the even case.

We note that we also have solved problem VI. 3.15 in [1].

THEOREM 2. $S^r V_{n+1}$ has exactly one nonfree component when $n + r = p - 2$. In fact

$$S^{p-n-2} V_{n+1} = \text{free} \oplus \begin{cases} V_{n+1} & \text{if } n \text{ is even} \\ V_{p-n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. $e_{p-n-2,n} = e_{n,1} = 1$.

For the actual computation of the numbers $e_{n,\nu}$ we can get a formula involving the number of restricted partitions. By II. 4.6 in [1] we have

$$G_{n+\nu,n}(\gamma, \gamma^{-1}) = \sum_{m=0}^{\nu n} A(m, \nu, n) \gamma^{\nu n - 2m}$$

where $A(m, \nu, n)$ is the number of partitions of m into at most ν parts all of size $\leq n$.

PROPOSITION 3. We have

$$(-1)^{n\nu} e_{n,\nu} = \sum_{\substack{m=0 \\ 2m \equiv \nu n}}^{\nu n} A(m, \nu, n) - \sum_{\substack{m=0 \\ 2m \equiv \nu n + 1}}^{\nu n} A(m, \nu, n)$$

where the congruences are mod p .

Proof. By the proof of Theorem 1 we get when n is even

$$\begin{aligned} e_{n,\nu} &= \frac{1}{p} \sum_{\gamma \in H} (1 - \gamma) G_{n+\nu,n}(\gamma, \gamma)^{-1} \\ &= \frac{1}{p} \sum_{\gamma \in H} (1 - \gamma) \sum_{m=0}^{\nu n} A(m, \nu, n) \gamma^{\nu n - 2m} \\ &= \frac{1}{p} \sum_{m=0}^{\nu n} A(m, \nu, n) \sum_{\gamma \in H} (\gamma^{\nu n - 2m} - \gamma^{\nu n + 1 - 2m}). \end{aligned}$$

But

$$\sum_{\gamma \in H} \gamma^j = \begin{cases} 0 & \text{if } j \not\equiv 0 \pmod{p} \\ p & \text{if } j \equiv 0 \pmod{p} \end{cases}$$

finishes the proof. The case when n is odd is similar.

EXAMPLE 4. Combining Theorem 1 and Proposition 3 we can write down a purely combinatorial identity equivalent to Theorem 1.

Let n be fixed and define $\nu' = p - n - \nu - 1$. Then $e_{n,\nu} = e_{n,\nu'}$ or

$$\sum_{2m \equiv \nu n} A(m, \nu, n) - \sum_{2m \equiv \nu n + 1} A(m, \nu, n) = \sum_{2m \equiv \nu' n} A(m, \nu', n) - \sum_{2m \equiv \nu' n + 1} A(m, \nu', n)$$

where the sums run over $0 \leq m \leq \nu n$ or $0 \leq m \leq \nu' n$ respectively.

REMARK 5. Since $A^r V_{r+n} \cong S^r V_{n+1}$ we have also computed the number of nonfree components of the exterior powers for which similar symmetry relations are valid.

EXAMPLE 6. Let us show how to compute the central number $e_{6,6} = 18$ in the triangle for $p = 19$ (this is the worst case). By the formula in the proposition

$$e_{6,6} = A(18, 6, 6) - A(9, 6, 6) - A(28, 6, 6) = 58 - 22 - 18 = 18.$$

As a check we also compute the decomposition

$$S^6 V_7 = 2V_1 + 3V_5 + 2V_7 + 4V_9 + V_{11} + 3V_{13} + 2V_{15} + V_{17} + 40V_{19}$$

and we read off $e_{6,6} = 2 + 3 + 2 + 4 + 1 + 3 + 2 + 1 = 18$.

ACKNOWLEDGMENT. I would like to thank Robert Fossum who in spite of my ironic comments insisted in computing the triangles up to $p = 11$. Thus he discovered the nice-looking pattern.

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