ON JOINT NUMERICAL RANGES

John J. Buoni and Bhushan L. Wadhwa
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JOHN J. BUONI AND BHUSHAN L. WADHWA

The joint numerical status of commuting bounded operators \(A_1\) and \(A_2\) on a Hilbert space is defined as \(\{(\phi(A_1), \phi(A_2))\} \) such that \(\phi\) is a state on the \(C^*\)-algebra generated by \(A_1\) and \(A_2\). It is shown that if \(A_1\) and \(A_2\) are commuting normal operators then their joint numerical status equals the closure of their joint numerical range. It is also shown that certain points in the boundary of the joint numerical range are joint approximate reducing eigenvalues.

The joint numerical range of \(A_1\) and \(A_2\) denoted by \(w(A_1, A_2)\) is \(\{((A_1x, x), (A_2x, x)) : x \in H, \|x\| = 1\}\). Thus \(w(A_1, A_2)\) is a bounded subset of \(C^2\). It is not known whether this set is convex, Dash [4, 6]. In this note, we shall show that there is faithful \(*\)-representation of the \(C^*\)-algebra generated by \(A_1\) and \(A_2\), \(C^*(A_1, A_2)\), under which the joint numerical range of \(A_1\) and \(A_2\) is convex. Following Berberian and Orland [1], we study the joint numerical status of \(A_1\) and \(A_2\), \(\Sigma(A_1, A_2) = \{(\phi(A_1), \phi(A_2))\} \) such that \(\phi\) is a state on \(C^*(A_1, A_2)\). If \(A_1\) and \(A_2\) are commuting normal operators then \(\Sigma(A_1, A_2) = \overline{w}(A_1, A_2)\). We also show that certain points in the boundary of \(w(A_1, A_2)\) are joint approximate reducing eigenvalues.

For the sake of notational convenience, all the results are being stated for two commuting operators. However, the results hold for any finite family of commuting operators.

Let \(B(H)\) denote the algebra of all bounded linear operators on the Hilbert space \(H\). Let \(C^*(A_1, A_2)\) denote the \(C^*\)-algebra generated by \(I, A_1,\) and \(A_2\). Let \(\Sigma\) denote the set of all states on \(C^*(A_1, A_2)\). Any state \(\phi\) in \(\Sigma\) induces a representation \(\Pi_\phi\) of \(C^*(A_1, A_2)\) which acts on a Hilbert space \(H_\phi\) and has a canonical cyclic vector \(\xi_\phi\). Also any maximal left ideal of \(C^*(A_1, A_2)\) is of the form \(K(\psi) = \{A \in C^*(A_1, A_2) : \psi(A^*A) = 0\}\) for some pure state \(\psi\) on \(C^*(A_1, A_2)\). For details concerning this the reader is referred to Dixmier [7]. The joint approximate point spectrum of \(A_1\) and \(A_2\), denoted by \(a(A_1, A_2)\), is \(\{(z_1, z_2) : (A_1 - z_1)x_n \to 0\text{ and } (A_2 - z_2)x_n \to 0\}\) where \(x_n \in H, \|x_n\| = 1\) such that \(\|A_1 - z_1\| \to 0\) and \(\|A_2 - z_2\| \to 0\) which is the same as \(\{(z_1, z_2) : B(H)(A_1 - z_1) + B(H)(A_2 - z_2) \neq B(H)\}\).

First we shall show that \(a(A_1, A_2)\) depends only on the \(C^*\)-algebra generated by \(A_1\) and \(A_2\). Our proof is similar to Bunce [2].
PROPOSITION 1.

\[ a(A_1, A_2) = \{(\lambda_1, \lambda_2) : C^*(A_1, A_2)(A_1 - \lambda_1) + C^*(A_1, A_2)(A_2 - \lambda_2) = C^*(A_1, A_2)\} . \]

Proof. If \((\lambda_1, \lambda_2) \in a(A_1, A_2)\) then there exists \(x_n \in H, ||x_n|| = 1\) such that \(||(A_1 - \lambda_1)x_n|| \to 0\) and \(||(A_2 - \lambda_2)x_n|| \to 0\). Suppose there are \(B_1\) and \(B_2\) in \(C^*(A_1, A_2)\) such that \(B_1(A_1 - \lambda_1) + B_2(A_2 - \lambda_2) = I\) then \(||x_n|| \leq ||B_1(A_1 - \lambda_1)x_n|| + ||B_2(A_2 - \lambda_2)x_n||\) and hence \(x_n \to 0\), which is a contradiction. Conversely, if \(C^*(A_1, A_2)(A_1 - \lambda_1) + C^*(A_1, A_2)(A_2 - \lambda_2) \neq C^*(A_1, A_2)\), then \(C^*(A_1, A_2)(A_1 - \lambda_1) + C^*(A_1, A_2)(A_2 - \lambda_2)\) being a proper left ideal of \(C^*(A_1, A_2)\) is contained in \(K(\phi) = \{B \in C^*(A_1, A_2) : \phi(B*B) = 0\}\) where \(\phi\) is a pure state on \(C^*(A_1, A_2)\) [7].

Thus, in particular \(\phi(D^*) = 0\) where \(D = (A_1 - \lambda_1)(A_1 - \lambda_1) + (A_2 - \lambda_2)(A_2 - \lambda_2)\). If \((\lambda_1, \lambda_2) \in a(A_1, A_2)\) then \(0 \notin a(D)\) i.e., there exists \(m > 0\) such that \((Dx, Dx) \geq (m^2x, x)\) for all \(x \in H\), i.e., \(\phi(D^*) \geq m^2I\) which is a contradiction.

The joint numerical status of \(A_1\) and \(A_2\), \(\Sigma(A_1, A_2)\) is defined to be \(\{(\phi(A_1), \phi(A_2))\}\) such that \(\phi\) is a state on \(C^*(A_1, A_2)\). Since \(\Sigma\) is a weak * compact convex subset of the dual space of \(C^*(A_1, A_2)\), it can be easily shown that \(\Sigma(A_1, A_2)\) is a compact, convex subset of \(C^*\). Since each unit vector in \(H\) induces a state on \(C^*(A_1, A_2)\), \(w(A_1, A_2) \subset \Sigma(A_1, A_2)\). Also the extreme points of \(\Sigma(A_1, A_2)\) correspond to the pure states in \(\Sigma\). Let \(\Pi_0 : C^*(A_1, A_2) \to B(K)\), for some Hilbert space \(K\), be the universal representation (Gelfand-Naimark representation) of \(C^*(A_1, A_2)\). We shall show that \(a(A_1, A_2) = a(\Pi_0(A_1), \Pi_0(A_2))\) and also the closure of \(w(\Pi_0(A_1), \Pi_0(A_2)) = \Sigma(A_1, A_2)\).

LEMMA 2. If \((\lambda_1, \lambda_2) \in a(A_1, A_2)\) then there exists a pure state \(\phi\) on \(C^*(A_1, A_2)\) such that \(\Pi_0(\phi(A_i)) = \lambda_i \xi_\phi\) for \(i = 1\) and \(2\), where \(\Pi_0\) is the irreducible representation induced by \(\phi\) with canonical cyclic vector \(\xi_\phi\).

Proof. If \((\lambda_1, \lambda_2) \in a(A_1, A_2)\) then by Proposition 1 there exists a pure state \(\phi\) on \(C^*(A_1, A_2)\) such that \(C^*(A_1, A_2)(A_1 - \lambda_1) + C^*(A_1, A_2)(A_2 - \lambda_2)\) is contained in \(K(\phi) = \{B \in C^*(A_1, A_2) : \phi(B*B) = 0\}\). Thus \(A_1 - \lambda_1I \in K(\phi)\) and so \(A_2 - \lambda_2I \in K(\phi)\). Hence \(\Pi_0(\phi(A_i)) = \lambda_i I\) and hence \(\Pi_0(\phi(A_i)) = \lambda_i \xi_\phi\) for \(i = 1\) and \(2\).

LEMMA 3. If \((\lambda_1, \lambda_2) \notin a(A_1, A_2)\) then \((\lambda_1, \lambda_2) \notin a(\Pi(A_1), \Pi(A_2))\) for any * representation \(\Pi\) of \(C^*(A_1, A_2)\).

Proof. If \((\lambda_1, \lambda_2) \notin a(A_1, A_2)\) then by Proposition 1, there exist \(D_1\) and \(D_2\) in \(C^*(A_1, A_2)\) such that \(D_1(A_1 - \lambda_1) + D_2(A_2 - \lambda_2) = I\). Therefore \(\Pi(D_1)(\Pi(A_1) - \lambda_1) + \Pi(D_2)(\Pi(A_2) - \lambda_2) = I\) and hence \((\lambda_1, \lambda_2) \notin a(\Pi(A_1), \Pi(A_2))\).
**Proposition 4.** \( a(A_1, A_2) = a(\Pi_0(A_1), \Pi_0(A_2)) \).

**Proof.** By Lemma 3, \( a(\Pi_0(A_1), \Pi_0(A_2)) \subset a(A_1, A_2) \). On the other hand, \( \Pi_0 \) is a faithful * representation, by Lemma 2, \( a(A_1, A_2) \subset a(\Pi_0(A_1), \Pi_0(A_2)) \).

**Proposition 5.** The closure of \( w(\Pi_0(A_1), \Pi_0(A_2)) = \Sigma(A_1, A_2) \).

**Proof.** If \( \phi \in \Sigma \), then there exists \( \xi_\phi \in K \) such that \( \Pi_0(A_i)\xi_\phi = \lambda_i \xi_\phi \) for \( i = 1 \) and \( 2 \). Thus \( \Sigma(A_1, A_2) \subset w(\Pi_0(A_1), \Pi_0(A_2)) \). On the other hand, for any \( x \in K, \|x\| = 1 \), if \( \lambda_i = \Pi_0(A_i)x \), we can define \( \phi: C^{*}(A_1, A_2) \rightarrow C \) by \( \phi(A) = (\Pi_0(A)x, x) \). Thus \( \phi \in \Sigma \) and \( \phi(A_i) = \lambda_i \) for \( i = 1 \) and \( 2 \).

**Proposition 6.** If \( A_1 \) and \( A_2 \) are commuting normal operators then \( \Sigma(A_1, A_2) = \overline{w}(A_1, A_2) = \text{convex hull } a(A_1, A_2) \).

**Proof.** If \( A_1 \) and \( A_2 \) are commuting normal operators, \( a(A_1, A_2) = \{(\phi(A_1), \phi(A_2)) \} \) such that \( \phi \) is a character on \( C^{*}(A_1, A_2) \). Let \( (\lambda_i, \lambda_2) \in \Sigma(A_1, A_2) \) be any extreme point of \( \Sigma(A_1, A_2) \), thus there is a pure state \( \phi \in \Sigma \) such that \( \phi(A_i) = \lambda_i \) for \( i = 1 \) and \( 2 \). Also \( C^{*}(A_1, A_2) \) is commutative \( C^{*} \)-algebra and hence \( \phi \) is a character on \( C^{*}(A_1, A_2) \) and thus \( (\lambda_i, \lambda_2) \in a(A_1, A_2) \subset \overline{w}(A_1, A_2) \). Since in this case \( w(A_1, A_2) \) is convex (Dash [6]), the result is proved.

Juneja [8] showed that if \( (z_1, z_2) \) is an extreme point of \( w(A_1, A_2) \) where \( A_1 \) and \( A_2 \) are commuting normal operators then \( (z_1, z_2) \) is a reducing approximate eigenvalue. We generalize this result to the case of arbitrary commuting operators.

**Lemma 7.** Let \( S_1 \) and \( S_2 \) be convex bounded subsets of \( C \). Let \( (z_1, z_2) \in (\partial S_1) \times (\partial S_2) \) where \( \partial S_i \) denotes the boundary of \( S_i \). Then there exist \( \lambda_i \) in the complement of the closure of \( S_i, i = 1 \) and \( 2 \), such that \( |z_i - \lambda_i| = \text{dist.}(\lambda_i, S_i) \) and \( |(z_1, z_2) - (\lambda_1, \lambda_2)| = \text{dist.}(\lambda_i, S_i) \times S_2) \).

**Lemma 8.** Let \( A \in B(H) \). If there exists a sequence of unit vectors \( x_n \) in \( H \) such that \( \|[(A - \lambda)^{-1} - (\mu - \lambda)^{-1}]x_n\| \rightarrow 0 \) and \( \|[(A^* - \lambda)^{-1} - (\mu^* - \lambda)^{-1}]x_n\| \rightarrow 0 \) then \( \|\lambda(A - \mu)x_n\| \rightarrow 0 \) and \( \|\lambda(A^* - \mu^*)x_n\| \rightarrow 0 \).

**Proposition 9.** If \( (z_1, z_2) \in [\partial w(A_1) \times \partial w(A_2)] \cap \overline{w}(A_1, A_2) \cap \sigma(A_1, A_2) \), then there exists a sequence \( x_n \) of unit vectors in \( H \) such that \( \|\lambda_i(A_i - z_i)x_n\| \rightarrow 0 \) and \( \|\lambda_i(A_i^* - \bar{z}_i)x_n\| \rightarrow 0 \) for \( i = 1 \) and \( 2 \).

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PROPOSITION 4. \( a(A_1, A_2) = a(\Pi_0(A_1), \Pi_0(A_2)) \).

**Proof.** By Lemma 3, \( a(\Pi_0(A_1), \Pi_0(A_2)) \subset a(A_1, A_2) \). On the other hand, \( \Pi_0 \) is a faithful * representation, by Lemma 2, \( a(A_1, A_2) \subset a(\Pi_0(A_1), \Pi_0(A_2)) \).

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**Proof.** If \( \phi \in \Sigma \), then there exists \( \xi_\phi \in K \) such that \( \Pi_0(A_i)\xi_\phi = \lambda_i \xi_\phi \) for \( i = 1 \) and \( 2 \). Thus \( \Sigma(A_1, A_2) \subset w(\Pi_0(A_1), \Pi_0(A_2)) \). On the other hand, for any \( x \in K, \|x\| = 1 \), if \( \lambda_i = \Pi_0(A_i)x \), we can define \( \phi: C^*(A_1, A_2) \rightarrow C \) by \( \phi(A) = (\Pi_0(A)x, x) \). Thus \( \phi \in \Sigma \) and \( \phi(A_i) = \lambda_i \) for \( i = 1 \) and \( 2 \).

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**Proof.** If \( A_1 \) and \( A_2 \) are commuting normal operators, \( a(A_1, A_2) = \{(\phi(A_1), \phi(A_2)) \} \) such that \( \phi \) is a character on \( C^*(A_1, A_2) \). Let \( (\lambda_i, \lambda_2) \in \Sigma(A_1, A_2) \) be any extreme point of \( \Sigma(A_1, A_2) \), thus there is a pure state \( \phi \in \Sigma \) such that \( \phi(A_i) = \lambda_i \) for \( i = 1 \) and \( 2 \). Also \( C^*(A_1, A_2) \) is commutative \( C^* \)-algebra and hence \( \phi \) is a character on \( C^*(A_1, A_2) \) and thus \( (\lambda_i, \lambda_2) \in a(A_1, A_2) \subset \overline{w}(A_1, A_2) \). Since in this case \( w(A_1, A_2) \) is convex (Dash [6]), the result is proved.

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**Proposition 9.** If \( (z_1, z_2) \in [\partial w(A_1) \times \partial w(A_2)] \cap \overline{w}(A_1, A_2) \cap \sigma(A_1, A_2) \), then there exists a sequence \( x_n \) of unit vectors in \( H \) such that \( \|\lambda_i(A_i - z_i)x_n\| \rightarrow 0 \) and \( \|\lambda_i(A_i^* - \bar{z}_i)x_n\| \rightarrow 0 \) for \( i = 1 \) and \( 2 \).
Proof. Since $\sigma(A_1, A_2) \subseteq w(A_1) \times w(A_2)$, by Lemma 7, there exists $\lambda = (\lambda_1, \lambda_2) \in w(A_1) \times w(A_2)$ such that $|\lambda_i - z_i| = \text{dist}(\lambda_i, w(A_i)) = \text{dist}(\lambda, w(A_1) \times w(A_2)) = \text{dist}(\lambda, w(A_1, A_2)) = \text{dist}(\lambda, \sigma(A_1, A_2))$, where $z = (z_1, z_2)$. Thus $\|{(A_i - \lambda_i)^{-1}}\| = 1/|\lambda_i - z_i|$ and $1/(z_i - \lambda_i) \in \sigma(A_i - \lambda_i)$ for $i = 1$ and $2$. Now using Proposition 3.3 of Dash [5], there exists a sequence $x_n$ of unit vectors in $H$ such that $\|{(A_i - \lambda_i)^{-1}} - (z_i - \lambda_i)^{-1}\| \to 0$ and $\|{(A_i^* - \lambda_i)^{-1}} - (z_i - \lambda_i)^{-1}\| \to 0$. The result follows from Lemma 8.

**COROLLARY 10.** If $(z_1, z_2)$ is an extreme point of $w(A_1, A_2)$ where $A_1$ and $A_2$ are commuting normal operators then there exists a sequence of unit vectors $x_n$ in $H$ such that $\|{(A_i - z_i)}x_n\| \to 0$ and $\|{(A_i^* - z_i)}x_n\| \to 0$.

Proof. Since $(z_1, z_2)$ is an extreme point of $w(A_1, A_2)$, $A_1$ and $A_2$ are commuting normal operators, as in the proof of Proposition 6, there exists a character $\phi$ on $C^*(A_1, A_2)$ such that $\phi(A_i) = z_i$ for $i = 1$ and $2$. Also $\phi$ restricted $C^*(A_i)$ is also a character and hence $\phi(A_i)$ is an extreme point of $w(A_i)$. Similarly $\phi(A_2)$ is an extreme point of $w(A_2)$. Thus $\langle \lambda_1, \lambda_2 \rangle \in (\partial w(A_1)] \times \partial w(A_2)) \cap \overline{w(A_1, A_2)} \cap \sigma(A_1, A_2)$.

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YOUNGSTOWN STATE UNIVERSITY
YOUNGSTOWN, OH 44555
AND
CLEVELAND STATE UNIVERSITY
CLEVELAND, OH 44115
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