Pacific Journal of Mathematics

A VANISHING THEOREM FOR THE mod p MASSEY-PETERSON SPECTRAL SEQUENCE

MASAMITSU MORI

Vol. 77, No. 2 February 1978

PACIFIC JOURNAL OF MATHEMATICS
Vol. 77, No. 2, 1978

A VANISHING THEOREM FOR THE MOD p MASSEY-PETERSON SPECTRAL SEQUENCE

MASAMITSU MORI

A vanishing theorem and periodicity theorem for the classical mod 2 Adams spectral sequence were originally proved by Adams [1]. The results were extended to the unstable range by Bousfield [2]. The purpose of this paper is to show the analogue of Bousfield's work for the mod p unstable Adams spectral sequence of Massey-Peterson type (called the mod p Massey-Peterson spectral sequence), where p is an odd prime. The results generalized those obtained by Liulevicius [5], [6] to the unstable range. As an immediate topological application we have the estimation of the upper bounds of the orders of elements in the p-primary component of the homotopy groups of, for example, an odd dimensional sphere, Stiefel manifold, or H-space.

1. The vanishing theorem. Let A denote the mod p Steenrod algebra. Let $A\mathscr{M}$ the category of unstable left A-modules and $\mathscr{M}A$ thecategory of unstable right A-modules. We may define $\mathrm{Ext}^s_{A\mathscr{M}}$, $s \geqq 0$, as the sth right derived functor of $\mathrm{Hom}_{A\mathscr{M}}$, and similarly define $\mathrm{Ext}^s_{\mathscr{M}A}$, since $A\mathscr{M}$ and $\mathscr{M}A$ are abelian categoriesw ith enough projectives. Note that, if $M \in A\mathscr{M}$ is of finite type, then

$$\mathrm{Ext}_{A\mathscr{M}}(M, Z_p) = \mathrm{Ext}_{\mathscr{M}A}(Z_p, M^*) .$$

Recall the mod p Massey-Peterson spectral sequence (see, for example, [4]). Let X be a simply connected space with $\pi_*(X)$ of finite type. Suppose that $H^*(X; Z_p) \cong U(M)$, $M \in A\mathscr{M}$, where $U(M)$ is the free unstable A-algebra generated by M. Then there is a spectral sequence $\{E_r(X)\}$ with

$$d_r \colon E_r^{s,t}(X) \longrightarrow E_r^{s+r,t+r-1}(X) ,$$

such that

$$E_2^{s,t}(X) \cong \mathrm{Ext}^{s,t}_{A\mathscr{M}}(M, Z_p) ,$$

and

$$E_\infty(X) \cong \mathrm{Gr}\, \pi_*(X)/(\text{torsion prime to } p) .$$

Let Λ be the bigraded differential algebra over Z_p introduced by Bousfield et al [3], which has multiplicative generators λ_i of bidegree $(1, 2i(p-1)-1)$ for $i>0$ and μ_i of bidegree $(1, 2i(p-1))$ for $i \geqq 0$.

For $N \in \mathscr{M}A$, let $V(N)^s$ be the subspace of $N \otimes \Lambda^s$ generated by all $x \otimes \nu_I$ with $\nu_I = \nu_{i_1} \cdots \nu_{i_s}$ allowable and $\deg x \geqq 2i$ if $\nu_{i_1} = \lambda_{i_1}$ and $\deg x \geqq 2i_1 + 1$ if $\nu_{i_1} = \mu_{i_1}$. Then $V(N)$ is the cochain complex with

$$\delta(x \otimes \nu_I) = (-1)^{\deg x} \sum_{i>0} x\rho^i \otimes \lambda_i \nu_I + \sum_{i \geqq 0} x\beta\rho^i \otimes \mu_i \nu_I + (-1)^{\deg x} x \otimes \partial \nu_I .$$

Here $x \otimes \nu_I$ is of bidegree (s, t) with $t = s + \deg x + \deg \nu_I$. Recall that for $N \in \mathscr{M}A$

$$\mathrm{Ext}^{s,t}_{\mathscr{M}A}(Z_p, N) \cong H^s(V(N))_{t-s} .$$

Let $O(N)$ be the subcomplex of $V(N)$ generated by all $x \otimes \nu_I \in V(N)^s$ with $\nu_I = \nu_{i_1} \cdots \nu_{i_s}$ allowable and $\nu_{i_s} = \lambda_{i_s}$. Let $T(N)$ be the quotient complex of $V(N)$ such that

$$T(N)^s = \begin{cases} N \otimes \mu_0^s & \text{for } s = 0, 1 , \\ N \otimes \mu_0^s + \sum_{t>0} N_{2t}^s \otimes \lambda_t \mu_0^{s-1} & \text{for } s \geqq 2 . \end{cases}$$

Then we have a long exact sequence

$$\cdots \longrightarrow H^{s-1}(T(N)) \xrightarrow{\delta} H^s(O(N)) \xrightarrow{j^*} H^s(V(N)) \xrightarrow{q^*} H^s(T(N)) \longrightarrow \cdots ,$$

which is induced from the natural isomorphism

$$H^*(O(N)) \cong H^*(\mathrm{Ker}\, q) ,$$

where $j: O(N) \to V(N)$ and $q: V(N) \to T(N)$ are the natural maps. Remark that $H^*(T(N))$ consists of towers in the sense that

$$H^s(T(N)) \cong H^{s+1}(T(N)) ,$$

for $s \geqq 2$, and thus $H^*(T(N))$ is easily determined.

DEFINITION. A function $\varphi_n(k)$, $n \geqq 2$, $k \geqq 0$, is defined as follows. If $n = 2, 3, 4$,

$$\varphi_n(k) = \begin{cases} [(k+2)/2(p-1)] & \text{for } k \geqq 2(p-1)-1 \\ 0 & \text{for } k < 2(p-1)-1 \end{cases}$$

where $[x]$ is the integer part of x, and if $n \geqq 5$,

$$\varphi(k) = \varphi_n(k) = i ,$$

where

$$\begin{aligned}
2i(p-1) &\leqq k < 2(i+1)(p-1)-1 &\quad \text{if}\quad i \not\equiv -1, 0 \bmod p\,,\\
2i(p-1) &\leqq k < 2(i+1)(p-1)-2 &\quad \text{if}\quad i \equiv -1 \bmod p\,,\\
2i(p-1)-2 &\leqq k < 2(i+1)(p-1)-1 &\quad \text{if}\quad i \equiv 0 \bmod p\,.
\end{aligned}$$

Now we state our main theorem.

THEOREM 1 (*Vanishing*). *Let $N \in \mathscr{M}A$ with $N_i = 0$ for $i < n$, where $n \geqq 2$. Then*

$$\mathrm{Ext}^{s,s+k+n}_{\mathscr{M}A}(Z_p, N) \cong H^s(V(N))_{k+n} \xrightarrow{q^*} H^s(T(N))\,,$$

is an isomorphism for $s > \varphi_n(k)$.

This will be proved in §4.

By virtue of our vanishing theorem the calculation of $H^*(V(N))$ is reduced to that of $H^*(O(N))$ in a large extent. Note that q^* is epimorphic when $U(M)$ is generated by a single element, where $M=N.^*$

As an immediate topological corollary we have.

COROLLARY 2. *Let X be a simply connected space with $\pi_*(X)$ of finite type. Suppose that $H^*(X; Z_p) \cong U(M)$, where M is an unstable A-module. If $M^i = 0$ for $i < n$, then the orders of elements in the p-primary component of $\pi_{k+n}(X)$ are at most $p^{\varphi_n(k)}$.*

This may be applied, for example, when X is an odd dimensional sphere, Stiefel manifold, or H-space.

REMARK. If $N_i = 0$, $i > m$, for some m, then $H^s(N)_t$ is zero for dimensional reason when t is large with respect to s. Hence in this case Corollary 2 is slightly improved.

2. Periodicity theorems. For a module $M \in A\mathscr{M}$ we define the β-cohomology by $H_\beta(M) = \mathrm{Ker}\,\beta/\mathrm{Im}\,\beta$.

DEFINITION. A module $M \in A\mathscr{M}$ is called β-trivial if

$$\rho^i \colon M^{2i} \longrightarrow H^{2ip}_\beta(M)\,,$$

is an isomorphism for all i and $H^k_\beta(M) = 0$ for $k \not\equiv 0 \bmod 2p$.

Remark that $M \in A\mathscr{M}$ is β-trivial if and only if $N = M^* \in \mathscr{M}A$ is towerless, i.e., $H^s(T(N)) = 0$ for $s > 0$.

Let $\mathscr{G}$ denote the category of graded Z_p-modules. Let L_sF denote the sth left derived functor of a functor $F\colon A\mathscr{M} \to \mathscr{G}$.

THEOREM 3 (*Periodicity*). *Let $F\colon A\mathscr{M} \to \mathscr{G}$ be a functor such that $F(M)=M/\beta M + \rho^1 M$. If $M \in A\mathscr{M}$ is β-trivial, then there is a natural map*

$$P\colon L_s F(M)^t \longrightarrow L_{s+p} F(M)^{t+2p(p-1)+p},$$

such that P is an isomorphism for $s \geqq 2$ and a monomorphism for $s=1$.

This will be proved in §3.

Additionally, we give here such a kind of periodicity theorems.

THEOREM 4. *Let $G_i(M) = M/\rho^i M$ for $M \in A\mathscr{M}$, where $0<i<p$. Then there is a natural map*

$$Q\colon L_s G_i(M)^t \longrightarrow L_{s+2} G_i(M)^{t+2p(p-1)},$$

such that Q is an isomorphism for $s \geqq 2$ and a monomorphism for $s=1$.

THEOREM 5. *Let $G_i(M) = M/\beta M + \rho^i M$ for $M \in A\mathscr{M}$, where $0< i<p$. If $M \in A\mathscr{M}$ is β-trivial, then there is a natural map*

$$Q\colon L_s G_i(M)^t \longrightarrow L_{s+2} G_i(M)^{t+2p(p-1)},$$

such that Q is an isomorphism for $s \geqq 2$ and a monomorphism for $s=1$.

THEOREM 6. *Let $G(M)=M/\rho^1+\beta\rho^1 M$ for $M \in A\mathscr{M}$. If $M \in A\mathscr{M}$ is β-trivial, then there is a natural map*

$$R\colon L_s G(M)^t \longrightarrow L_{s+2} G(M)^{t+4p-2},$$

such that R is an isomorphism for $s \geqq 2$ and a monomorphism for $s=1$.

3. **Proofs of periodicity theorems.** Suppose given a circular sequence of functors from $A\mathscr{M}$ to $\mathscr{G}$ and natural transformations,

(♯)
$$\begin{tikzcd}
A_{k-2} \arrow[r, "R_{k-2}"] & A_{k-3} \arrow[r] & \cdots \arrow[r] & A_1 \arrow[r, "R_1"] & A_0=A_k \arrow[dll, "R_0=R_k"] \\
 & & A_{k-1} \arrow[ull, "R_{k-1}"] & &
\end{tikzcd}$$

satisfying $R_i R_{i+1} = 0$ for $i=0, \cdots, k-1$. Define functors $\operatorname{Ker} R_i$, $\operatorname{Im} R_i$, $\operatorname{Coker} R_i$, $H_i = \operatorname{Ker} R_i/\operatorname{Im} R_{i+1}$ in a usual way.

DEFINITION. A module $M \in A\mathscr{M}$ is called trivial for the diagram (♯) if

$$L_sA_i(M) = L_sH_i(M) = 0 ,$$

for all $s > 0$ and $i = 0, \cdots, k-1$.

LEMMA. *If $M \in A\mathscr{M}$ is trivial for the diagram (#), then there is a natural map*

$$P: L_s \operatorname{Coker} R_0(M)^t \longrightarrow L_{s+k} \operatorname{Coker} R_0(M)^{t+h} ,$$

such that P is an isomorphism for $s \geqq 2$ and a monomorphism for $s = 1$. Here $h = \sum_{i=0}^{k-1} h_i$, $h_i = \deg R_i$.

Proof. Let $h(a) = \sum_{i=a}^{k-1} h_i$. Since M is trivial for (#), we have the following natural isomorphism

$$\begin{aligned} L_{s+k} \operatorname{Coker} R_0(M)^{t+h} &\cong L_{s+k-1} \operatorname{Im} R_0(M)^{t+h} \\ &\cong L_{s+k-2} \operatorname{Ker} R_0(M)^{t+h(1)} \cong \cdots \\ &\cong L_s \operatorname{Ker} R_{k-2}(M)^{t+h(k-1)} \cong L_s \operatorname{Im} R_{k-1}(M)^{t+h(k-1)} . \end{aligned}$$

On the other hand the natural map

$$L_s \operatorname{Coker} R_0(M)^t \longrightarrow L_s \operatorname{Im} R_{k-1}(M)^{t+h(k-1)} ,$$

is an isomorphism for $s \geqq 2$ and a monomorphism for $s = 1$.

We shall use the following circular sequence due to Toda [9] (see, also, Oka [8]) to prove the periodicity theorems.

$$\text{(3.1)} \qquad \begin{tikzcd} M \arrow[r, "R_{p-2}"] & M \arrow[r] & \cdots\cdots \arrow[r] & M \arrow[r, "R_1"] & M \arrow[dl, "R"] \\ & & M/\beta M + M/\beta M \arrow[ulll, "R'"] & & \end{tikzcd}$$

where $R_i = (i+1)\beta\rho^1 - i\rho^1\beta$, $R = (\rho^1\beta, \rho^1)$ and $R' = \rho^1\beta - \beta\rho^1\beta$.

$$\text{(3.2)} \qquad M \underset{\rho^{p-i}}{\overset{\rho^i}{\rightleftarrows}} M \qquad \text{for} \quad 0 < i < p ,$$

$$\text{(3.3)} \qquad M/R_iM \underset{\rho^{p-i}}{\overset{\rho^i}{\rightleftarrows}} M/\beta M \qquad \text{for} \quad 0 < i < p ,$$

$$\text{(3.4)} \qquad M/\rho^1M \underset{\rho^1}{\overset{\rho^1}{\rightleftarrows}} M/\rho^1M .$$

Here $M \in A\mathscr{M}$ and the maps are induced from the left actions.

Proof of Theorem 3. We shall use the diagram (3.1). For convenience, we put $R_0 = R_p = R$, $R_{p-1} = R'$. Let $H_i(M)$ denote the cohomology $\operatorname{Ker} R_i/\operatorname{Im} R_{i+1}$. If M is a free unstable A-module, then:

(i) $\rho^s: (M/\beta M)^{2s} \cong H_0^{2sp}(M)$ if $s \equiv -1 \bmod p$,
$\rho^s: M \cong H_0^{2sp}(M)$ if $s \not\equiv -1 \bmod p$,
$\rho^s: (\beta M)^{2s+1} \cong H_0^{2tp+1}(M)$ if $s \not\equiv -1 \bmod p$,

(ii) for $i = 1, \cdots, p-2$,

$$\rho^s\colon (M/\beta M)^{2s} \cong H_i^{2sp}(M)\,,$$
$$\rho^s + \beta\rho^s\colon (\beta M)^{2s+2} + (M/\beta M)^{2s+1} \cong H_i^{2sp+2}(M)\,,$$
$$\rho^s\colon (M/\beta M)^{2s+2} \cong H_i^{2sp+3}(M)$$

(iii) $\rho^s + 0\colon (M/\beta M)^{2s} + 0 \cong H_{p-1}^{2sp}(M)\,,$

$$0 + \rho^s\colon 0 + (M/\beta M)^{2s+j} \cong H_{p-1}^{2sp+j+1}(M) \quad \text{for } j = 0, 1 \text{ if } s \equiv 0 \bmod p\,,$$
$$\rho^s + \rho^s\colon (\beta M)^{2s+j+1} + (M/\beta M)^{2s+j} \cong H_{p-1}^{2sp+j+1}(M) \quad \text{for } j = 0, 1 \text{ if } s \not\equiv 0 \bmod p\,,$$
$$0 + \rho^s\colon 0 + (M/\beta M)^{2s+2} \cong H_{p-1}^{2sp+3}(M)\,.$$

(iv) otherwise $H_i^k(M) = 0$.

This unstable version of Toda's exactness theorem is shown by long but straightforward computations. Now Theorem 3 is proved by applying lemma.

By using the diagrams (3.2), (3.3) and (3.4), Theorems 4, 5 and 6 follow in a similar way, and thus we only state the following facts.

Let $M \in A\mathscr{M}$ be a free unstable module. Fix i such that $0 < i < p$.

If $H(M) = \mathrm{Ker}\,(\rho^i\colon M \to M)/\mathrm{Im}\,(\rho^{p-i}\colon M \to M)$, then:

(i) $\rho^s\colon M^{2s+j} \cong H^{2sp+j}(M)$ for $j = 0, 1$,

$\beta\rho^s + \rho^s\colon M^{2s+j-1} + M^{2s+j} \cong H^{2sp+j}(M)$ for $j = 2, \cdots, 2i-1$,

$\beta\rho^s\colon M^{2s+j-1} \cong H^{2sp+j}(M)$ for $j = 2i, 2i+1$.

(ii) otherwise $H^k(M) = 0$.

If $H(M) = \mathrm{Ker}\,(\rho^{p-i}\colon M/\beta M \to M/R_iM)/\mathrm{Im}\,(\rho^i\colon M/R_iM \to M/\beta M)$, then:

(i) $\rho^s\colon (M/\beta M)^{2s+j} \cong H^{2sp+j}(M)$

for $j = 0, 1, \cdots, i-1$ and $s \equiv 0, 1, \cdots, i-1 \bmod p$,

(ii) otherwise $H^k(M) = 0$.

Next put $H(M) = \mathrm{Ker}\,(\rho^i\colon M/R_iM \to M/\beta M)/\mathrm{Im}\,(\rho^{p-i}\colon M/\beta M \to M/R_iM)$, then:

(i) if $s \equiv 0 \bmod p$,

$$\rho^s\colon (M/\beta M)^{2s+j} \cong H^{2sp+j}(M) \quad \text{for } j = 0, 1\,,$$
$$\beta\rho^s + \rho^s\colon (M/\beta M)^{2s+j-1} + (M/\beta M)^{2s+j} \cong H^{2sp+j}(M) \quad \text{for } j = 2, \cdots, 2i-1\,,$$
$$\beta\rho^s\colon (M/\beta M)^{2s+j-1} \cong H^{2sp+j}(M) \quad \text{for } j = 2i, 2i+1\,,$$

(ii) if $s \equiv 1, \cdots, p-i-1 \bmod p$,

$$\rho^s\colon (M/\beta M)^{2s+j} \cong H^{2sp+j}(M) \qquad \text{for} \quad j = 0, 1 ,$$
$$\beta\rho^s + \rho^s\colon M^{2s+j-1} + (M/\beta M)^{2s+j} \cong H^{2p+i}(M) \quad \text{for} \quad j = 2, \cdots, 2i - 1 ,$$
$$\beta\rho^s\colon M^{2s+j-1} \cong H^{2sp+j}(M) \qquad \text{for} \quad j = 2i, 2i + 1 ,$$

(iii) if $s \equiv p - i \bmod p$,

$$\beta\rho^s\colon M^{2s+j-1} \cong H^{2sp+j}(M) \qquad \text{for} \quad j = 2, \cdots, 2i - 1 ,$$

(iv) otherwise $H^k(M) = 0$.

Finally, if $H(M) = \operatorname{Ker}(\beta\rho^1\colon M/\rho^1 M \to M/\rho^1 M)/\operatorname{Im}(\beta\rho^1\colon M/\rho^1 M \to M/\rho^1 M)$, then:

(i) $$\rho^{ps}\colon M^{2s+j} \cong H^{2sp^2+j}(M) \qquad \text{for} \quad j = 0, 1 ,$$
$$\beta\rho^{ps} + \rho^{ps}\colon (M/\beta M)^{2s+1} + M^{2s+2} \cong H^{2sp^2+2}(M) ,$$

(ii) otherwise $H^k(M) = 0$.

4. **Proof of the vanishing theorem.** Let $F(n)$ denote a free unstable A-module on one generator ι_n. We define an unstable A-module $N(n)$ to be the quotient of $F(n)$ by the relation $\beta\iota_n = 0$. Next define $M(n)$ to be the subcomplex of $N(n)$ by ommitting the ι_n from $N(n)$ if n odd and ommitting the $\iota_n, (\iota_n)^p, \cdots, (\iota_n)^{p^t}, \cdots$ from $N(n)$ if n even. Note that $M(n)$ is β-trivial.

First we suppose that n is odd. Then by the long exact sequence induced from a short exact sequence

$$0 \longrightarrow M(n) \longrightarrow N(n) \longrightarrow Z_p \longrightarrow 0 ,$$

we have an isomorphism

$$E_2^{s,t+n}(S^n) = \operatorname{Ext}_{A\mathscr{M}}^{s,t+n}(Z_p, Z_p) \cong \operatorname{Ext}_{A\mathscr{M}}^{s-1,t+n}(M(n), Z_p) ,$$

for $t \neq s$. Let $C(n)$ be a minimal resolution of $M(n)$. By virtue of Theorem 3 we can prove the vanishing theorem for Z_p by analysing $C(n)$. Namely, $\operatorname{Ext}_{A\mathscr{M}}^{s-1,t+n}(M(n), Z_p)(t \neq s)$ vanishes for $s > \varphi_n(t-s)$. Furthermore we can observe the periodicity phenomenon in a range near the vanishing line. In fact, by Theorems 3, 4 and 5 we have two periodicity operators P and Q of bidegree $(p, 2p(p-1)+p)$ and $(2, 2p(p-1))$, respectively.

For lower dimensional sphere we shall give periodic families. Let $1 < m \leqq p+1$. In $E_2^{s,t+2m-1}(S^{2m-1})$ there appear nontrivial elements when $(s, t-s)$ is as follows:

(i) $(1, q-1)$

$(1, pq-1)$ for $m = p+1$,

(ii) $(s, sq-1), (s, (m+s-2)q-2)$

for $s = 2, \cdots, p-m+1$ and $m \neq p, p+1$,

$$(s, sq-1), (s, pq-2), (s, pq-1)$$
$$\text{for} \quad s = p-m+2 \quad \text{and} \quad m \neq p+1,$$
$$(s, sq-1), (s, pq-2), (s, pq-1), (s, (m+s-2)q-2)$$
$$\text{for } s = p-m+3, \cdots, p-1 \quad \text{and } p \neq 3,$$
$$(p, pq-2), (p, pq-1), (p, (p+m-2)q-2),$$
$$(p+1, (p+1)q-1), (p+1, (p+m-1)q-2),$$

where $q = 2(p-1)$. Applying the periodicity operators P and Q repeatedly, we can determine the behavior of all $E_2(S^{2m-1})$ near the vanishing line. (Possibly other elements appear in a range apart from the vanishing line when we apply the iteration of the operator Q.)

We next suppose that n is even. Let $L(n; t)(0 < t \leqq \infty)$ be an unstable A-module with elements $\sigma_n, (\sigma_n)^p, \cdots, (\sigma_n)^{p^t}$ where $\deg \sigma_n = n$. By the long exact sequence induced from short exact sequences

$$0 \longrightarrow M(n) + L(p^{t+1}; \infty) \longrightarrow N(n) \longrightarrow L(n; t) \longrightarrow 0,$$
$$0 \longrightarrow M(p^{t+1}n) \longrightarrow N(p^{t+1}n) \longrightarrow L(p^{t+1}n; \infty) \longrightarrow 0,$$

we have an isomorphism

$$\begin{aligned}&\mathrm{Ext}_{A\mathscr{M}}^{s,t+n}(L(n; t), Z_p)\\ &\quad \cong \mathrm{Ext}_{A\mathscr{M}}^{s-1,t+n}(M(n), Z_p) + \mathrm{Ext}_{A\mathscr{M}}^{s-2,t+n}(M(p^{t+1}n), Z_p),\end{aligned}$$

for $t \neq s, s + (p^{t+1}-1)n - 1$. Thus in a similar way we have the required results for $L(n; t)$.

Now we have shown that

$$q^*: H^s(V(N))_{k+n} \longrightarrow H^s(T(N)_{k+n},$$

is an isomorphism for $s > \varphi_n(k)$, when $N^* = H^*(S^n; Z_p) = Z_p(n \text{ odd})$ and $N^* = L(n; t)(n \text{ even})$. The general case follows inductively using the five lemma.

References

[1] J. F. Adams, *Stable homotopy theory, third edition*, Lecture Notes in Math. No. 3, Springer-Verlag, 1969.

[2] A. K. Bousfield, *A vanishing theorem for the unstable Adams spectral sequence*, Topology, **9** (1970), 337–344.

[3] A. K. Bousfield, E. B. Curtis, D. M. Kan, D. G. Quillen, D. L. Rector and J. W. Schlesinger, *The* mod p *lower central series and the Adams spectral sequence*, Topology, **5** (1966), 331–342.

[4] A. K. Bousfield and D. M. Kan, *The homotopy spectral sequence of a space with coefficients in a ring*, Topology, **11** (1972), 79–106.

[5] A. Liulevicius, *Zeros of the cohomology of the Steenrod algebra*, Proc. Amer. Math. Soc., **14** (1963), 972–976.

[6] ———, *The cohomology of a subalgebra the Steenrod algebra*, Trans. Amer. Math. Soc., **104** (1962), 443–449.

[7] W. S. Massey and F. P. Peterson, *The* mod 2 *cohomology structure of certain fiber spaces*, Mem. Amer. Math. Soc., **74** (1967).
[8] S. Oka, *Some exact sequences of modules over the Steenrod algebra*, Hiroshima Math. J., **1** (1971), 109-121.
[9] H. Toda, *p-primary components of homotopy groups I*, Mem. Coll. Sci. Univ. Kyoto, **31** (1958), 129-142.

Received January 30, 1977.

UNIVERSITY OF THE RYUKYUS
NAHA, OKINAWA, JAPAN

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Older back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 77, No. 2 February, 1978