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LOCALLY UNIVALENT FUNCTIONS AND COEFFICIENT DISTORTIONS

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LOCALLY UNIVALENT FUNCTIONS AND COEFFICIENT DISTORTIONS

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 AND D. N. TELAGE

We look at functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfying $\sum_{n=2}^{\infty} n |a_n| > 1$ and determine conditions for which the arguments of the coefficients may vary without affecting the univalence of the function. A bound on the radius of starlikeness for the convolution of functions taken from the closed convex hull of convex functions and a special subclass of starlike functions is also obtained.

1. **Introduction.** Let LS denote the class of functions of the form $z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic and locally univalent in the unit disk U , and let S denote the subclass of univalent functions. It is well known that a sufficient condition for $z + \sum_{n=2}^{\infty} a_n z^n$ to be in S is that

$$(1) \quad \sum_{n=2}^{\infty} n |a_n| \leq 1.$$

For functions of the form

$$(2) \quad z - \sum_{n=2}^{\infty} |a_n| z^n$$

the condition (1) is also necessary. This follows because functions that fail to satisfy (1) are not even in LS . The necessary and sufficient condition (1) for functions of the form (2) to be in S makes extremal problems much more manageable. Very little is known for functions in S of the form

$$(3) \quad f_{\lambda}(z) = z - e^{i\lambda} \sum_{n=2}^{\infty} |a_n| z^n,$$

where the coefficients are not necessarily real but have constant argument. In [6] it is asked if a function $g(\lambda, n)$ can be found for which the inequality $|a_n| \leq g(\lambda, n)$ is sharp. Note that $g(0, n) = 1/n$ and $g(\pi, n) = n$, with extremal functions $z - z^n/n$ and $z/(1 - z)^2$ respectively.

In this paper we show that a function in LS must satisfy (1) when its coefficients are "close to" negative. Since the degree of closeness depends on both λ and the coefficients in (3), rather than on λ alone, we cannot conclude that $g(\lambda, n) \leq 1/n$ for any positive λ . We also examine the extent to which a violation of condition (1)

enables us to distort the arguments of some of the coefficients to construct functions that are not in LS .

The Hadamard product or convolution of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. For

$$(4) \quad h_{\lambda}(z) = z + e^{i\lambda} \sum_{n=2}^{\infty} z^n,$$

we may express (3) as $f_0 * h_{\lambda}$. When λ is sufficiently small $h_{\lambda}(z)$ is starlike and we find a bound on the radius of starlikeness for $h_{\lambda} * f$, where $\operatorname{Re} f(z)/z > 1/2$ ($z \in U$). This generalizes a result of MacGregor [4].

2. Coefficient distortions. Given a function $f(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the function $g(z) = z + \sum_{n=2}^{\infty} e^{i\lambda_n} b_n z^n$ is said to be in $F_{\varepsilon}(f)$ for some $\varepsilon > 0$ if $-\varepsilon \leq \lambda_n \leq \varepsilon$ for all n .

LEMMA 1. *If $f(z) = z + \sum_{n=2}^{\infty} b_n z^n \notin LS$, then there exists an $\varepsilon > 0$ such that $g \in F_{\varepsilon}(f)$ implies that $g \notin LS$.*

Proof. Suppose, on the contrary, that there is a sequence $\varepsilon(n)$ tending to 0 for which we can find a corresponding sequence of functions $g_n \in F_{\varepsilon(n)}$ such that $g_n \in LS$ for all n . Since $f \notin LS$, there exists a point $z_0 \in U$ such that $f'(z_0) = 0$. Note that $\{g'_n\}$ converges uniformly to f' in some neighborhood D of z_0 . Since $g'_n \neq 0$ in D for any n , it follows by Hurwitz's theorem that $f' \neq 0$ in D . This contradicts our assumption that $f'(z_0) = 0$.

THEOREM 1. *If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ with $\sum_{n=2}^{\infty} n |a_n| > 1$, then there exists an $\varepsilon > 0$ such that $g \in F_{\varepsilon}(f)$ implies that $g \notin LS$.*

Proof. Since $f'(0) = 1$ and $f'(r) < 0$ for $r (< 1)$ sufficiently close to 1, there must exist a point r_0 , $0 < r_0 < 1$, such that $f'(r_0) = 0$. The result now follows from the lemma, with $b_n = -|a_n|$.

COROLLARY. *If $\sum_{n=2}^{\infty} n |a_n| > 1$, then $f_{\lambda}(z)$, defined by (3), is not in LS for λ sufficiently small.*

Our next theorem shows that if (1) is violated, then we can always construct a nonunivalent function by distorting the arguments of finitely many coefficients.

THEOREM 2. *If $\sum_{n=2}^N n |a_n| > 1$, then there exist real numbers $\alpha_2, \dots, \alpha_N$ ($-\pi < \alpha_j \leq \pi$) such that*

$$f(z) = z + \sum_{n=2}^N a_n e^{i\alpha_n} z^n + \sum_{n=N+1}^{\infty} a_n z^n \notin LS .$$

Proof. Let $C = z(t)$ be an arc of increasing modulus from the origin to the boundary of U such that

$$g(z) = \sum_{n=N+1}^{\infty} n a_n z^{n-1} \leq 0 \quad \text{for } z \in C .$$

For each point $z = z(t) \in C$, $|z| = r$, choose $\alpha_n(t)$ so that

$$n a_n e^{i\alpha_n(t)} z^{n-1} = -n |a_n| r^{n-1} \quad (n = 2, \dots, N) .$$

Next define $f_t(z)$ by

$$f_t(z) = z + \sum_{n=2}^N a_n e^{i\alpha_n(t)} z^n + \sum_{n=N+1}^{\infty} a_n z^n ,$$

where $\alpha_n(t)$ varies continuously with t . Since $g(z) \leq 0$ for $z \in C$, $|z| = r$, we have for all t that

$$f'_t(z) \leq 1 - \sum_{n=2}^N n |a_n| r^{n-1} \quad (z \in C) .$$

Since $f'_t(0) = 1$ and $f'_t(z) < 0$ for $z \in C$ sufficiently close to the boundary of U , it follows for some $\xi = z(t_0)$ on C that $f'_{t_0}(\xi) = 0$. Setting $f(z) = f_{t_0}(z)$ and $\alpha_n = \alpha_n(t_0)$, the result follows.

COROLLARY 1. *If $\sum_{n=2}^N n |a_n| > 1$, then there exist real numbers $\alpha_2, \dots, \alpha_N$ ($-\pi < \alpha_j \leq \pi$) and an $\varepsilon > 0$ such that for each λ , $-\varepsilon \leq \lambda \leq \varepsilon$,*

$$f_\lambda(z) = z + e^{i\lambda} \left(\sum_{n=2}^N a_n e^{i\alpha_n} z^n + \sum_{n=N+1}^{\infty} a_n z^n \right) \notin LS .$$

Proof. The corollary is established upon applying Lemma 1 to Theorem 2.

COROLLARY 2. *If $\sum_{k=2}^N n_k |a_{n_k}| > 1$, then there exist real numbers $\alpha_2, \dots, \alpha_N$ ($-\pi < \alpha_j \leq \pi$) and $\varepsilon > 0$ such that for each λ , $-\varepsilon \leq \lambda \leq \varepsilon$,*

$$f_\lambda(z) = z + e^{i\lambda} \left(\sum_{k=2}^N a_{n_k} e^{i\alpha_k} z^{n_k} + \sum_{\substack{n=2 \\ n \neq n_k}}^{\infty} a_n z^n \right) \notin LS .$$

Proof. The proof is the same as that of Corollary 1 except for a rearrangement of terms.

3. Extremal examples. One might ask for conditions under which $\sum_{n=2}^{\infty} n |a_n| > 1$ guarantees the existence of a real number λ

such that $f_\lambda(z) = z + e^{i\lambda} \sum_{n=2}^{\infty} a_n z^n \notin S$. Before answering this question, we need

LEMMA 2. For $n \geq 3$ there exist real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ such that $|\sum_{k=1}^n e^{i\alpha_k z^k}| < n$ ($|z| \leq 1$).

Proof. We have $|\sum_{k=1}^n e^{i\alpha_k e^{ik\theta}}| = n$ if and only if there is a θ for which

$$\alpha_1 + \theta = \alpha_k + k\theta \pmod{2\pi}, \quad k = 2, \dots, n.$$

Clearly the α_k 's can be chosen to preclude the existence of such a θ .

THEOREM 3. (i) If $\sum_{k=2}^3 n_k |a_{n_k}| > 1$, then $f_\lambda(z) = z + e^{i\lambda}(a_{n_2} z^{n_2} + a_{n_3} z^{n_3}) \notin LS$ for some λ .

(ii) If $N > 3$, then we can find $\{\alpha_{n_k} | k = 2, 3, \dots, N\}$ such that $\sum_{k=2}^N n_k |a_{n_k}| > 1$ and $f_\lambda(z) = z + e^{i\lambda} \sum_{k=2}^N a_{n_k} z^{n_k} \in S$ for all λ .

Proof. To prove (i), assume $\arg a_{n_2} = \alpha_2 + \pi$ and $\arg a_{n_3} = \alpha_3 + \pi$. Then for $\theta = (\alpha_2 - \alpha_3)/(n_3 - n_2)$, we have

$$f'_\lambda(r e^{i\theta}) = 1 - e^{i\lambda}(n_2 |a_{n_2}| r^{n_2-1} + n_3 |a_{n_3}| r^{n_3-1}) e^{i\beta},$$

where

$$\beta = \alpha_2 + \frac{(n_3 - 1)(\alpha_2 - \alpha_3)}{n_3 - n_2}.$$

Hence $f_\lambda(z) \notin LS$ for $\lambda = -\beta$.

To prove (ii), we choose $\{\alpha_k\}$ so that

$$\left| \sum_{k=2}^N e^{i\alpha_k} z^{n_k-1} \right| \leq A < N - 1 \quad (|z| \leq 1).$$

For $1 < B \leq (N - 1)/A$, set

$$f_\lambda(z) = z + e^{i\lambda} \sum_{k=2}^N \frac{B e^{i\alpha_k}}{(N - 1)n_k} z^{n_k}.$$

Then

$$f'_\lambda(z) = 1 + \frac{B}{N - 1} e^{i\lambda} \sum_{k=2}^{N-1} e^{i\alpha_k} z^{n_k-1},$$

so that

$$\operatorname{Re} f'_\lambda(z) \geq 1 - \frac{B}{N - 1} \left| \sum_{k=2}^{N-1} e^{i\alpha_k} z^{n_k-1} \right| > 1 - \frac{AB}{N - 1} \geq 0 \quad (z \in U).$$

By a criterion of Kaplan [3], $f_\lambda(z) \in S$ for all λ .

4. A radius of starlikeness theorem. Denote by F functions of the form $f(z) = z + \sum_{n=2}^\infty a_n z^n$ that are analytic in U and satisfy $\operatorname{Re} f(z)/z > 1/2$. MacGregor has shown [4] that the radius of starlikeness of F is $1/\sqrt{2}$. In this section we generalize this result. It is known [1] that the family F is the closed convex hull of convex functions, and that a function $f(z)$ is in F if and only if it can be expressed as

$$(5) \quad f(z) = \int_x \frac{z}{1 - xz} d\mu(x)$$

for some probability measure μ defined on the unit circle X .

In [2] Campbell proves

LEMMA A. Let M be a class of starlike functions with $b(r) = \max \{\arg f(z)/z: |z| = r, f \in M\}$. If $|z(f'(z)/f(z)) - a(r)| \leq d(r)$ for $f \in M$, where $a(r)$ and $d(r)$ are continuous functions of r satisfying $\lim_{r \rightarrow 1} a(r)/d(r) > 0$, then the radius of starlikeness of the closed convex hull of M is at least as large as the first positive root of $a(r) - d(r) \sec b(r) = 0$.

In the sequel, we let

$$(6) \quad H_\varepsilon = \left\{ z + e^{i\lambda} \sum_{n=2}^\infty z^n \| e^{i\lambda} - 1 \| \leq \varepsilon < 1 \right\}$$

and

$$(7) \quad G_\varepsilon = \left\{ \frac{h(xz)}{x} \mid h \in H_\varepsilon, |x| = 1 \right\}.$$

We shall also need

LEMMA 3. Set $a(r) = 1/(1 - r^2) - \varepsilon^2 r^2 / (1 - \varepsilon^2 r^2)$ and $d(r) = r/(1 - r^2) + \varepsilon r / (1 - \varepsilon^2 r^2)$. For $g \in G_\varepsilon$, the values for $(zg'(z)/g(z))$, $|z| \leq r$, lie in a disk centered at $a(r)$ and having radius $d(r)$.

Proof. It suffices to prove the result for $h \in H_\varepsilon$ since the class G_ε consists of rotations of these functions. Writing $h(z) = (z + (e^{i\lambda} - 1)z^2)/(1 - z)$ for $h \in H_\varepsilon$, we see that $zh'(z)/h(z) = 1/(1 - z) + ((e^{i\lambda} - 1)z)/(1 + (e^{i\lambda} - 1)z)$. Since

$$\frac{-\varepsilon r}{1 - \varepsilon r} \leq \operatorname{Re} \frac{(e^{i\lambda} - 1)z}{1 + (e^{i\lambda} - 1)z} \leq \frac{\varepsilon r}{1 + \varepsilon r} \quad (|z| \leq r),$$

it follows that

$$(8) \quad \frac{1}{1+r} - \frac{\varepsilon r}{1-\varepsilon r} \leq \operatorname{Re} \frac{zh'(z)}{h(z)} \leq \frac{1}{1-r} + \frac{\varepsilon r}{1+\varepsilon r}.$$

Thus the values for $zh'(z)/h(z)$ are contained in a disk centered at

$$\frac{1}{2} \left[\left(\frac{1}{1-r} + \frac{\varepsilon r}{1+\varepsilon r} \right) + \left(\frac{1}{1+r} - \frac{\varepsilon r}{1-\varepsilon r} \right) \right] = a(r)$$

whose radius is

$$\frac{1}{2} \left[\left(\frac{1}{1-r} + \frac{\varepsilon r}{1+\varepsilon r} \right) - \left(\frac{1}{1+r} - \frac{\varepsilon r}{1-\varepsilon r} \right) \right] = d(r).$$

LEMMA 4. *A function $g \in G_\varepsilon$ defined by (7) is starlike if and only if $\varepsilon \leq 1/3$.*

Proof. It suffices to consider $h \in H_\varepsilon$ defined by (6). Letting $r \rightarrow 1$ in the left-hand side of (8), we see that $\operatorname{Re} zh'(z)/h(z) \geq 0$ when $\varepsilon \leq 1/3$. For $e^{i\lambda} - 1 = \varepsilon e^{i\sigma}$ and $\beta = \pi - \sigma$, we have $e^{i\beta} h'(e^{i\beta})/h(e^{i\beta}) = 1/2 - \varepsilon/(1-\varepsilon) < 0$ when $\varepsilon > 1/3$.

REMARK. Since the Pólya-Schoenberg conjecture is true [5], we know that for f convex and $h \in H_\varepsilon$ ($\varepsilon \leq 1/3$), the Hadamard product $h * f$ is starlike. However if f is only required to be starlike, then $h_\lambda * f$ need not be in S for any h_λ defined by (4), $\lambda \neq 0$. To see this, observe that $h_\lambda * z/(1-z)^2 = z + e^{i\lambda} \sum_{n=2}^{\infty} n z^n$ is not in S for $\lambda \neq 0$ because $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $|a_2| = 2$, is in S only if $f(z) = z/(1-xz)^2$, $|x| = 1$.

We now give a bound for the radius of starlikeness for $h * f$, $f \in F$, $h \in H_\varepsilon$ with $\varepsilon \leq 1/3$.

THEOREM 4. *Suppose $h(z) = z + e^{i\lambda} \sum_{n=2}^{\infty} z^n \in H_\varepsilon$, $\varepsilon \leq 1/3$, and $f \in F$, with $a(r)$ and $d(r)$ defined in Lemma 3. Then $h * f$ is starlike in a disk $|z| < r_0$, where r_0 is the first positive root of*

$$(9) \quad a(r) - \frac{d(r)}{\sqrt{(1-\varepsilon^2 r^2)(1-r^2)} - \varepsilon r^2} = 0.$$

Proof. In view of (5),

$$(h * f)(z) = \int_x h * \frac{z}{1-xz} d\mu(x) = \int_x \frac{h(xz)}{x} d\mu(x), \quad |x| = 1.$$

By Lemma 4, the kernel functions are all starlike. An application of Lemma 3 to Lemma A shows that $h * f$ is starlike in a disk whose radius is at least as large as the first positive root of

$$(10) \quad a(r) - d(r) \sec b(r) = 0 ,$$

where

$$b(r) = \max_{|z|=r} \left\{ \arg \frac{g(z)}{z} \mid g \in G_\varepsilon \right\} .$$

Since

$$\begin{aligned} b(r) &\leq \max_{|z|=r} |\arg(1 + \varepsilon z)| + \max_{|z|=r} |\arg(1 - z)| \\ &= \sin^{-1}(\varepsilon r) + \sin^{-1}(r) = t(r) , \end{aligned}$$

$$(11) \quad \sec b(r) \leq \sec t(r) = \frac{1}{\cos t(r)} = \frac{1}{\sqrt{(1 - \varepsilon^2 r^2)(1 - r^2) - \varepsilon r^2}} .$$

A substitution of the right-hand side of (11) into (10) yields the desired result.

REMARK. If $\varepsilon = 0$ then (9) reduces to $1/(1 - r^2) - r/((1 - r^2)^{3/2}) = 0$, whose smallest positive root is $1/\sqrt{2}$. This coincides with a sharp result of MacGregor [4].

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