SUFFICIENT CONDITIONS FOR THE SET OF HAUSDORFF COMPACTIFICATIONS TO BE A LATTICE

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Let \( K(X) \) be the complete upper semilattice of compactifications of a completely regular Hausdorff space \( X \). We show that if \( \beta X \setminus X \) is \( C^* \)-embedded in \( \beta X \) and if either \( \alpha X \setminus X \) is realcompact or is a \( P \)-space for some \( \alpha X \) in \( K(X) \), then \( K(X) \) is a lattice.

1. Introduction. Throughout this paper, all topological spaces under consideration are supposed to be completely regular and Hausdorff, unless stated otherwise.

A compactification of a space \( X \) is a compact space \( \alpha X \) which contains \( X \) as a dense subspace. We say \( \alpha_i X \) and \( \alpha_j X \) are equivalent compactifications of \( X \) if there is a homeomorphism \( h \) from \( \alpha_i X \) onto \( \alpha_j X \) such that \( h \) restricted to \( X \) in \( \alpha_i X \) is the identity map onto \( X \) in \( \alpha_j X \). We do not distinguish between equivalent compactifications. For compactifications \( \alpha_i X \) and \( \alpha_j X \), we say that \( \alpha_i X \succeq \alpha_j X \) if and only if there is continuous function from \( \alpha_i X \) onto \( \alpha_j X \) such that \( h \) restricted to \( X \) is the identity. Thus, \( \alpha_i X \) is equivalent to \( \alpha_j X \) if and only if \( \alpha_i X \succeq \alpha_j X \) and \( \alpha_j X \succeq \alpha_i X \). Let \( K(X) \) denote the set of all compactifications of \( X \). Then \( K(X) \) with the order \( \succeq \) defined as above is a complete upper semilattice. Lubben [3] proved that \( X \) is locally compact if and only if \( K(X) \) is a complete lattice. Next, Shirota [6] showed that if \( X \) is first countable then \( K(X) \) is a lattice if and only if \( X \) is locally compact. Thus, \( Q \) (=rationals) provides us with the simplest example for which \( K(Q) \) is not a lattice. Visliseni and Flaksmaier [9] showed that if there exists a sequence in \( \beta X \setminus X \) which converges to a point in \( X \), then \( K(X) \) cannot be a lattice. In the same paper they also constructed a non-locally compact space \( X \) for which \( K(X) \) is a lattice.

In this paper we determine two classes of spaces which properly contain the class of locally compact spaces and for which \( K(X) \) is a lattice, whenever \( X \) is a member of either of them. Examples are constructed to show that none of these conditions are necessary.

2. Preliminaries. The terminology of [1] and [11] are used throughout. The following will be needed for subsequent development.
DEFINITION 2.1. Let $\alpha X \in K(X)$, $f_\alpha: \beta X \to \alpha X$ be continuous and $f_\alpha|_X = \text{id}$. Then $f_\alpha$ is closed and hence we can consider $\alpha X$ as the quotient space of $\beta X$ induced by $f_\alpha$. Define

$$\mathcal{F}(\alpha X) = \{f_\alpha^{-1}(p) | p \in \alpha X \setminus X\}.$$

THEOREM 2.2 (Magill [4]). Let $\alpha, X, \alpha_2, X \in K(X)$. Then $\alpha, X \leq \alpha_2, X$ if and only if each set in $\mathcal{F}(\alpha_2 X)$ is a subset of a set in $\mathcal{F}(\alpha X)$.

DEFINITION 2.3. A space $X$ is said to be of countable type if and only if every compact subset is contained in a compact set of countable character (i.e., one having a countable neighborhood system).

THEOREM 2.4 ([2], page 115). A space $X$ is of countable type if and only if $\beta X \setminus X$ is Lindelöf.

THEOREM 2.5 ([1], page 115). Lindelöf spaces are realcompact.

DEFINITION 2.6. A space $X$ is of point countable type if and only if every point is contained in a compact set of countable character.

THEOREM 2.7 ([8], page 341). If $X$ is a space of point countable type then $\beta X \setminus X$ is realcompact.

THEOREM 2.8 ([9], page 1424). If, in the subspace $\beta X \setminus X$ of the space $\beta X$, there exists a countable sequence of points converging to some point in $X$, then $K(X)$ is not a lattice.

3. Major results.

LEMMA 3.1 ([10], page 28). $\beta X \setminus X$ is $C^*$-embedded in $\beta X$ if and only if $C_{l_{\beta X}}(\beta X \setminus X) = \beta(\beta X \setminus X)$.

DEFINITION 3.2. For $\alpha X \in K(X)$, let $f_\alpha: \beta X \to \alpha X$ be the quotient map, define

$$\mathcal{M}_\alpha = \{p \in \beta X \setminus X | |f_\alpha^{-1}(f_\alpha(p))| > 1\},$$

and

$$\mathcal{S}_\alpha = \{F \subseteq \mathcal{M}_\alpha | F = f_\alpha^{-1}(y), \text{ some } y \in \alpha X\}.$$

LEMMA 3.3. If $C_{l_{\beta X}}(\mathcal{M}_\alpha) \subseteq \beta X \setminus X$ for every $\alpha X \in K(X)$, then $K(X)$ is a lattice.
Proof. Since $K(X)$ is a complete upper semi-lattice, it is sufficient to show any two elements of $K(X)$ have a lower bound. Let $\alpha X, \alpha_2 X \in K(X)$. $A = C_{\beta X}(\mathcal{U}_a) \cup C_{\beta X}(\mathcal{U}_b)$ is compact in $\beta X \setminus X$. Obtain $\alpha X$ by identifying $A$ to a point, then $\alpha X$ is a compactification of $X$. Clearly, each set in $\mathcal{F}(\alpha X)$ is a subset of a set in $\mathcal{F}(\alpha X)$ for $i = 1, 2$. By Theorem 2.2, $\alpha X \subseteq \alpha X, \alpha X$. Hence, $K(X)$ is a lattice.

Lemma 3.4 ([1], page 62). Let $f: X \to Y$ be continuous, $A$ be dense in $X$. If $f|_A$ is a homeomorphism, then $f(X \setminus A) \subseteq Y \setminus f(A)$.

Definition 3.5. Let $Y$ be a quotient space of $X$ with the quotient map $P$. Let $\{A_i\}_{i=1}^k$ be a collection of disjoint, nonempty subsets in $X$ with $k \geq 2$. We say $\{A_i\}_{i=1}^k$ is a section partition induced by $P$ if and only if there exists $B \subseteq Y$ such that $P(A_i) = B$ and $P^{-1}(b) \cap A_i$ is a singleton for $1 \leq i \leq k, b \in B$. $P$ induces a partition on $A = \bigcup_{i=1}^k A_i$, namely, $A = \bigcup_{b \in B} A_b$, $A_{i_1} \cap A_{i_2} = \emptyset$ if $i_1 \neq i_2$, where $A_b = \bigcup_{i=1}^k (P^{-1}(b) \cap A_i)$. This partition induces the section correspondence induced by $P$ on $A$.

Lemma 3.6. If $\beta X \setminus X$ is $C^*$-embedded in $\beta X$ then for every $\alpha X \in K(X)$, $\mathcal{M}_\alpha$ contains no copy of $N$ which is $C$-embedded in $\beta X \setminus X$.

Proof. Let $\alpha X \in K(X)$ such that $\mathcal{M}_\alpha$ contains a copy of $N$ which is $C$-embedded in $\beta X \setminus X$. $F$ is compact for each $F \in \mathcal{G}_a$, so it can contain only finitely many points of $N$. Form $A$ by choosing one point from each nonempty $F \cap N$, then $A$ is infinite. Let $h \in C(\beta X \setminus X)$ such that $h(A) = N \subseteq R$. $h|_A$ carries $A$ homeomorphically onto a closed set in $R$, so $A$ is $C$-embedded in $\beta X \setminus X$ by 1.19 of [1]. Therefore, $A$ is a copy of $N$, which is $C$-embedded in $\beta X \setminus X$. If $F = f^{-1}(f(a))$ for some $a \in A$ then since $a \in \mathcal{M}_\alpha$, we have $F \in \mathcal{G}_a$. Let $\mathcal{A} = \{F \in \mathcal{G}_a | F \cap A = \emptyset\}$. Form $B$ by choosing one point from each $F \setminus A$, $F \in \mathcal{A}$. $\{A, B\}$ is a section partition induced by $f_a$. We want to show that $B$ is closed in $\beta X \setminus X$. Let $(b_2)$ be an ultranet in $B$, and $b_2 \to b \in (\beta X \setminus X) \setminus B$. Let $(a_2)$ be the corresponding ultranet in $A$ through the section correspondence induced by $f_a$ on $f_a(A)$. Since $\beta X$ is compact, $a_2 \to a \in \beta X$. Clearly, $(a_2)$ is nontrivial, since $(b_2)$ is nontrivial. Also, $a \in X$, since $A$ is closed and discrete in $\beta X \setminus X$. It is known that $f_a$ is continuous, so $f_a(a_2) \to f_a(a)$ and $f_a(b_2) \to f_a(b)$. Since $f_a(a_2) = f_a(b_2)$ for all $\lambda$, and the limit points of these nets are unique, it follows that $f_a(a) = f_a(b)$. This is not possible since $f_a(\beta X \setminus X) \subseteq \alpha X \setminus f_a(X)$ by Lemma 3.4. Thus $B$ is closed in $\beta X \setminus X$. Since $A$ is a $C$-embedded copy of $N$ and $B$ is a closed set disjoint
from $A$, so $A$ and $B$ are completely separated in $\beta X \setminus X$ by $3B$ of [1]. As $\beta X \setminus X$ is $C^*$-embedded in $\beta X$, therefore $A$ and $B$ are completely separated in $\beta X$ by 1.17 of [1]. It follows that $\mathcal{C}l_{\beta X}(A) \cap \mathcal{C}l_{\beta X}(B) = \phi$. Choose $(a_i)$ in $A$ and $(b_i)$ in $B$ as before, with $a_i \to a \in X$, $b_i \to b \in X$. Then $f_{a_i}(a) = f_{a_i}(b)$. This is a contradiction, since $f_a|_X$ is one-to-one. Hence $\mathcal{H}_a$ contains no copy of $N$, which is $C$-embedded in $\beta X \setminus X$ for all $\alpha X$ in $K(X)$.

**Theorem 3.7.** If $\beta X \setminus X$ is $C^*$-embedded in $\beta X$, and if $\alpha X \setminus X$ is realcompact for some $\alpha X$ in $K(X)$ then $K(X)$ is a lattice.

**Proof.** If $\alpha X \setminus X$ is realcompact for some $\alpha X$, then $\beta X \setminus X$ is realcompact by 8.13 of [1].

**Claim.** $\mathcal{C}l_{\beta X}(\mathcal{M}_a) \subseteq \beta X \setminus X$ for every $\alpha X \in K(X)$. Suppose not, then there exists $\alpha X \in K(X)$ such that $\mathcal{M}_a$ has a limit point $x_0 \in X$. Let $Y = \{x_0\} \cup (\beta X \setminus X)$ endowed with the relative topology as a subspace of $\beta X$. $\beta X \setminus X$ is realcompact and dense in $Y$, so $\beta X \setminus X$ is not $C$-embedded in $Y$. Let $f \in C(\beta X \setminus X)$ such that $f$ cannot be extended to $Y$. Let $[-\infty, \infty]$ be the two-point compactification of $R$. Clearly, $f$ can be considered as a continuous function of $\beta X \setminus X$ into $[-\infty, \infty]$. $f$ has an extension $\tilde{f}$ from $\beta (\beta X \setminus X) = \mathcal{C}l_{\beta X}(\beta X \setminus X)$ into $[-\infty, \infty]$. Without loss of generality, we may assume $\tilde{f}(x_0) = \infty$. Since $x_0 \in \mathcal{C}l_{\beta X \setminus X}(\mathcal{M}_a)$, so $f$ is unbounded on $\mathcal{M}_a$. By 1.20 of [1], $\mathcal{M}_a$ contains a copy of $N$ which is $C$-embedded in $\beta X \setminus X$. This contradicts Lemma 3.6, and hence $\mathcal{C}l_{\beta X}(\mathcal{M}_a) \subseteq \beta X \setminus X$ for every $\alpha X \in K(X)$. Lemma 3.3 shows that $K(X)$ is a lattice.

**Corollary 3.8.** If $X$ is a space of point countable type and $\beta X \setminus X$ is $C^*$-embedded in $\beta X$ then $K(X)$ is a lattice.

**Theorem 3.9.** If $\beta X \setminus X$ is $C^*$-embedded in $\beta X$ and if $\alpha X \setminus X$ is a $P$-space for some $\alpha X \in K(X)$, then $K(X)$ is a lattice.

**Proof.** We claim that $f_a(\mathcal{M}_a)$ is finite. For if $f_a(\mathcal{M}_a)$ is infinite then it contains a countably infinite subset $A$. By 4K of [1], we see that $A$ is a copy of $N$, which is $C$-embedded in $\alpha X \setminus X$. Let $f \in C(\alpha X \setminus X)$ such that $f(A) = N \subseteq R$. Hence, $f \circ f_a \in (\beta X \setminus X)$ is unbounded on $f_a^{-1}(A) \subseteq \mathcal{M}_a$. Thus $f_a^{-1}(A)$ contains a copy of $N$ which is $C$-embedded in $\beta X \setminus X$. Since $\beta X \setminus X$ is $C^*$-embedded in $\beta X$, this contradicts Lemma 3.6. Therefore, $f_a(\mathcal{M}_a)$ is finite. Let $\gamma X \in K(X)$.

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1 Yusuf Ünlü proved independently in his doctoral thesis [7] that $K(X)$ is a lattice if either (1) $\beta X \setminus X$ is realcompact and $C^*$-embedded in $\beta X$, or (2) $\beta X \setminus X$ is a $P$-space and $\mathcal{C}l_{\beta X}(\beta X \setminus X)$ is an $F$-space.
Claim. $f_a(\mathcal{M}_a \setminus \mathcal{M}_\alpha)$ is finite. Suppose $f_a(\mathcal{M}_a \setminus \mathcal{M}_\alpha)$ is infinite then $\mathcal{M}_a \setminus \mathcal{M}_\alpha$ contains a copy of $\mathbb{N}$ which is $C$-embedded in $\beta X \setminus X$. This is a contradiction. $\mathcal{M}_a = \cup \{ f_a'(p) | p \in f_a(\mathcal{M}_a) \}$ so that $\mathcal{M}_a$ is a finite union of closed (hence compact) subsets of $\beta X$. Thus $\mathcal{M}_a$ is compact. Similarly, $\mathcal{M}_a \setminus \cup \{ f_a'(p) | p \in f_a(\mathcal{M}_a \setminus \mathcal{M}_\alpha) \} \cup \mathcal{M}_\alpha$ and both of these sets are compact. Therefore, $\text{Cl}_{\beta X}(\mathcal{M}_a) \subseteq \beta X \setminus X$. Since this is for an arbitrary $\gamma X \in K(X)$, the theorem follows from Lemma 3.3.

We summarize the major results of this section in the following theorem:

**Theorem 3.10.** If $\beta X \setminus X$ is $C^*$-embedded in $\beta X$ then any of the following conditions implies that $K(X)$ is a lattice:

(i) $\alpha X \setminus X$ is realcompact for some $\alpha X \in K(X)$,

(ii) $\alpha X \setminus X$ is a $P$-space for some $\alpha X \in K(X)$,

(iii) $X$ is of countable type,

(iv) $X$ is of point-countable type.

Note that the class of spaces $X$ for which $\beta X \setminus X$ is $C^*$-embedded in $\beta X$ and for which $\alpha X \setminus X$ is realcompact for some $\alpha X$ in $K(X)$ contains the class of locally compact spaces. ($\beta X \setminus X$ is compact so that it is both realcompact and $C^*$-embedded in $\beta X$.) Likewise, the class of spaces $X$ for which $\beta X \setminus X$ is $C^*$-embedded in $\beta X$ and for which $\alpha X \setminus X$ is a $P$-space for some $\alpha X$ in $K(X)$ contains the class of locally compact spaces. ($\beta X \setminus X$ is $C^*$-embedded in $\beta X$ since it is compact and $\omega X \setminus X = \{ p \}$ is a $P$-space.) Thus our results here can be considered as generalizations of those of Lubben [3].

4. Examples. Let $\Omega$ denote the class of ordinals. For $\alpha \in \Omega$, $W(\alpha) = \{ \alpha \in \Omega | \sigma < \alpha \}$. $\omega$ will denote the smallest member of $\Omega$ with infinitely many predecessors: $W(\omega)$ is infinite and for all $\alpha < \omega$, $W(\alpha)$ is finite. $\omega_1$ will denote the smallest member of $\Omega$ with uncountably many predecessors.

**Theorem 4.1** ([1], page 138). If $X$ is compact, with $|X| < \mathfrak{N}$, $\alpha \neq 0$, then $\beta(X \times W(\omega_\alpha)) = X \times W(\omega_\alpha + 1)$.

**Proof.** See ([10], page 92).

**Theorem 4.2** ([1], page 89). $X \subseteq Y \subseteq \beta X$, then $\beta Y = \beta X$.

**Lemma 4.3.** For $\alpha Y \in K(Y)$, there exists $X$ such that $Y = \beta X \setminus X$ and $\text{Cl}_{\beta X}(Y) = \alpha Y$.

**Proof.** Let $\lambda \neq 0$ be chosen, so that $|\alpha Y| < \mathfrak{N}_2$. By Theorem
4.1, we have $\beta(\alpha Y \times W(\omega_x)) = \alpha Y \times W(\omega_x + 1)$. Let $X = \beta(\alpha Y \times W(\omega_x))(Y \times \{\omega_i\})$, then $\alpha Y \times W(\omega_x) \subseteq X \subseteq \beta(\alpha Y \times W(\omega_x))$ and hence $\beta X = \beta(\alpha Y \times W(\omega_x)) = \alpha Y \times W(\omega_x + 1)$. Since $\alpha Y \times \{\omega_i\}$ is compact and contains $Y \times \{\omega_i\} = Y$ as a dense subspace, $X$ is the space desired.

**Corollary 4.4.** For any space $Y$ there is an $X$ such that $\beta X \setminus X = Y$ and $Y$ is $C^*$-embedded in $X$.

**Theorem 4.5.** Given any two spaces $X$ and $Y$, there is an $\alpha X \in K(X)$ such that $Y$ is homeomorphic to $Cl_{\alpha X}(\alpha X \setminus X)$ iff there is a continuous map $h$ from $Cl_{\beta X}(\beta X \setminus X)$ onto $Y$ such that $h(\beta X \setminus X) \subseteq Y \setminus h(R(X))$ and $h$ is one-to-one on $R(X)$, where $R(X)$ is the set of points at which $X$ is not locally compact.

**Example 4.6.** (1) Let $\omega N$ be the one-point compactification of $N$. Then there exists $X$ such that $\beta X \setminus X = N$ and $Cl_{\beta X}(N) = \omega N$. There exists a sequence, namely $N$, which converges to $(\omega, \omega_i) \in X$. Thus $K(X)$ is not a lattice by 2.8.

In the above example, $\beta X \setminus X$ is realcompact and a $P$-space but not $C^*$-embedded in $\beta X$.

**Example 4.7.** (2) If $Y = W(\omega_1)$, then $\beta Y = W(\omega_1 + 1)$. Let $X = (\beta Y \times \beta Y)(Y \times \{\omega_i\})$, then $\beta X \setminus X = Y$. Let $\mathcal{D}$ be the collection of subsets of $\beta X$ of the form $\{((\lambda + 2j, \omega_1), (\lambda + 2j + 1, \omega_1))\}$ for $\lambda$ a limit ordinal, $j = 0, 1, 2, \ldots$, and all other singletons. Then $\mathcal{D}$ is a decomposition space of $X$. Let $P: X \to \mathcal{D}$ be the quotient map, then $\mathcal{D}$ can be considered as the quotient space of $X$ induced by $P$. Clearly $P(\beta X \setminus X)$ is compact Hausdorff. By 4.5 we have $\mathcal{D} = \alpha X \in K(X)$. Similarly, let $\mathcal{D}'$ be the collection of subsets of $\beta X$ of the form $\{(\alpha + 2j - 1, \omega_1), (\alpha + 2j, \omega_1)\}$ for $\alpha$ a limit ordinal, $j = 1, 2, \ldots$, and all other singletons, then $\mathcal{D}' = \alpha_2 X \in K(X)$. If $\alpha X \in K(X)$ and $\alpha X \leq \alpha_1 X, \alpha_2 X$, then the following diagram commutes:

$$
\begin{array}{c}
\beta X \\
\alpha X
\end{array}
\begin{array}{c}
\downarrow f_{\alpha_1} \quad \quad \quad \quad \quad \downarrow f_{\alpha_2}
\end{array}
\begin{array}{c}
\alpha_1 X \\
\alpha_2 X
\end{array}
\begin{array}{c}
\downarrow f_1 \quad \quad \quad \quad \quad \downarrow f_2
\end{array}
$$

Thus, if $f_\alpha((\lambda, \omega)) = y$, for some $\lambda$ a limit ordinal then $f((\lambda + j,$
\( f_\alpha(W \times \{\omega_j\}) = y = f_\alpha((\omega, \omega_j)) \), which is a contradiction since \( f_\alpha(\beta X \setminus X) \subseteq f_\alpha(\beta X) \setminus f_\alpha(X) \). Hence \( K(X) \) is not a lattice.

In this example, the subspace \( \beta X \setminus X \) is \( C^* \)-embedded but not realcompact nor a \( P \)-space. We also claim that \( \alpha X \setminus X \) is not a \( P \)-space for any \( \alpha X \in K(X) \). For if \( \alpha X \setminus X \) is a \( P \)-space, then \( \alpha X \setminus X \) contains a \( C \)-embedded copy of \( N \), which implies \( Y \) contains a \( C \)-embedded copy of \( N \). But this is not possible since \( Y \) is pseudo-compact.

**Example 4.9.** (3) Let \( Y \) be the subspace of \( W(\omega_1) \) obtained by deleting all nonisolated points having a countable base, then \( Y \) is a \( P \)-space that is not realcompact ([11], page 138).

Let \( X \) be chosen so that \( \beta X \setminus X = Y \) and \( Y \) \( C^* \)-embedded in \( \beta X \), then \( K(X) \) is a lattice by Theorem 3.9, \( \beta X \setminus X \) is not realcompact.

**Example 4.3.** (4) Let \( Q \) be the set of rationals. Choose \( X \) so that \( \beta X \setminus X = Q \) and \( Q \) is \( C^* \)-embedded in \( \beta X \). Since \( Q \) is realcompact, \( K(X) \) is a lattice. We claim that \( \alpha X \setminus X \) is not a \( P \)-space for any \( \alpha X \in K(X) \). For if \( \alpha X \setminus X \) is a \( P \)-space, then \( \alpha X \setminus X \) contains a \( C \)-embedded copy of \( N \) which contradicts Lemma 3.6.

**Example 4.10.** (5) \( E = \{2n | n \in N\} \) and \( 0 = \{2n+1 | n \in N\} \). Then \( N = E \cup 0 \) and \( E \cap 0 = \phi \). Define \( t : N \rightarrow N \) by \( t(2n) = 2n + 1 \) and \( t(2n + 1)2n, n \in N \). Thus, \( t(E) = 0 \) and \( t(0) = E \). For each \( p \in \beta N \setminus N \), there exists a unique free ultrafilter \( U_p \) on \( N \) such that \( U_p \rightarrow p \).

Let \( \mathcal{U} = \{U_p\}_{p \in \beta N \setminus N} \). It is clear that \( \mathcal{U} \) is exactly the set of free ultrafilters on \( N \). Define \( \mathcal{U}_E = \{U_p \in \mathcal{U} | E \in U_p\} \) and \( \mathcal{U}_0 = \{U_p \in \mathcal{U} | 0 \in U_p\} \). Obviously, \( \mathcal{U}_E \) and \( \mathcal{U}_0 \) form a partition of \( \mathcal{U} \). If \( U_p \in \mathcal{U}_E \), then \( t(U_p) \) the ultrafilter generated by \( \{t(u) | u \in U_p\} \) is identical to \( \{t(u) | u \in U_p\} \), furthermore, \( t(U_p) \in \mathcal{U}_0 \). Similarly, \( t(U_p) \in \mathcal{U}_E \) if \( U_p \in \mathcal{U}_0 \). Thus, \( t \) induces a one-to-one correspondence between \( \mathcal{U}_E \) and \( \mathcal{U}_0 \). Each \( p \) in \( \beta N \setminus N \) corresponds to unique \( U_p \) in \( \mathcal{U} \), therefore the partition \( \mathcal{U} = \mathcal{U}_E \cup \mathcal{U}_0 \), \( \mathcal{U}_E \cap \mathcal{U}_0 = \phi \) induces a partition on \( \beta N \setminus N \). The induced partition is \( \beta N \setminus N = (Cl_{\beta N}(E) \setminus E) \cup (Cl_{\beta N}(0) \setminus 0) \) with \( (Cl_{\beta N}(E) \setminus E) \cap (Cl_{\beta N}(0) \setminus 0) = \phi \).

Define a relation \( \sim \) on \( \beta N \) as follows: \( p_1 \sim p_2 \) if and only if \( p_1 \neq p_2 \) or \( t(U_{p_1}) = U_{p_2} \). Then \( \sim \) is an equivalence relation on \( \beta N \). Let \( \mathcal{D} \) be the identification space \( \beta N / \sim \) with the quotient map \( P \). Clearly \( \mathcal{D} \) is compact and \( T_1 \). We want to show \( \mathcal{D} \) is Hausdorff. For \( x \in P(N) \), \( P^{-1}(x) \) is a singleton in \( N \), so \( P^{-1}(x) \) is both open and closed in \( \beta N \). It follows that \( \{x\} \) is both open and closed in \( \mathcal{D} \). Thus \( x \) can be separated from any other point by open sets in \( \mathcal{D} \). Let \( p, q \in P(\beta N \setminus N) \). Then
\( P^{-1}(p) = \{p_1, p_2\} \), and \( P^{-1}(q) = \{q_1, q_2\} \) for \( p_1, q_1 \in \text{Cl}_{\beta N}(E) \setminus E \) and \( p_2, q_2 \in \text{Cl}_{\beta N}(0) \setminus 0 \). Let \( u, v \) be open in \( \beta N \) such that \( u, v \subseteq \text{Cl}_{\beta n}(E) \), \( p_1 \in u \), \( q_1 \in v \) and \( u \cap v = \emptyset \). Let \( \bar{t} \) be the extension of \( t \) from \( \beta N \) to \( \beta N \). Obviously, \( \bar{t} \) is a homeomorphism, so \( \bar{t}(u) \) and \( \bar{t}(v) \) are open in \( \beta N \), moreover \( \bar{t}(u), \bar{t}(v) \subseteq \text{Cl}_{\beta N}(0) \) and \( p_2 \in \bar{t}(u), q_2 \in \bar{t}(v) \). Let \( G = P(u \cup (\bar{t}(u)), H = P(v \cup (\bar{t}(v))) \). Clearly, \( P^{-1}(G) = u \cup (\bar{t}(u)) \) and \( P^{-1}(H) = v \cup (\bar{t}(v)) \), so \( G \) and \( H \) are open in \( \mathcal{D} \). Since \( p \in G, q \in H, G \cap H = \emptyset \), so \( p, q \) can be separated by open sets. Thus \( \mathcal{D} \) is Hausdorff. Thus there is a \( \gamma N \in K(N) \) such that \( \gamma N = \mathcal{D} \).

Let \( X \) be obtained as in Lemma 4.3 such that \( \beta X \setminus X = N \) and \( \text{Cl}_{\beta X}(N) = \gamma N \). For \( \alpha X \in K(X) \), we claim \( \alpha X \) has the following properties.

(1) \( \mathcal{S}_1^\alpha = \{ F \in \mathcal{S}_\alpha \mid |F| \geq 3 \} \) is finite,

(2) \( \mathcal{S}_2^\alpha = \{ F \in \mathcal{S}_\alpha \mid |F| = 2, F \subseteq E \} \) and

\( \mathcal{S}_3^\alpha = \{ F \in \mathcal{S}_\alpha \mid |F| = 2, F \subseteq E \} \) are finite,

(3) Let \( \mathcal{S}_4^\alpha = \{ F \in \mathcal{S}_\alpha \mid |F \cap E| = |F \cap 0| = 1 \} \), then

\( \mathcal{S}_4^\alpha = \{ F \in \mathcal{S}_\alpha \mid F \neq \{2n, 2n + 1\} \} \) for any \( n \in N \) is finite.

Proof of (1). If \( \mathcal{S}_1^\alpha \) is infinite, then \( \mathcal{S}_1^\alpha \) contains three copies of \( N \), say \( \{A_i\}_{i=1}^3 \), which are C-embedded in \( N \subseteq \beta X \) such that \( \{A_i\}_{i=1}^3 \) is a section partition induced by \( f_\alpha \). Clearly, \( \{f_\alpha^{-1}(A_i)\}_{i=1}^3 \) is a section partition induced by \( g_\alpha \circ f_\beta \) where \( g_\alpha \) is the restriction of \( f_\alpha \) to \( \text{Cl}_{\beta X}(N) = \gamma N \). Let \( (a_i^{(1)}) \) be an ultranet in \( A_1 \) and \( a_i^{(1)} \to a_1 \in \beta N \setminus N \). Let \( (a_i^{(2)}) \subseteq A_2, (a_i^{(3)}) \subseteq A_3 \) be ultranets induced by the section correspondences which are induced by \( g_\alpha \circ f_\beta \) on \( \text{Cl}_{\beta X}(N) \). Let \( a_i^{(2)} \to a_2, a_i^{(3)} \to a_3 \), where \( a_2, a_3 \in \beta N \setminus N \). Obviously \( a_1, a_2, a_3 \) are distinct. By the definition of \( \gamma N, |f_\beta^{-1}((a_i^{(1)})_{i=1}^3)| \geq 2 \). \( f_\beta^{-1} \) is one-to-one, so \( |(g_\alpha \circ f_\beta)((a_i^{(1)})_{i=1}^3)| \geq 2 \). This is not possible, since \( (g_\alpha \circ f_\beta)(a_i^{(1)}) = (g_\alpha \circ f_\beta)(a_i^{(2)}) = (g_\alpha \circ f_\beta)(a_i^{(3)}) \) for all \( \lambda \) which implies \( |(g_\alpha \circ f_\beta)((a_i^{(1)})_{i=1}^3)| = 1 \). Thus (1) holds.

Proof of (2). It is sufficient to show \( \mathcal{S}_2^\alpha \) cannot be infinite. Suppose \( \mathcal{S}_2^\alpha \) is infinite, then \( E \) contains two copies of \( N \), say \( A_1, A_2 \), which are C-embedded in \( N = \beta X \setminus X \) such that \( \{A_i\}_{i=1}^2 \) is a section partition induced by \( f_\alpha \). This is not possible, since no two-points in \( \text{Cl}_{\beta N}(E) \) are equivalent with respect to \( \sim \), and \( f_\beta^{-1} \) is one-to-one. Thus (2) holds.

Proof of (3). If \( \mathcal{S}_3^\alpha \) is infinite, then there exists \( A = \{a_n\}_{n=0}^\infty \subseteq E, B = \{b_n\}_{n=0}^\infty \subseteq 0 \) such that \( \{A, B\} \) is a section partition induced by \( f_\alpha, \{a_n, b_n\} \in \mathcal{S}_3^\alpha \) for \( n \in N \), and \( t(A) \cap B = \emptyset \). Let \( a \in \text{Cl}_{\beta N}, \) then \( t(\mathcal{Z}_\alpha) \to \bar{t}(a) \in \text{Cl}_{\beta N}(B), \) since \( B \in t(\mathcal{Z}_\alpha) \). Let \( (a_2) \) be the ultranet in \( A \) based on \( A \cap \mathcal{Z}_\alpha \) such that \( a_2 \to a \). Let \( (b_2) \) be the ultranet in \( B \) induced by the map \( a_2 \to b_2 \). Then \( b_2 \to b \in \text{Cl}_{\beta N}(B) \). \( a \) and \( b \) are not
SUFFICIENT CONDITIONS FOR THE SET OF HAUSDORFF

Thus \( f_1(a) \neq f_2(b) \). However, \( (g \circ f_1)(a) = (g \circ f_2)(b) \). This is a contradiction. Hence (3) holds.

Let \( \mathcal{S}_a = \{ F \in \mathcal{G}_a | F = \{2n, 2n + 1\} \text{ for some } n \in \mathbb{N}\} \), \( G_a = \{ x \in \mathcal{M}_a | x \in F \text{ for some } F \in \mathcal{S}_a \} \). Let \( K_a = \{ x \in \mathcal{M}_a | x \in \bigcup_{i=1}^{n} \mathcal{S}_a \} \). Then \( \mathcal{M}_a = G_a \cup K_a \).

Using these notations, for \( \alpha_1 X, \alpha_2 X \in K(X) \), we write \( \mathcal{M}_a = G_a \cup K_a \). We want to show that \( \alpha_1 X \) and \( \alpha_2 X \) have a lower bound in \( K(X) \). Let \( \tau X \) be obtained by identifying subsets of \( \beta X \) of the form \( \{2n, 2n + 1\} \) to a point for each \( n \in \mathbb{N} \). It is clear that \( \tau X \in K(X) \). Let \( K = f_1(K_a \cup K_a) \). Obtain \( \alpha X \) by identifying \( K \) to a point, then \( \alpha X \in K(X) \). Each set in \( \mathcal{F}(\alpha X) \) is a subset of a set in \( \mathcal{F}(\alpha X) \), thus \( K(X) \) is a lattice by Theorem 2.2.

This example shows that the condition \( Cl_{\beta X}(\mathcal{M}_a) \subseteq \beta X \setminus X \) for every \( \alpha X \in K(X) \) in Lemma 3.3 is not necessary for \( K(X) \) to be a lattice.

REFERENCES


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