Pacific Journal of Mathematics

THE 2-CLASS GROUP OF BIQUADRATIC FIELDS. II

EZRA BROWN AND CHARLES JOHN PARRY

Vol. 78, No. 1 March 1978

PACIFIC JOURNAL OF MATHEMATICS
Vol. 78, No. 1, 1978

THE 2-CLASS GROUP OF BIQUADRATIC FIELDS, II

EZRA BROWN AND CHARLES J. PARRY

We describe methods for determining the exact power of 2 dividing the class number of certain cyclic biquadratic number fields. In a recent article, we developed a relative genus theory for cyclic biquadratic fields whose quadratic subfields have odd class number; we considered the case in which the quadratic subfield is $Q(\sqrt{l})$ with $l \equiv 5 \pmod 8$ a prime. Here we shall extend our methods to the cases in which the subfield is $Q(\sqrt{2})$ or $Q(\sqrt{l})$ with $l \equiv 1 \pmod 8$ a prime. We consider all such cases for which the 2-class group of the biquadratic field is of rank at most 3.

2. Notation and preliminaries.

Q: the field of rational numbers.
l: a rational prime satisfying $l = 2$ or $l \equiv 1 \pmod 8$.
p, q, p_i: rational primes.
k: the quadratic field $Q(\sqrt{l})$.
$\varepsilon = (u + v\sqrt{l})/2$, the fundamental unit of k, with $u, v > 0$.
m: a square-free positive rational integer, relatively prime to l.
$d = -m\sqrt{l}\,\varepsilon$.
K: the biquadratic field $k(\sqrt{d})$.
h, h_0: the class numbers of K and k, respectively.
$\left(\frac{x, y}{\pi}\right)$: the quadratic norm residue symbol over k.
$\left[\frac{\alpha}{\beta}\right]$: the quadratic residue symbol for k.
$\left(\frac{a}{b}\right)$: the rational quadratic residue (Legendre) symbol.
$\left(\frac{a}{b}\right)_4$: the rational 4th power residue symbol (defined if and only if $(a/b) = 1$).
$N(\ )$: the relative norm for K/k.
H: the 2-Sylow subgroup of the class group of K.

It is easy to see that K is a cyclic extension of Q of degree 4 which contains k. Recall that ε has (absolute) norm -1, that h_0 is odd and that H has rank $t - 1$, where t is the number of prime ideals of k which ramify in K.

3. Class number divisibility: The case $l \equiv 1 \pmod 8$.

THEOREM 1. *Let* $m = p \equiv 3 \pmod 4$. *Then*

$$h \equiv 2 \pmod 4) \quad \textit{if} \quad \left(\frac{p}{l}\right) = -1 ;$$
$$\equiv 4 \pmod 8 \quad \textit{if} \quad \left(\frac{p}{l}\right)_4 = -1 ;$$
$$\equiv 0 \pmod{16} \quad \textit{if} \quad \left(\frac{p}{l}\right)_4 = 1 .$$

Proof. The number t of prime ideals of k which ramify in K is equal to 2 or 3 according as $(p/l) = -1$ or 1. In the first case,

$$\left(\frac{p, d}{\sqrt{l}}\right) = \left[\frac{p}{\sqrt{l}}\right] = \left(\frac{p}{l}\right) = -1 ,$$

so that only the principal ambiguous class is in the principal genus. By Theorem 1 of [1] we have $H \simeq Z_2$.

If $(p/l) = 1$, then $p = \pi_1\pi_2$, where π_1 and π_2 are prime ideals of k. The ideals $\pi_1^{h_0}$ and $\pi_2^{h_0}$ are principal ideals, and

$$\pi_1^{h_0} = a + b\sqrt{l} > 0 ,$$
$$\pi_2^{h_0} = a - b\sqrt{l} > 0 .$$

Thus,
$$\left(\frac{a + b\sqrt{l}, d}{\sqrt{l}}\right) = \left[\frac{a + b\sqrt{l}}{\sqrt{l}}\right] = \left(\frac{a}{l}\right)$$
$$= \left(\frac{a^2}{l}\right)_4 = \left(\frac{p}{l}\right)_4 .$$

Also,
$$\left(\frac{a + b\sqrt{l}, d}{\pi_2}\right) = \left[\frac{a + b\sqrt{l}}{\pi_2}\right] = \left(\frac{2a}{p}\right) .$$

Because $p \equiv 3 \pmod 4$ and h_0 is odd, a is even; if $a = 2^i c$ with c odd, then $i = 1$ if and only if $p \equiv 3 \pmod 8$. Thus,

$$\left(\frac{2a}{p}\right) = \left(\frac{2}{p}\right)^{i+1}\left(\frac{c}{p}\right) = \left(\frac{c}{p}\right) = \left(\frac{-p}{c}\right)$$
$$= \left(\frac{l}{c}\right) = \left(\frac{c}{l}\right) = \left(\frac{c^2}{l}\right)_4 = \left(\frac{a^2}{l}\right)_4 = \left(\frac{p}{l}\right)_4 .$$

We then have the following table of characters:

Norm\Character	$\sqrt{l}$	π_1	π_2
$\varepsilon\sqrt{l}$	1	$\left(\frac{p}{l}\right)_4$	$\left(\frac{p}{l}\right)_4$
$a + b\sqrt{l}$	$\left(\frac{p}{l}\right)_4$	1	$\left(\frac{p}{l}\right)_4$
$a - b\sqrt{l}$	$\left(\frac{p}{l}\right)_4$	$\left(\frac{p}{l}\right)_4$	1

If $(p/l)_4 = -1$, then only the principal ambiguous class is in the principal genus; by Theorem 1 of [1], we have $H \simeq Z_2 \times Z_2$, so that $h \equiv 4 \pmod 8$.

If $(p/l)_4 = 1$, then all four ambiguous classes are in the principal genus, so that $h \equiv 0 \pmod{16}$.

THEOREM 2. *Let* $m = p_1 p_2 \cdots p_t \equiv 3 \pmod 4$ *with* $(p_i/l) = -1$ *for all* i. *Then*

$$h \equiv 2^t \pmod{2^{t+1}} .$$

Proof. H has rank t, so we just need to show that the only ambiguous class in the principal genus is the principal class. Now

$$\left(\frac{p_i, d}{\sqrt{l}}\right) = \left[\frac{p_i}{\sqrt{l}}\right] = \left(\frac{p_i}{l}\right) = -1 , \quad \text{and}$$
$$\left(\frac{p_i, d}{p_j}\right) = \left[\frac{p_i}{p_j}\right] = 1 \quad \text{for} \quad i \neq j .$$

It follows that $(p_i, d/p_i) = -1$ and $(\varepsilon\sqrt{l}, d/p_i) = -1$, by the product rule. Thus, no two of the ramified prime ideals belong to the same genus, and so the desired result follows.

THEOREM 3. *Let* $m = pq \equiv 3 \pmod 4$ *with* $(p/l) = 1$ *and* $(q/l) = -1$. *Then*

$$h \equiv 8 \pmod{16} \quad \text{if} \quad \left(\frac{p}{l}\right)_4 \neq \left(\frac{q}{p}\right) ;$$
$$\equiv 16 \pmod{32} \quad \text{if} \quad p \equiv 1 \pmod 4 \quad \text{and} \quad \left(\frac{p}{l}\right)_4 = \left(\frac{q}{p}\right) \neq \left(\frac{l}{p}\right)_4 ;$$
$$\equiv 0 \pmod{32} \quad \text{if either} \quad p \equiv 3 \pmod 4 \quad \text{and} \quad \left(\frac{p}{l}\right)_4 = \left(\frac{q}{p}\right) ,$$
$$\text{or} \quad p \equiv 1 \pmod 4 \quad \text{and} \quad \left(\frac{p}{l}\right)_4 = \left(\frac{q}{p}\right) = \left(\frac{l}{p}\right)_4 .$$

Proof. Here H has rank 3. Using the notation of Theorem 1, we have that

$$\left(\frac{a + b\sqrt{l}, d}{\pi_2}\right) = \left[\frac{a + b\sqrt{l}}{\pi_2}\right] = \left[\frac{2a}{\pi_2}\right] = \left(\frac{2a}{p}\right) .$$

If $p \equiv 3 \pmod 4$, then $(2a/p) = (p/l)_4$, as before. However, if $p \equiv 1 \pmod 4$, then

$$\left(\frac{2a}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{a}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{a^2}{p}\right)_4 = \left(\frac{2}{p}\right)\left(\frac{b}{p}\right)\left(\frac{l}{p}\right)_4 .$$

Now $b = 2^i c$ with c odd; furthermore, $i = 1$ if and only if $p \equiv 5 \pmod 8$. Hence,

$$\left(\frac{2}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{2}{p}\right)^{i+1}\left(\frac{c}{p}\right) = \left(\frac{c}{p}\right) = \left(\frac{p}{c}\right) = \left(\frac{a^2}{c}\right) = 1\,;$$

we deduce that $(2a/p) = (l/p)_4$. Furthermore,

$$\left(\frac{a + b\sqrt{l}, d}{q}\right) = \left[\frac{a + b\sqrt{l}}{q}\right] = \left(\frac{p}{q}\right) = \left(\frac{q}{p}\right), \quad \text{and}$$

$$\left(\frac{q, d}{\pi_1}\right) = \left[\frac{q}{\pi_1}\right] = \left(\frac{q}{p}\right).$$

The remaining characters are easily evaluated; if we set $(l/p)_4 = (p/l)_4$ if $p \equiv 3 \pmod 4$, we have the following table of characters:

Norm\Character	$\sqrt{l}$	q	π_1	π_2
$\varepsilon\sqrt{l}$	-1	-1	$\left(\frac{p}{l}\right)_4$	$\left(\frac{p}{l}\right)_4$
q	-1	-1	$\left(\frac{q}{p}\right)$	$\left(\frac{q}{p}\right)$
$a + b\sqrt{l}$	$\left(\frac{p}{l}\right)_4$	$\left(\frac{q}{p}\right)$	$\left(\frac{q}{p}\right)\left(\frac{p}{l}\right)_4\left(\frac{l}{p}\right)_4$	$\left(\frac{l}{p}\right)_4$
$a - b\sqrt{l}$	$\left(\frac{p}{l}\right)_4$	$\left(\frac{q}{p}\right)$	$\left(\frac{l}{p}\right)_4$	$\left(\frac{q}{p}\right)\left(\frac{p}{l}\right)_4\left(\frac{l}{p}\right)_4$

The theorem follows, as before, from an analysis of the various cases.

THEOREM 4. *Let* $m = p \equiv 1 \pmod 4$ *with* $(p/l) = -1$. *Then*

$$h \equiv 8 \pmod{16} \quad \text{if} \quad \left(\frac{2}{l}\right)_4 \neq \left(\frac{2}{p}\right);$$

$$\equiv 16 \pmod{32} \quad \text{if} \quad \left(\frac{2}{l}\right)_4 = \left(\frac{2}{p}\right) = (-1)^{(l+7)/8};$$

$$\equiv 0 \pmod{32} \quad \text{if} \quad \left(\frac{2}{l}\right)_4 = \left(\frac{2}{p}\right) = (-1)^{(l-1)/8}.$$

Proof. Here, the two prime divisors of 2 in k ramify in K. Put $2 = 2_1 2_2$ in k, with

$$2_1^{h_0} = \alpha = \frac{a + b\sqrt{l}}{2} > 0\,,$$

and

$$2_1^{h_0} = \bar{\alpha} = \frac{a - b\sqrt{l}}{2} > 0 .$$

Then

$$\left(\frac{\alpha, d}{\sqrt{l}}\right) = \left[\frac{\alpha}{\sqrt{l}}\right] = \left[\frac{a/2}{\sqrt{l}}\right] = \left(\frac{2a}{l}\right)$$
$$= \left(\frac{4a^2}{l}\right)_4 = \left(\frac{2}{l}\right)_4 ,$$
$$\left(\frac{\alpha, d}{p}\right) = \left[\frac{\alpha}{p}\right] = \left(\frac{2}{p}\right) , \quad \text{and}$$
$$\left(\frac{p, d}{2_1}\right) = (-1)^{(p-1)/2} = 1. \quad \text{Now}$$
$$\left[\frac{a + b\sqrt{l}}{2}\right]^2 = \frac{1}{2}(a^2 - 2^{h_0+1} + ab\sqrt{l}) , \quad \text{so that}$$
$$a\bar{\alpha} \equiv \frac{1}{2}(a^2 - ab\sqrt{l}) \equiv a^2 - 2^{h_0} \pmod{2_1^2} . \quad \text{Thus,}$$
$$\left(\frac{\bar{\alpha}, d}{2_1}\right) = \left(\frac{a, d}{2_1}\right)\left(\frac{a^2 - 2^{h_0}, d}{2_1}\right)$$
$$= (-1)^{(a-1)/2}(-1)^{(a^2-2^{h_0}-1)/2}$$
$$= \left(\frac{-1}{a}\right)(-1)^{2^{h_0-1}} .$$

To evaluate $(-1/a)$, note that

$$\left(\frac{a}{l}\right) = \left(\frac{a^2}{l}\right)_4 = \left(\frac{2}{l}\right)_4$$

and

$$\left(\frac{2}{a}\right) = \left(\frac{-l}{a}\right) = \left(\frac{-1}{a}\right)\left(\frac{l}{a}\right) = \left(\frac{-1}{a}\right)\left(\frac{a}{l}\right) .$$

Hence,

$$\left(\frac{-1}{a}\right) = \left(\frac{2}{a}\right)\left(\frac{a}{l}\right) = \left(\frac{2}{a}\right)\left(\frac{2}{l}\right)_4 .$$

Since $(2/b) = 1$, we have $b^2 \equiv 1 \pmod{16}$, so that

$$a^2 - lb^2 \equiv a^2 - l \equiv 2^{h_0+2} \pmod{16} .$$

If $h_0 = 1$, then $a^2 \equiv l + 8 \pmod{16}$, so that

$$\left(\frac{2}{a}\right) = 1 \quad \text{if and only if} \quad l \equiv 9 \pmod{16} ;$$

if $h_0 > 1$, then $a^2 \equiv l \pmod{16}$, so that

$$\left(\frac{2}{a}\right) = 1 \quad \text{if and only if} \quad l \equiv 1 \pmod{16}\,.$$

In either case,

$$\left(\frac{\bar{\alpha}, d}{2_1}\right) = (-1)^{2^{h_0-1}}\left(\frac{-1}{a}\right) = (-1)^{(l-1)/8}\left(\frac{2}{l}\right)_4\,.$$

Finally, we note that

$$\left(\frac{p, d}{\sqrt{l}}\right) = \left(\frac{p, d}{p}\right) = -1\,.$$

This yields the following table of generic characters:

Norm\Characters	$\sqrt{l}$	p	2_1	2_2
p	-1	-1	$+1$	$+1$
α	$\left(\frac{2}{l}\right)_4$	$\left(\frac{2}{p}\right)$	$(-1)^{(l-1)/8}\left(\frac{2}{p}\right)$	$(-1)^{(l-1)/8}\left(\frac{2}{l}\right)_4$
$\bar{\alpha}$	$\left(\frac{2}{l}\right)_4$	$\left(\frac{2}{p}\right)$	$(-1)^{(l-1)/8}\left(\frac{2}{l}\right)_4$	$(-1)^{(l-1)/8}\left(\frac{2}{p}\right)$

If $(2/l)_4 \neq (2/p)$, then all three lines of the table are distinct and only the principal ambiguous class lies in the principal genus; this implies that $h \equiv 8 \pmod{16}$.

If $(2/l)_4 = (2/p) \neq (-1)^{(l-1)/8}$, then the last two lines are identical, but different from the first. Here, exactly two ambiguous classes lie in the principal genus, and so $h \equiv 16 \pmod{32}$.

In the case $(2/l)_4 = (2/p) = (-1)^{(l-1)/8}$, there are 4 ambiguous classes in the principal genus. Thus $h \equiv 0 \pmod{32}$.

COROLLARY. *If* $m = 1$, *then*

$$\begin{aligned} h &\equiv 4 \pmod 8 \quad \textit{if} \quad \left(\frac{2}{l}\right)_4 = -1\,; \\ &\equiv 8 \pmod{16} \quad \textit{if} \quad l \equiv 9 \pmod{16} \quad \textit{and} \quad \left(\frac{2}{l}\right)_4 = 1\,; \\ &\equiv 0 \pmod{16} \quad \textit{if} \quad l \equiv 1 \pmod{16} \quad \textit{and} \quad \left(\frac{2}{l}\right)_4 = 1\,. \end{aligned}$$

Proof. Here $t = 3$ and so H has rank 2. The table of generic characters is obtained by setting $(2/p) = 1$ in the last two lines of the table in Theorem 4. There are 1, 2 or 4 ambiguous classes in the principal genus according as the condition of the first, second or third line of the corollary holds.

THEOREM 5. *If $m = 2$, then*

$$h \equiv 4 \pmod 8)\,, \quad \text{if } \left(\frac{2}{l}\right)_4 = -1\,;$$
$$\equiv 0 \pmod{16}\,, \quad \text{if } \left(\frac{2}{l}\right)_4 = 1\,.$$

Proof. Using the notation of the preceding theorem, we have

$$\left(\frac{\bar{\alpha}, d}{2_1}\right) = \left(\frac{\bar{\alpha}, -2\varepsilon\sqrt{l}}{2_1}\right) = \left(\frac{\bar{\alpha}, 2}{2_1}\right)\left(\frac{\bar{\alpha}, -\varepsilon\sqrt{l}}{2_1}\right)$$
$$= \left(\frac{\bar{\alpha}, 2}{2_1}\right)(-1)^{(l-1)/8}\left(\frac{2}{l}\right)_4\,,$$

the last step following from the calculations of Theorem 4. Now

$$\alpha^3 = \left(\frac{a + b\sqrt{l}}{2}\right)^3 = \left(\frac{1}{2}\right)(a(a^2 - 3\cdot 2^{h_0}) + b(a^2 - 2^{h_0})\sqrt{l}\,)\,,$$

so that

$$\left(\frac{\bar{\alpha}, 2}{2_1}\right) = \left(\frac{a^2 - 2^{h_0}, 2}{2_1}\right)\left(\frac{a(a^2 - 2^{h_0+1}), 2}{2_1}\right)$$
$$= \left(\frac{2}{a^2 - 2^{h_0}}\right)\left(\frac{2}{a}\right)\left(\frac{2}{a^2 - 2^{h_0+1}}\right)$$
$$= (-1)^{2^{h_0-1}}\left(\frac{2}{a}\right) = (-1)^{(l-1)/8}\,.$$

Hence,

$$\left(\frac{\bar{\alpha}, d}{2_1}\right) = (-1)^{(l-1)/8}(-1)^{(l-1)/8}\left(\frac{2}{l}\right)_4 = \left(\frac{2}{l}\right)_4\,.$$

We obtain the following table of characters and the result follows by considerations similar to those previously mentioned:

Norm\Character	$\sqrt{l}$	2_1	2_2
$\varepsilon\sqrt{l}$	1	$\left(\frac{2}{l}\right)_4$	$\left(\frac{2}{l}\right)_4$
α	$\left(\frac{2}{l}\right)_4$	1	$\left(\frac{2}{l}\right)_4$
$\bar{\alpha}$	$\left(\frac{2}{l}\right)_4$	$\left(\frac{2}{l}\right)_4$	1

THEOREM 6. *If $m = 2p$ with $(p/l) = -1$, then*

$$\begin{aligned} h &\equiv 8 \pmod{16} \quad \textit{if} \quad \left(\frac{2}{l}\right)_4 \neq \left(\frac{2}{p}\right); \\ &\equiv 16 \pmod{32} \quad \textit{if} \quad \left(\frac{2}{l}\right)_4 = \left(\frac{2}{p}\right) \neq (-1)^{(l-1)/8}, \\ &\qquad \textit{and } p \equiv 3 \pmod 8; \\ &\equiv 0 \pmod{32}, \quad \textit{otherwise}. \end{aligned}$$

Proof. First we note that

$$\left(\frac{\bar\alpha, d}{2_1}\right) = \left(\frac{\bar\alpha, -2p\varepsilon\sqrt{l}}{2_1}\right) = \left(\frac{\bar\alpha, 2}{2_1}\right)\left(\frac{\bar\alpha, -\varepsilon p\sqrt{l}}{2_1}\right)$$
$$= (-1)^{(l-1)/8}\left(\frac{\bar\alpha, -\varepsilon p\sqrt{l}}{2_1}\right).$$

If $p \equiv 1 \pmod 4$, then the last symbol was evaluated in the proof of Theorem 4 and reduces to $(-1)^{(l-1)/8}(2/l)_4$.

If $p \equiv 3 \pmod 4$, then 2_1 is unramified in the extension $Q(\sqrt{d_1})$, where $d_1 = -\varepsilon p\sqrt{l}$. Thus, the last symbol is equal to 1. Hence

$$\left(\frac{\bar\alpha, d}{2_1}\right) = \left(\frac{\alpha, d}{2_2}\right) = \left(\frac{2}{l}\right)_4 \quad \text{or} \quad (-1)^{(l-1)/8}$$

according as $p \equiv 1$ or $3 \pmod 4$. Evaluation of the remaining symbols is routine, and we have the following table for $p \equiv 3 \pmod 4$:

Norm\Character	$\sqrt{l}$	p	2_1	2_2
$\varepsilon\sqrt{l}$	-1	-1	$\left(\frac{2}{l}\right)_4$	$\left(\frac{2}{l}\right)_4$
p	-1	-1	$\left(\frac{2}{p}\right)$	$\left(\frac{2}{p}\right)$
α	$\left(\frac{2}{l}\right)_4$	$\left(\frac{2}{p}\right)$	$(-1)^{(l-1)/8}\left(\frac{2}{p}\right)\left(\frac{2}{l}\right)_4$	$(-1)^{(l-1)/8}$
$\bar\alpha$	$\left(\frac{2}{l}\right)_4$	$\left(\frac{2}{p}\right)$	$(-1)^{(l-1)/8}$	$(-1)^{(l-1)/8}\left(\frac{2}{p}\right)\left(\frac{2}{l}\right)_4$

If $p \equiv 1 \pmod 4$, the four entries in the lower right-hand corner are replaced by

$$\begin{matrix} \left(\frac{2}{p}\right) & \left(\frac{2}{l}\right)_4 \\ \left(\frac{2}{l}\right)_4 & \left(\frac{2}{p}\right) \end{matrix}$$

and the desired results follow as before.

4. Class numbers divisibility: The case $l = 2$.

THEOREM 7. *If* $m = p$, *then*

$$\begin{aligned} h &\equiv 2 \pmod 4, \quad \textit{if} \quad p \equiv \pm 3 \pmod 8; \\ &\equiv 4 \pmod 8, \quad \textit{if} \quad p \equiv \pm 7 \pmod{16}; \\ &\equiv 8 \pmod{16}, \quad \textit{if} \quad p \equiv 1 \pmod{16} \quad \textit{and} \quad \left(\frac{2}{p}\right)_4 = -1; \\ &\equiv 0 \pmod{16}, \quad \textit{if} \quad p \equiv 1 \pmod{16} \quad \textit{and} \quad \left(\frac{2}{p}\right)_4 = 1, \quad \textit{or} \\ &\qquad\qquad\qquad \textit{if} \quad p \equiv 15 \pmod{16}. \end{aligned}$$

Proof. If $p \equiv \pm 3 \pmod 8$ then H is cyclic and

$$\left(\frac{p, d}{\sqrt{2}}\right) = \left(\frac{2}{p}\right) = -1.$$

Hence, the only ambiguous class in the principal genus is the principal class, and so $H \simeq Z_2$.

If $p \equiv \pm 1 \pmod 8$ then H has rank 2. Let $p = \pi_1\pi_2 = (a + b\sqrt{2})(a - b\sqrt{2})$ with $\pi_1 = a + b\sqrt{2} > 0$. If $p \equiv 7 \pmod 8$, then

$$\begin{aligned} \left(\frac{\pi_1, d}{\pi_2}\right) &= \left[\frac{\pi_1}{\pi_2}\right] = \left[\frac{2a}{\pi_2}\right] = \left(\frac{2a}{p}\right) = \left(\frac{a}{p}\right) \\ &= \left(\frac{-1}{a}\right)\left(\frac{p}{a}\right) = \left(\frac{-1}{a}\right)\left(\frac{-2b^2}{a}\right) \\ &= \left(\frac{2}{a}\right) = (-1)^{(a^2-1)/8} = (-1)^{(p+2b^2-1)/8} \\ &= (-1)^{(p+1)/8}, \end{aligned}$$

since b must be odd. Furthermore,

$$b\varepsilon\sqrt{2} = 2b + b\sqrt{2} \equiv 2b - a \pmod{\pi_1},$$

so that

$$b^2\varepsilon\sqrt{2} \equiv 2b^2 - ab \equiv a^2 - ab \equiv a(a - b) \pmod{\pi_1}.$$

Thus,

$$\left(\frac{\varepsilon\sqrt{2}, d}{\pi_1}\right) = \left[\frac{\varepsilon\sqrt{2}}{\pi_1}\right] = \left(\frac{a(a-b)}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{a-b}{p}\right).$$

But $(a - b)(a + b) = a^2 - b^2 = p + b^2$, so if $a - b = 2^t c$ with c odd, we have

$$\left(\frac{a-b}{p}\right)=\left(\frac{2}{p}\right)^i\left(\frac{c}{p}\right)=\left(\frac{c}{p}\right)=\left(\frac{-p}{c}\right)=\left(\frac{b^2}{c}\right)=1\,.$$

Hence,

$$\left(\frac{\varepsilon\sqrt{2}\,,d}{\pi_1}\right)=\left(\frac{a}{p}\right)=(-1)^{(p+1)/8}\,.$$

Thus, for $p\equiv 7 \pmod 8$, we have the following table of generic characters:

Norm\Character	$\sqrt{2}$	π_1	π_2
$\varepsilon\sqrt{2}$	1	$(-1)^{(p+1)/8}$	$(-1)^{(p+1)/8}$
π_1	$(-1)^{(p+1)/8}$	1	$(-1)^{(p+1)/8}$
π_2	$(-1)^{(p+1)/8}$	$(-1)^{(p+1)/8}$	1

If $p\equiv 7 \pmod{16}$, then none of the above lines are the same, so that $h\equiv 4 \pmod 8$; if $p\equiv 15 \pmod{16}$, then all of the above lines are the same, so that $h\equiv 0 \pmod{16}$.

Now let $p\equiv 1 \pmod 8$. Then

$$\left(\frac{\pi_1, d}{\pi_2}\right)=\left(\frac{a}{p}\right)=\left(\frac{a^2}{p}\right)_4=\left(\frac{2b^2}{p}\right)_4$$
$$=\left(\frac{2}{p}\right)_4\left(\frac{b}{p}\right)\,.$$

Setting $b=2^ic$ with c odd, we have

$$\left(\frac{b}{p}\right)=\left(\frac{2}{p}\right)^i\left(\frac{c}{p}\right)=\left(\frac{c}{p}\right)=\left(\frac{p}{c}\right)=\left(\frac{a^2}{c}\right)=1\,.$$

Hence,

$$\left(\frac{\pi_1, d}{\pi_2}\right)=\left(\frac{\pi_2, d}{\pi_1}\right)=\left(\frac{2}{p}\right)_4\,.$$

Now

$$\left(\frac{\varepsilon\sqrt{2}\,, d}{\pi_2}\right)=\left(\frac{a}{p}\right)\left(\frac{a-b}{p}\right)=\left(\frac{2}{p}\right)_4\left(\frac{a-b}{p}\right)\,.$$

Since $(a-b)(a+b)=p+b^2$, we have

$$\left(\frac{a-b}{p}\right)=\left(\frac{p}{a-b}\right)=\left(\frac{-b^2}{a-b}\right)=\left(\frac{-1}{a-b}\right)\,.$$

A paper of G. Pall [2] contains a table, part of which we reproduce here:

$$p = a^2 - 2b^2 = u^2 + v^2 , \quad v \text{ even}$$

$p \pmod{16}$	$v \pmod 8$	$a \pmod 8$	$b \pmod 4$
1	4	7	0
1	4	5	2
1	0	3	2
1	0	1	0
9	0	1	2
9	0	3	0
9	4	5	0
9	4	7	2

Thus, if $p \equiv 1 \pmod{16}$, then $(-1/(a-b)) = 1$ if and only if $v \equiv 0 \pmod 8$, and if $p \equiv 9 \pmod{16}$, then $(-1/(a-b)) = 1$ if and only if $v \equiv 4 \pmod 8$, so

$$\left(\frac{-1}{a-b}\right) = (-1)^{v/4}(-1)^{(p-1)/8} .$$

Now, Dirichlet's necessary and sufficient condition that $(2/p)_4 = 1$ is that $v \equiv 0 \pmod 8$. Hence, $(2/p)_4 = (-1)^{v/4}$;

$$\begin{aligned}\left(\frac{\varepsilon\sqrt{2}\,, d}{\pi_1}\right) &= \left(\frac{a}{p}\right)\left(\frac{a-b}{p}\right) = \left(\frac{2}{p}\right)_4\left(\frac{-1}{a-b}\right) \\ &= \left(\frac{2}{p}\right)_4(-1)^{v/4}(-1)^{(p-1)/8} \\ &= \left(\frac{2}{p}\right)_4\left(\frac{2}{p}\right)_4(-1)^{(p-1)/8} = (-1)^{(p-1)/8} .\end{aligned}$$

We thus have the following table:

Norm\Character	$\sqrt{2}$	π_1	π_2
$\varepsilon\sqrt{2}$	1	$(-1)^{(p-1)/8}$	$(-1)^{(p-1)/8}$
π_1	$(-1)^{(p-1)/8}$	$(-1)^{(p-1)/8}\left(\frac{2}{p}\right)_4$	$\left(\frac{2}{p}\right)_4$
π_2	$(-1)^{(p-1)/8}$	$\left(\frac{2}{p}\right)_4$	$(-1)^{(p-1)/8}\left(\frac{2}{p}\right)_4$

If $p \equiv 9 \pmod{16}$, then each line is different; thus, only the principal ambiguous class belongs to the principal genus, and so $H \simeq Z_2 \times Z_2$, $h \equiv 4 \pmod 8$.

If $p \equiv 1 \pmod{16}$, then there are either two or four ambiguous classes in the principal genus, according as $(2/p)_4 = -1$ or 1. In these cases, $h \equiv 8$ or $0 \pmod{16}$, respectively.

THEOREM 8. *If $m = p_1 \cdots p_t$ with $(2/p_i) = -1$ for all i, then*

$$h \equiv 2^t \pmod{2^{t+1}} .$$

Comment. The proof is quite similar to the proof of Theorem 2, so we omit it.

THEOREM 9. *Let $m = pq$ with $(2/p) = 1$ and $(2/q) = -1$. If $p \equiv 1 \pmod 8$, then*

$$\begin{aligned} h &\equiv 8 \pmod{16}, \quad \text{if } \left(\frac{p}{q}\right) \neq (-1)^{(p-1)/8} ; \\ &\equiv 16 \pmod{32}, \quad \text{if } \left(\frac{2}{p}\right)_4 \neq (-1)^{(p-1)/8} = \left(\frac{p}{q}\right) ; \\ &\equiv 0 \pmod{32}, \quad \text{otherwise} . \end{aligned}$$

If $p \equiv 7 \pmod 8$, then

$$\begin{aligned} h &\equiv 8 \pmod{16}, \quad \text{if } \left(\frac{p}{q}\right) \neq (-1)^{(p+1)/8} ; \\ &\equiv 16 \pmod{32}, \quad \text{if } q \equiv 3 \pmod 4 \text{ and } \left(\frac{p}{q}\right) = (-1)^{(p+1)/8} = -1 ; \\ &\equiv 0 \pmod{32}, \quad \text{otherwise} . \end{aligned}$$

Comment. The proof involves straightforward extensions of the tables, constructed in the proof of Theorem 7, so we will omit it.

5. **Numerical results.** A slight modification of the methods described in [3] allow us to compute the relative class number $h^* = h/h_0$ of K. As $h_0 = 1$ for most small values of l, we have $h^* = h$ for almost all values within the range of our computations. In the tables below we list all fields within the range of our calculations, where the maximum power of dividing h^* exceeds the power predicted in §3. We have only computed values of h^* for the fields discussed in Theorems 1, 4, 5, 6, and 7. The column of the table headed by f gives the prime factorization of h^*.

Table 1				Table 1 (con't)			
($d = -\varepsilon\sqrt{l}\ p,\ p \equiv 3 \bmod 4$)				($d = -\varepsilon\sqrt{l}\ p,\ p \equiv 3 \bmod 4$)			
l	p	h^*	f	l	p	h^*	f
17	67	160	$2^5 \cdot 5$	73	71	640	$2^7 \cdot 5$
	103	32	$2^5 \cdot$	89	67	128	2^7
	251	1088	$2^6 \cdot 17$	97	47	64	2^6
	463	160	$2^5 \cdot 5$		103	544	$2^5 \cdot 17$
41	23	32	2^5	113	7	160	$2^5 \cdot 5$
	59	288	$2^5 \cdot 9$	193	3	160	$2^5 \cdot 5$
	83	1184	$2^5 \cdot 37$		47	576	$2^6 \cdot 3^2$
	139	832	$2^6 \cdot 13$	233	71	5696	$2^6 \cdot 89$
	163	1312	$2^5 \cdot 41$		107	800	$2^5 \cdot 5^2$
	223	256	2^8	257*	11	64	2^6
	271	160	$2^5 \cdot 5$		23	640	$2^6 \cdot 5$
	283	3328	$2^8 \cdot 13$		67	416	$2^5 \cdot 13$
	379	2080	$2^5 \cdot 5 \cdot 13$	281	59	160	$2^5 \cdot 5$
	491	2592	$2^5 \cdot 3^4$				

(*) $h_0 = 3$ when $l = 257$.

Table 2				Table 2 (con't)			
($d = -\varepsilon\sqrt{l}\ p,\ p \equiv 1 \bmod 4$)				($d = -\varepsilon\sqrt{l}\ p,\ p \equiv 1 \bmod 4$)			
l	p	h^*	f	l	p	h^*	f
17	149	320	$2^6 \cdot 5$	41	173	1856	$2^6 \cdot 29$
	157	512	2^9		181	1088	$2^6 \cdot 17$
	229	640	$2^7 \cdot 5$		197	2048	2^{11}
	293	640	$2^7 \cdot 5$		229	1600	$2^6 \cdot 5^2$
	353	1024	2^{10}		269	1600	$2^6 \cdot 5^2$
	389	1600	$2^6 \cdot 5^2$		293	3200	$2^7 \cdot 5^2$
	409	832	$2^6 \cdot 13$		373	4096	2^{12}
41	53	832	$2^6 \cdot 13$		389	2176	$2^7 \cdot 17$
	61	320	$2^6 \cdot 5$		433	5248	$2^7 \cdot 41$
	109	576	$2^6 \cdot 3^2$	73	41	320	$2^6 \cdot 5$

Table 2 (con't)

($d = -\varepsilon\sqrt{l}\ p,\ p \equiv 1 \bmod 4$)

l	p	h^*	f
78	89	512	2^9
	109	2368	$2^6\cdot 37$
89	73	2560	$2^9\cdot 5$
	97	2560	$2^9\cdot 5$
97	53	512	2^9
	101	832	$2^6\cdot 13$
	109	3904	$2^6\cdot 61$
113	17	320	$2^6\cdot 5$
	41	1088	$2^6\cdot 17$
	53	832	$2^6\cdot 13$
	73	1600	$2^6\cdot 5^2$
	89	3712	$2^7\cdot 29$
	97	4352	$2^8\cdot 17$
	109	1664	$2^7\cdot 13$
137	5	128	2^7
	53	1664	$2^7\cdot 13$

Table 2 (con't)

($d = -\varepsilon\sqrt{l}\ p,\ p \equiv 1 \bmod 4$)

l	p	h^*	f
137	73	1280	$2^8\cdot 5$
	109	3136	$2^6\cdot 7^2$
193	101	10816	$2^6\cdot 13^2$
233	29	1280	$2^8\cdot 5$
	37	2304	$2^8\cdot 3^2$
241	5	128	2^7
	61	4608	$2^9\cdot 3^2$
	97	16000	$2^7\cdot 5^3$
257	17	832	$2^6\cdot 13$
	41	2560	$2^9\cdot 5$
	73	3200	$2^7\cdot 5^2$
	89	4672	$2^6\cdot 73$
281	29	1600	$2^6\cdot 5^2$
	101	2176	$2^7\cdot 17$
	109	6400	$2^8\cdot 5^2$

Note: For tables 1 and 2, $p < 500$ when $l = 17$ or 41 and $p < 110$ otherwise.

Table 3

($d = -m\varepsilon\sqrt{l},\ m = 1$ or 2)

l	m	h^*	f
257	1	32	2^5
337	1	256	2^8
89	2	64	2^6
113	2	32	2^5
233	2	128	2^7

Table 4

$(d = -2\varepsilon\sqrt{l}\, p)$

l	p	h^*	f
17	5	32	2^5
	37	320	$2^6 \cdot 5$
	47	320	$2^6 \cdot 5$
	61	256	2^8
41	3	32	2^5
	11	256	2^8
	13	128	2^7
	19	512	2^9
	23	256	2^8
	31	640	$2^7 \cdot 5$
	53	576	$2^6 \cdot 3^2$
	67	512	2^9
73	17	832	$2^6 \cdot 13$
	37	576	$2^6 \cdot 3^2$
	41	3200	$2^7 \cdot 5^2$
	71	4352	$2^8 \cdot 17$
89	11	512	2^9
	17	320	$2^6 \cdot 5$
	67	1600	$2^6 \cdot 5^2$
	73	1600	$2^6 \cdot 5^2$
97	5	320	$2^6 \cdot 5$
	13	320	$2^6 \cdot 5$
	47	3200	$2^7 \cdot 5^2$

Table 4 (con't)

$(d = -2\varepsilon\sqrt{l}\, p)$

l	p	h^*	f
113	7	320	$2^6 \cdot 5$
	23	640	$2^7 \cdot 5$
	31	1152	$2^7 \cdot 3^2$
	41	2368	$2^6 \cdot 3^7$
	53	1600	$2^6 \cdot 5^2$
	71	1664	$2^7 \cdot 13$
	73	3712	$2^7 \cdot 29$
137	13	512	2^9
	43	2624	$2^6 \cdot 41$
	67	3904	$2^6 \cdot 61$
	73	3904	$2^6 \cdot 61$
193	5	320	$2^6 \cdot 5$
	7	1152	$2^7 \cdot 3^2$
	13	3328	$2^8 \cdot 13$
	37	3392	$2^7 \cdot 53$
	53	1664	$2^7 \cdot 13$
	61	11072	$2^6 \cdot 173$
233	19	1280	$2^8 \cdot 5$
	23	3328	$2^8 \cdot 13$
	37	3712	$2^7 \cdot 29$
	71	5248	$2^7 \cdot 41$
	73	3328	$2^8 \cdot 13$

Table 5

$(d = -\varepsilon\sqrt{2}\, p)$

p	h^*	f
47	32	2^5
127	160	$2^5 \cdot 5$
223	160	$2^5 \cdot 5$

Table 5 (con't)

$(d = -\varepsilon\sqrt{2}\, p)$

p	h^*	f
239	320	$2^6 \cdot 5$
257	160	$2^5 \cdot 5$
271	160	$2^5 \cdot 5$

Table 5 (con't)			Table 5 (con't)		
($d = -\varepsilon\sqrt{2}\,p$)			($d = -\varepsilon\sqrt{2}\,p$)		
p	h^*	f	p	h^*	f
367	160	$2^5 \cdot 5$	1279	640	$2^7 \cdot 5$
431	320	$2^6 \cdot 5$	1423	1088	$2^6 \cdot 17$
463	640	$2^7 \cdot 5$	1439	1600	$2^6 \cdot 5^2$
479	160	$2^5 \cdot 5$	1553	800	$2^5 \cdot 5^2$
577	416	$2^5 \cdot 13$	1601	640	$2^7 \cdot 5$
751	576	$2^6 \cdot 3^2$	1663	1088	$2^6 \cdot 17$
1039	800	$2^5 \cdot 5^2$	1759	1664	$2^7 \cdot 13$
1151	640	$2^7 \cdot 5$	1823	1184	$2^5 \cdot 5 \cdot 17$
1153	544	$2^5 \cdot 17$	1889	1184	$2^5 \cdot 37$
1201	1088	$2^6 \cdot 17$	1951	1312	$2^5 \cdot 41$
1217	512	2^9			

REFERENCES

1. Ezra Brown and Charles J. Parry, *The 2-class group of certain biquadratic number fields*, J. reine und angew, Math., **295** (1977), 61–71.

2. Gordon Pall, *Discriminantal divisors of binary quadratic forms*, J. Number Theory, **1** (1969), 525–533.

3. Charles J. Parry, *Real quadratic fields with class number divisible by* 5, Math. of Comp., **31** (1977), 1019-1029.

Received February 23, 1977.

VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY
BLACKSBURG, VA 24061

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

C. W. CURTIS
University of Oregon
Eugene, OR 97403

C. C. MOORE
University of California
Berkeley, CA 94720

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. FINN AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH B. H. NEUMANN F. WOLF K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 78, No. 1 March, 1978