Pacific Journal of Mathematics

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Vol. 78, No. 1 March 1978

PACIFIC JOURNAL OF MATHEMATICS
Vol. 78, No. 1, 1978

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This is a study of the manifold structure of the focal set of an immersed submanifold in a real space form $\tilde{M}$. A typical result is the following:

THEOREM. *Let M be an orientable (immersed) hypersurface in $\tilde{M}$ which is complete with respect to the induced metric. Let λ be a differentiable principal curvature of constant multiplicity $\nu > 1$ on M. Then the focal map f_λ factors through an immersion of the $(n-\nu)$-dimensional manifold M_λ into $\tilde{M}$. In this way, $f_\lambda(M)$ is an immersed submanifold of $\tilde{M}$.*

We explain the notation used in the theorem. Under the hypotheses, the principal vectors corresponding to λ form a smooth ν-dimensional distribution T_λ on M whose leaves are umbilic submanifolds of $\tilde{M}$. On each leaf, λ is constant. f_λ is the map from part of M onto the set of focal points arising from the principal curvature λ. M_λ is the space of leaves of T_λ which intersect the domain of f_λ. The proof relies on the work of Palais [13] on foliations.

This theorem generalizes that of Nomizu [10] who proved a similar result for hypersurfaces in the sphere with constant principal curvatures. Because of the abundance of examples of such hypersurfaces, one can produce (through stereographic projection) many examples of hypersurfaces in Euclidean and hyperbolic space to which our theorem applies (see §3.c).

If λ has constant multiplicity one, then $f_\lambda(M)$ is not an $(n-1)$-dimensional manifold without additional hypotheses. This case is handled by Theorem 3.2 which is a generalization of the classical determination of conditions under which a sheet of the focal set of a surface in E^3 is a curve. (See, for example, Eisenhart [6, p. 310–314].)

A key ingredient in the proofs of the above results is the computation of the rank of f_λ. Our result in this area (Theorem 2.1) applies to submanifolds of arbitrary codimension.

The classical version of these theorems was used by Banchoff [1] and Cecil [3] in characterizing taut immersions of surfaces in E^3. Applications of the results of the present paper to the classification of taut immersions of $S^k \times S^{n-k}$ into E^{n+1} may be found in our forthcoming paper [5].

1. Preliminaries.

In this paper, all manifolds and maps are

taken to be C^∞ unless explicitly stated otherwise. The words "smooth" and "differentiable" are synonyms for C^∞. We will always be considering an immersion $f: M \to \tilde{M}$, but we will treat f locally as an embedding. Thus, we will often identify x with $f(x)$ and suppress the mention of f.

1.a. Space forms. We will assume that $\tilde{M}$ is a real space form of dimension $n + k$. Thus $\tilde{M}$ is a Euclidean space, E^{n+k}, a sphere, S^{n+k}, or a hyperbolic space, H^{n+k}, which has constant sectional curvature 0, 1, -1, respectively.

We will use the following representation for the hyperbolic space, H^m (for more detail, see [7, vol. II, p. 268]). Consider $\boldsymbol{R}^{m+1}$ with a natural basis $e_1, \cdots, e_{m+1}$ and a nondegenerate quadratic form $\langle , \rangle$ defined by

$$\langle x, y\rangle = -x^{m+1}y^{m+1} + \sum_{i=1}^{m} x^i y^i ,$$

where

$$x = \sum_{i=1}^{m+1} x^i e_i \quad \text{and} \quad y = \sum_{i=1}^{m+1} y^i e_i .$$

Then H^m is the hypersurface,

$$\{x \in R^{m+1} \mid \langle x, x\rangle = -1,\ x^{m+1} \geqq 1\} ,$$

on which $\langle , \rangle$ restricts to a positive definite metric of constant curvature -1.

Similarly, S^m is defined using the usual Euclidean inner product in $\boldsymbol{R}^{m+1}$, that is,

$$\langle x, y\rangle = \sum_{i=1}^{m+1} x^i y^i ,$$

and

$$S^m = \{x \in \boldsymbol{R}^{m+1} \mid \langle x, x\rangle = 1\} ,$$

on which $\langle , \rangle$ restricts to a metric of constant sectional curvature 1.

1.b. Horizontal lifts. Let NM denote the normal bundle of M with natural projection π. Let s be a local cross-section of NM. We recall the fundamental relationship which holds for any $X \in T_x M$, for any $x \in M$,

$$\tilde{\nabla}_X s = -A_\xi X + \nabla^\perp_X s , \tag{1.1}$$

where $\tilde{\nabla}$ is the covariant differentiation in $\tilde{M}$, A is the shape operator determined by $\xi = s(x)$, and $\nabla^\perp$ is the connection in the normal bundle.

Let $\xi \in NM$. A vector $Z \in T_\xi NM$, the tangent space to NM at ξ, is said to be *horizontal* if there is a local section s of NM such that $\nabla^\perp s = 0$ at $x = \pi\xi$, and $Z = s_* X$, for some $X \in T_x M$. One can show that Z is uniquely determined by ξ and X. We will refer to Z as the *horizontal lift of X to ξ*, and denote this horizontal lift by X_ξ.

Differentiation in horizontal directions on NM can be related to differentiation on M as follows. Let $\lambda: NM \to \boldsymbol{R}$, $\xi \in NM$, $x = \pi\xi$, and $X \in T_x M$. Then

$$X_\xi \lambda = X(\lambda \circ s) ,$$

where s is any local section of NM with $\xi = s(x)$, and $\nabla^\perp s = 0$ at x.

1.c. Locating the focal points. For each $\xi \in NM$, let $F(\xi)$ be the point of $\tilde{M}$ reached by traversing a distance $|\xi|$ along the geodesic in $\tilde{M}$ with initial tangent vector ξ. If ξ is the zero element of the tangent space to $\tilde{M}$ at $f(x)$, let $F(\xi) = f(x)$.

DEFINITION. A point $p \in M$ is called a *focal point of (M, x) of multiplicity ν* if $p = F(\xi)$ where $\pi\xi = x$, and the Jacobian of F at ξ has nullity ν.

Assume now that ξ has unit length. The location of the focal points along a particular geodesic $F(t\xi)$ in $\tilde{M}$ is determined by the sectional curvature of $\tilde{M}$, and the eigenvalues of A_ξ as follows.

PROPOSITION 1.1. *Let M be a submanifold of a space form $\tilde{M}$. Let ξ be a unit vector in NM with $\pi\xi = x$. Then $F(t\xi)$ is a focal point of (M, x) of multiplicity ν if and only if there is an eigenvalue λ of A_ξ of multiplicity ν such that,*

(a) $\lambda = 1/t$ *if* $\tilde{M} = E^{n+k}$,

(b) $\lambda = \cot t$ *if* $\tilde{M} = S^{n+k}$,

(c) $\lambda = \coth t$ *if* $\tilde{M} = H^{n+k}$.

Proof. Statement (a) is well-known (see, for example, Milnor [8, p. 34]). Statement (c) is proven in Cecil [4, p. 343] and (b) can be proven similarly.

1.d. Principal curvature functions. The shape operator A defines a smooth map $\xi \to A_\xi$ of NM into the space of symmetric tensors of type (1, 1) on M. If $\lambda \in \boldsymbol{R}$ is an eigenvalue of A_ξ, λ is called a principal curvature of A_ξ. If the principal curvatures $\{\lambda_i(\xi)\}_{i=1}^n$ are ordered $\lambda_1 \geqq \cdots \geqq \lambda_n$ for each $\xi \in NM$, then each λ_i is a continuous on NM. Furthermore, a continuous principal curvature function

function which has constant multiplicity on an open set $U \subset NM$ must be smooth. Proofs of these two well-known facts may be found in Ryan [14, p. 371] and Nomizu [11], respectively.

1.e. Sheets of the focal set. Proposition 1.1 demonstrates that the location of focal points is determined by the eigenvalues of A_ξ where ξ is a unit normal vector to M. Thus, the natural domain of parametrization of the focal set of M is the bundle of unit normal vectors to M, which we denote by $UN(M)$.

Let U be an open set in $UN(M)$, and let $\lambda: U \to R$ be a differentiable principal curvature of constant multiplicity $\nu > 0$. We define the focal map $f_\lambda: U \to \tilde{M}$ by the following formulas where $x = \pi\xi$,

$$\text{(a)} \quad f_\lambda(\xi) = x + \frac{1}{\lambda(\xi)}\xi \,,$$

$$\text{(1.2)} \quad \text{(b)} \quad f_\lambda(\xi) = \cos\theta x + \sin\theta\xi \text{ where } \cot\theta = \lambda \,,$$

$$\text{(c)} \quad f_\lambda(\xi) = \cosh\theta x + \sinh\theta\xi \text{ where } \coth\theta = \lambda \,,$$

for $\tilde{M}$ equal to E^{n+k}, S^{n+k}, H^{n+k}, respectively. Then $f_\lambda(U)$ is called the *sheet of the focal set over U corresponding to* λ.

It is clear that f_λ is not defined if $\lambda(x) = 0$ in thecase $\tilde{M} = E^{n+k}$, nor if $|\lambda(x)| \leqq 1$ in the case $\tilde{M} = H^{n+k}$. In the spherical case, each principal curvature λ gives rise to two different focal points determined by substituting $\theta = \cot^{-1}\lambda$ and $\theta = \cot^{-1}\lambda + \pi$ into (1.2b).

1.f. Stereographic projection. Let q be an arbitrary point of S^m, and let E^m be the Euclidean subspace of E^{m+1} defined by,

$$E^m = \{x \in E^{m+1} \mid \langle x, q\rangle = 0\} \,,$$

where $\langle\ ,\ \rangle$ is the Euclidean inner product in E^{m+1}. Then stereographic projection from the pole q is the map P from $S^m - \{q\}$ onto E^m defined by,

$$P(x) = q + e^{\sigma(x)}(x - q) \,,$$

where

$$e^{-\sigma(x)} = 1 - \langle x, q\rangle \,.$$

One can easily show that P is conformal diffeomorphism with,

$$\langle P_* X, P_* Y\rangle = e^{2\sigma(x)}\langle X, Y\rangle \,,$$

for all X, Y tangent to S^m at x.

Similarly, consider hyperbolic space $H^m \subset \boldsymbol{R}^{m+1}$ with the indefinite inner product $\langle\ ,\ \rangle$ as defined in §1.a. Let q be a point of $\boldsymbol{R}^{m+1}$ such

that $-q \in H^m$. Let D^m be the m-dimensional disk defined by

$$D^m = \{x \in \boldsymbol{R}^{m+1} | \langle x, q\rangle = 0, \langle x, x\rangle < 1\},$$

on which the metric $\langle\ ,\ \rangle$ restricts to a Euclidean metric which we now denote for emphasis by g. (D^m is a unit disk with respect to the metric g, although in terms of the usual coordinates on $\boldsymbol{R}^{m+1}$, the boundary of D^m is, in general, an ellipsoid.)

Then, stereographic projection from the pole q is the map $P: H^m \to D^m$ which is defined by the formula,

$$P(x) = q + e^{\sigma(x)}(x - q),$$

where

$$e^{-\sigma(x)} = 1 + \langle x, q\rangle .$$

As in the spherical case, one easily shows that P is a conformal diffeomorphism onto the Euclidean disk D^m with,

$$g(P_*X, P_*Y) = e^{2\sigma(x)}\langle X, Y\rangle ,$$

for all X, Y tangent to H^m at x.

1.g. **Conformally related spaces.** In this section, we note some facts about conformal diffeomorphisms which can then be applied to the specific case of stereographic projection.

Let $\phi: (\tilde{M}, g) \to (\tilde{M}', g')$ be a conformal diffeomorphism of Riemannian manifolds with,

$$g'(\phi_*X, \phi_*Y) = e^{2\sigma(x)}g(X, Y),$$

for all X and Y tangent to $\tilde{M}$ at x.

For a submanifold M of $\tilde{M}$, let ξ be a local field of unit normals to M near x. Then $\xi' = \phi_*(e^{-\sigma}\xi)$ is a field of unit normals to $\phi(M)$ near $\phi(x)$, and the corresponding shape operators are related by

$$B_{\xi'} = e^{-\sigma}(A_\xi - g(\text{grad } \sigma, \xi)I).$$

This relationship, obtained by a straightforward computation, yields as a consequence the following statement in the codimension one case.

PROPOSITION 1.2. *Let M be a hypersurface in $\tilde{M}$ and let λ be a differentiable principal curvature of constant multiplicity. Then*

$$\mu = e^{-\sigma}(\lambda - g(\text{grad } \sigma, \xi))$$

is a differentiable principal curvature of the same constant multiplicity on $\phi(M)$, and the respective principal distributions of λ and μ coincide on M.

2. **The rank of f_λ.** As in §1, let M be an n-dimensional smooth manifold immersed in a real space form $\tilde{M}$ of dimension $n+k$. Suppose $\xi \in UN(M)$, and λ is an eigenvalue of A_ξ. Let $T_\lambda(\xi)$ be the eigenspace of λ.

THEOREM 2.1. *Let M be an n-dimensional submanifold of an $(n+k)$-dimensional real space form $\tilde{M}$. Let λ be a differentiable principal curvature of constant multiplicity ν near a point ξ in $UN(M)$. If there is an $X \in T_\lambda(\xi)$ whose horizontal lift to ξ satisfies $X_\xi\lambda \neq 0$, then the rank of f_λ at ξ is $n+k-\nu$. Otherwise, f_λ has rank $n+k-\nu-1$.*

The proof is a rather long but straightforward computation which we leave to the reader. It may be broken down into four steps. First, one shows that $n+k-\nu$ is an upper bound for the rank of f_λ. Next, one shows that $n+k-\nu-1$ is a lower bound. Thirdly, one proves the following result which distinguishes the two possibilities.

PROPOSITION 2.1. *The rank of f_λ is $n+k-\nu$ at ξ if and only if the range of $(f_\lambda)_*$ contains*

(a) ξ *when* $\tilde{M} = E^{n+k}$,
(b) $x - \lambda(\xi)$ *when* $\tilde{M} = S^{n+k}$,
(c) $x + \lambda(\xi)$ *when* $\tilde{M} = H^{n+k}$,

where $x = \pi\xi$.

Finally, one proves that the respective conditions of Proposition 2.1 hold if and only if there is $X \in T_\lambda(\xi)$ such that $X_\xi\lambda \neq 0$. Note that Proposition 2.1(a) generalizes the classical result that the normal to a surface is tangent to its evolute (focal set) when f_λ has maximal rank.

The question answered by Theorem 2.1 can formulated in more general ambient spaces. Along normal geodesics, the occurrence of focal points is still related to the principal curvatures, but the curvature of the ambient space must be taken into account. Given sufficient homogeneity properties for $\tilde{M}$, however, one should be able to define f_λ and compute its rank.

3. **Focal sets of hypersurfaces.** Let M be an orientable hypersurface of a real space form $\tilde{M}$. Let ξ be a global field of unit normals on M. The following result is basic to our discussion. See [14, p. 371-373], for example, for a proof.

PROPOSITION 3.1. *Suppose λ is a principal curvature of constant multiplicity $\nu > 1$ on M. Then T_λ is involutive and $X\lambda = 0$ for all $X \in T_\lambda$.*

If λ has constant multiplicity one, then one cannot conclude that $X\lambda = 0$ for $X \in T_\lambda$. Given the dependence of the rank of f_λ on $X\lambda$ as shown in Theorem 2.1, it is natural to consider the two cases $\nu > 1$ and $\nu = 1$ separately.

3.a. The case of multiplicity $\nu > 1$. Before we state the first proposition, we recall the following definition. A submanifold V of any space $\tilde{M}$ is said to be *umbilic* if for each $x \in V$, there is a real-valued linear function ω on $T_x^\perp V$ such that for any η, the shape operator B_η of V satisfies the equation $B_\eta = \omega(\eta)I$, where I is the identity endomorphism on $T_x V$.

PROPOSITION 3.2. *The leaves of the foliation T_λ are umbilic ν-dimensional submanifolds of $\tilde{M}$.*

Proof. Let V be any leaf of T_λ. The normal space to V in $\tilde{M}$ at x is

$$T_x^\perp V = T_x^\perp M \oplus T_\lambda^\perp(x) .$$

Let $\eta \in T_\lambda^\perp V$, and let B_η be the shape operator of V determined by η. If $\eta \in T_x^\perp M$, then clearly $B_\eta = \lambda I$. On the other hand, suppose $\eta \in T_\lambda^\perp(x)$ is a principal vector of the shape operator A of M. Extend η to a vector field $Y \in T_\lambda^\perp$ on W. There is a unique vector field $Z \in T_\lambda^\perp$ such that $\langle Z, Y\rangle = 0$ and

$$AY = \mu Y + Z \tag{3.1}$$

for some function μ on W. This is possible since $T_\lambda^\perp$ is invariant under A, even though the eigenvalues of A need not be smooth.

Let $X \in T_\lambda$ be a vector field on W. Since $Z = 0$ at x, one easily shows that $\nabla_X Z \in T_\lambda^\perp$ at x. Using equation (3.1), the Codazzi equation $(\nabla_X A)Y = (\nabla_Y A)X$ becomes

$$(X\mu)Y - (Y\lambda)X + \nabla_X Z = (A - \mu I)\nabla_X Y - (A - \lambda I)\nabla_Y X .$$

Since $\nabla_X Z \in T_\lambda^\perp$ at x, one finds from the above equation that the T_λ-component of $\nabla_X Y$ at x is

$$\frac{-(\eta\lambda)X}{\lambda - \mu} .$$

Since $\langle AX, Y\rangle = 0$, this is also the T_λ-component of $\tilde{\nabla}_X Y$, and one obtains

$$B_\eta X = \frac{(\eta\lambda)X}{\lambda - \mu} ,$$

thus proving that V is umbilic.

We now prove the main theorem of this section. In the statement below, M_λ is the space of leaves U/T_λ, where U is the domain of f_λ.

THEOREM 3.1. *Let M be an orientable (immersed) hypersurface in a real space form $\tilde{M}$ which is complete with respect to the induced metric. Let λ be a differentiable principal curvature of constant multiplicity $\nu > 1$ on M. Then the focal map f_λ factors through an immersion of the $(n-\nu)$-dimensional manifold M_λ into $\tilde{M}$. In this way, $f_\lambda(M)$ is an immersed submanifold of $\tilde{M}$.*

Proof. Since the leaves of T_λ are umbilic, T_λ is a regular foliation as defined by Palais [13, p. 13] and the space of leaves is an $(n-\nu)$-dimensional manifold in the sense of Palais, which may not be Hausdorff. Moreover, the computations involved in proving Theorem 2.1 show that $(f_\lambda)_* \equiv 0$ on T_λ. Thus by [13, p. 25], f_λ factors through a map g_λ from the space of leaves $M_\lambda = U/T_\lambda$ into $\tilde{M}$. Since rank g_λ = rank $f_\lambda = n - \nu$, g_λ is an immersion. Finally, the regularity of T_λ implies that each leaf is a closed subset of M [13, p. 18]. Thus if M is complete, each leaf is also complete with respect to its induced metric (see, for example, [7, vol. I, p. 179]). Hence each leaf which intersects U is a ν-dimensional metric sphere in $\tilde{M}$. Because such leaves are compact, M_λ is Hausdorff [13, p. 16]. This completes the proof of Theorem 3.1.

The following remark demonstrates that the assumption of completeness in Theorem 3.1 is necessary to guarantee that M_λ is a Hausdorff manifold.

REMARK 3.3. *An example in which rank f_λ is constant but M_λ is not a Hausdorff manifold.*

Let

$$f(t) = \begin{cases} e^{-1/t} & \text{if } t > 0 \\ 0 & \text{if } t \leqq 0 . \end{cases}$$

Let K be the envelope of a 1-parameter family of spheres of radius 1 in E^3 whose centers lie on the curve,

$$\gamma(t) = (t, 0, f(t)),\ t \in (-1, 1) .$$

Then $\gamma(t)$ is the sheet of the focal set of K corresponding to the constant principal curvatures $\lambda = 1$.

Let N be the intersection of K with the closed upper half-space, $z \geqq 0$, with the points satisfying $z = 0, x \geqq 0$ removed. Let M be the union of N with its reflection in the plane $z = 0$.

The leaf space M/T_λ is not Hausdorff since the two semi-circular leaves in the plane $x = 0$ cannot be separated in the quotient topology. This is consistent with the fact that the focal set,

$$f_\lambda(M) = \{(x, 0, z) \mid z = |f(x)|,\ -1 < x < 1\} ,$$

is not a 1-manifold in a neighborhood of the origin. Nevertheless, the rank of f_λ is identically equal to 1 on M.

3.b. The case of multiplicity $\nu = 1$. In this case, the fact that for hypersurfaces of H^{n+1}, the domain of f_λ, U, does not include those $x \in M$ where $|\lambda(x)| \leqq 1$, becomes quite significant. In fact, the conditions (a), (b), and (c) of Theorem 3.2 are equivalent on U, but not necessarily on all of M. Specifically, (a) and (b) are equivalent on M, and they imply (c). However, one can construct a surface M in H^3 such that the focal set $f_\lambda(M)$ is a curve, and yet not all the lines of curvature of M corresponding to λ are of constant curvature. This is done by beginning with a standard example K on which (a), (b), and (c) are satisfied and modifying K on the region where $|\lambda| < 1$, so as to destroy property (b), but introduce no new focal points and thus preserve (c).

THEOREM 3.2. *Let M be an orientable (immersed) hypersurface in $\tilde{M}$. Suppose λ is a differentiable principal curvature of constant multiplicity 1 on M. Then the following are equivalent on M if $\tilde{M} = E^{n+1}$ or S^{n+1}, and on U, the domain of f_λ, if $\tilde{M} = H^{n+1}$.*

(a) *λ is constant along each leaf of T_λ (the lines of curvature).*

(b) *The leaves of T_λ are plane curves of constant curvature.*

(c) *The rank of f_λ is identically equal to $n - 1$ on its domain.*

Unlike the case $\nu > 1$, one must use a different proof for the different ambient spaces. We first give a proof for the Euclidean case, then handle the others by stereographic projection.

Proof of Theorem 3.2 (*Euclidean case*):

(a) $\Leftrightarrow$ (c). This follows immediately from Theorem 2.1 and the connectedness of the leaves of T_λ.

(a) $\Rightarrow$ (b). We will give an outline of the proof, leaving details to the reader. Assuming (a), let W be a coordinate patch of the form $\phi(U \times V)$, where $U \subset \boldsymbol{R}$, $V \subset \boldsymbol{R}^{n-1}$ such that each leaf of T_λ is determined locally by $\{(u, v) \mid v = \text{constant}\}$.

We first assume that λ is a nonzero constant on each leaf which passes through W. By a proper choice of ξ, we may assume that $\lambda > 0$ on W. Using the fact that $(f_\lambda)_*$ annihilates T_λ and is injective on $T_\lambda^\perp$, we can factor

$$f_\lambda: W \longrightarrow E^{n+1}$$

through an immersion

$$p: V \longrightarrow E^{n+1} .$$

In addition, the real function $r = 1/\lambda$ is well-defined on V, and one can show that for $x = \phi(u, v)$,

$$\langle f(x) - p(v), p_*(\bar{v}) \rangle = -r(v)(\bar{v}r) , \tag{3.2}$$

for all $\bar{v} \in T_v V$, and

$$| f(x) - p(v) | = r(v) . \tag{3.3}$$

Thus, for a particular value of v, $f(x)$ lies on the circle determined by intersecting the sphere indicated in (3.3) with the 2-plane indicated in (3.2). Hence each leaf lies locally on a circle. By the connectedness of the leaves, the whole leaf must lie on the same circle and thus be a plane curve of constant curvature.

Finally, if γ is a leaf of T_λ on which $\lambda \equiv 0$, one can choose an inversion I of E^{n+1} such that $I(\gamma)$ is a line of curvature of $I(M)$ on which the associated principal curvature is a nonzero constant. By the above argument, $I(\gamma)$ lies on a circle so that γ itself lies on a circle or a straight line.

(b) $\Rightarrow$ (a). This is easily shown using the Frenet equations for plane curves.

Proof of Theorem 3.2 (*non-Euclidean case*): (a) $\Leftrightarrow$ (c). This follows as in the Euclidean case from Theorem 2.1, the fact that f_λ is defined at each $x \in U$, and from the connectedness of the leaves.

(a) $\Leftrightarrow$ (b). First one can easily show by explicit calculation that the leaves of T_λ are plane curves of constant curvature in S^{n+1}, respectively, H^{n+1}, if and only if P embeds the leaves of T_μ as plane curves of constant curvature in E^{n+1}, respectively, D^{n+1}. Here P is stereographic projection from any pole q, and (see Proposition 1.2)

$$\mu = e^{-\sigma}(\lambda - \langle \operatorname{grad} \sigma, \xi \rangle) ,$$

for σ as defined in 1.f, and for the appropriate choice of $\langle , \rangle$. The proof of the theorem will thus follow from the equivalence of (a) and (b) in the Euclidean case, and from the following result.

PROPOSITION 3.4. *Let M be hypersurface in S^{n+1}, respectively, H^{n+1}. Suppose λ is a principal curvature of constant multiplicity* 1 *on M, and let X denote the field of unit principal vectors of λ on M. Let $\mu = e^{-\sigma}(\lambda - \langle \operatorname{grad} \sigma, \xi \rangle)$ be the corresponding principal*

curvature on $P(M)$ in E^{n+1}, respectively, D^{n+1}. Then $X\lambda=0$ at $x \in M$ if and only if $X_\mu = 0$ at x.

Proof. For M^n in S^{n+1}, *and* $P: S^{n+1} - \{q\} \to E^{n+1}$, stereographic projection from the pole q, the function σ is defined by the equation,

$$e^{-\sigma(x)} = 1 - \langle x, q \rangle .$$

A straightforward computation yields that,

$$\operatorname{grad} \sigma = e^{\sigma}(q - \langle x, q \rangle x) .$$

Thus, using the fact that $\langle x, \xi \rangle = 0$, one obtains,

$$\mu = e^{-\sigma}(\lambda - \langle e^{\sigma} q, \xi \rangle) = e^{-\sigma}\lambda - \langle q, \xi \rangle ,$$

and

$$X\mu = -e^{-\sigma}(X\sigma)\lambda + e^{-\sigma}(X\lambda) - \langle q, D_X \xi \rangle ,$$

where D is the usual Euclidean connection on R^{n+2}. Since $\langle X, \xi \rangle = 0$, it follows that $D_X \xi = \tilde{\nabla}_X \xi$ where $\tilde{\nabla}$ is the Levi-Civita connection in S^{n+1}. However, $\tilde{\nabla}_X \xi = -\lambda X$, so $D_X \xi = -\lambda X$. This and the fact that,

$$X\sigma = \langle \operatorname{grad} \sigma, X \rangle = e^{\sigma} \langle q, X \rangle ,$$

allow the above expression for $X\mu$ to be written as,

$$X\mu = -\langle q, X \rangle \lambda + e^{-\sigma}(X\lambda) + \langle q, X \rangle \lambda = e^{-\sigma}(X\lambda) ,$$

and, clearly, $X\mu = 0$ if and only if $X\lambda = 0$.

Similarly, for M^n in H^{n+1}, the equation,

$$e^{-\sigma(x)} = 1 + \langle x, q \rangle ,$$

implies,

$$\operatorname{grad} \sigma = -e^{\sigma}(q + \langle x, q \rangle x) ,$$

and the result follows as in the spherical case.

As in the case of Theorem 3.1 for multiplicity $\nu > 1$, if one assumes, in addition, that M is complete with respect to the induced metric, then one can produce a natural manifold structure on $f_\lambda(M)$ by introducing the space of leaves $M_\lambda = U/T_\lambda$, where U is the domain of f_λ.

This case differs slightly from the $\nu > 1$ case. As in the $\nu > 1$ situation, the completeness of M implies that each leaf of T_λ is also complete with respect to the induced metric. This is sufficient to guarantee that each leaf in M_λ is a covering space of the metric circle on which it lies (see, for example, [7, vol. I, p. 176]). Since

the circle is not simply connected, however, one cannot conclude that the leaf itself is compact, as in the $\nu > 1$ case. However, using the fact that each leaf is a covering of the circle on which it lies, one can produce a direct argument that M_λ is Hausdorff, and we will omit the proof here. Thus, one obtains the following global version of Theorem 3.2.

THEOREM 3.3. *Let M be an orientable (immersed) hypersuface in $\tilde{M}$ which is complete with respect to the induced metric. Let λ be a differentiable principal curvature of constant multiplicity 1 on M. Suppose the equivalent conditions* (a), (b), (c) *of Theorem* 3.2 *are satisfied on the domain of f_λ. Then f_λ factors through an immersion of the $(n-1)$-dimensional manifold M_λ into $\tilde{M}$. In this way, $f_\lambda(M)$ is an immersed submanifold of $\tilde{M}$.*

3.c. Example of hypersurfaces whose focal sets are manifolds. There is a large class of hypersurfaces in S^{n+1} which have constant principal curvatures. These are the so-called isoparametric hypersurfaces which have been studied by Cartan [2], Nomizu [10] Takahashi and Takagi [15], Ozeki and Takeuchi [12], and Munzner [9]. Nomizu [10] showed that each sheet of the focal set of such a hyersurface is a minimal submanifold of S^{n+1}.

Let M be a hypersurface in S^{n+1} with constant principal curvatures, and let P denote stereographic projection from S^{n+1} to E^{n+1} from any pole q. Then by Propositions 1.2 and 3.4, the principal curvatures of the hypersurface $P(M)$ in E^{n+1} have constant multiplicities, and they are constant along their corresponding principal foliations. Thus each sheet of the focal set of $P(M)$ is a manifold by Theorems 3.1 and 3.3.

Examples of hypersurfaces in H^{n+1} whose focal sets are manifolds are now easily constructed. Let K be a hypersurface of E^{n+1} as constructed in the above paragraph. Let L be the image of K under a contraction of E^{n+1} such that L is contained in the unit disk D^{n+1} centered at the origin in E^{n+1}. Now, let P denote stereographic projection from H^{n+1} onto D^{n+1} from the pole $(0, \cdots, -1)$. Then $P^{-1}(L)$ is a hypersurface in H^{n+1} with the property that each sheet of its focal set is a manifold.

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Received May 2, 1977 and in revised form February 15, 1978. During the period of this research, the authors were supported by the following grants. NSF Grant Nos. MPS75-12825 and MCS76-07044, NSF Grant No. MCS76-06930, and an Indiana University Summer Faculty Fellowship.

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PACIFIC JOURNAL OF MATHEMATICS

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 78, No. 1 March, 1978