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To an analytic function $L(z)$ we associate the differential operator $L(D)$, D denoting differentiation with respect to a real variable x. We interpret L as the generator of a process with independent increments having exponential martingale $m(x(t), t) = \exp(zx(t) - tL(z))$. Observing that $m(x, -t) = e^{zC}1$ where $C = e^{tL}xe^{-tL}$, we study the operator calculus for C and an associated generalization of the operator xD, $A = CD$. We find what functions f have the property that $u_n = C^n f$ satisfy the evolution equation $u_t = Lu$ and the eigenvalue equations $Au_n = nu_n$, thus generalizing the powers x^n. We consider processes on R^N as well as R^1 and discuss various examples and extensions of the theory.

In the case that L generates a Markov semigroup, we have transparent probabilistic interpretations. In case L may not generate a probability semigroup, the general theory gives some insight into what properties any associated "processes with independent increments" should have. That is, the purpose is to elucidate the Markov case but in such a way that hopefully will lead to practicable definitions and will present useful ideas for defining more general processes—involving, say, signed and/or singular measures.

II. Probabilistic basis. Let $p_t(x)$ be the transition kernel for a process $\rho(t)$ with stationary independent increments. That is,

$$\int_A p_t(x) = \operatorname{Prob}(\rho(t) \in A \mid \rho(0) = 0) .$$

The Lévy-Khinchine formula says that, generally:

$$\int_R e^{i\xi x} p_t(x) = e^{tL(i\xi)}$$

where $L(i\xi) = ai\xi - \sigma^2\xi^2/2 + \int_{R-\{0\}} e^{i\xi u} - 1 - i\xi\eta(u) \cdot M(du)$ with

$$\eta(u) = u(|u| \leqq 1) + \operatorname{sgn} u(|u| \geqq 1) \text{ and } \int \frac{u^2}{1+u^2} M(du) < \infty .$$

Denoting d/dx by D, this states that $L(D)$ is the generator of the process $\rho(t)$. It follows that

$$m(t) = e^{z\rho(t) - tL(z)} \text{ is a martingale}$$

for every $z \in \boldsymbol{C}$, $\operatorname{Re} z = 0$. And this is clearly equivalent to the condition that L generates ρ. We define $h_k(x, t)$ by

$$e(z) = e^{zx+tL(z)} = \sum_0^\infty \frac{z^k}{k!} h_k(x, t) .$$

Observe that $\partial e/\partial z = (x + tL'(z))e(z)$ which is the basis for § IV. Note that this expansion is good only in case the distribution of $\rho(t)$ has all moments finite. In fact, define moment functions

$$h_k(x, t) = \int_R (x + y)^k p_t(y) = e^{tL(D)} x^k .$$

And check that,

$$\sum_0^\infty \frac{z^k}{k!} h_k(x, t) = e^{zx} \int_R e^{zy} p_t(y) = e^{zx+tL(z)} .$$

Thus if $p_t(x)$ has only moments up to order α, then define h_k for $0 \leqq k < \alpha$, say. Also, k need not be restricted to integer values, but generally $k \in \boldsymbol{R}$, $k \geqq 0$. We set $H_k(x, t) = h_k(x, -t)$.

Note that since for $t > 0$, $s < t$,

$$E(e^{z\rho(t)-tL(z)} \mid 0 \leqq \rho(u) \leqq s) = e^{z\rho(s)-sL(z)}$$

then $E(H_k(\rho(t), t) \mid 0 \leqq \rho(u) \leqq s) = H_k(\rho(s), s)$. That is, $H_k(\rho(t), t)$ is a martingale for every k. From the above remarks it follows:

(1) $h_k(x, t) = E(x + \rho(t))^k$, $t \geqq 0$.

(2) $Eh_k(x + \rho(t), s) = h_k(x, t + s)$, $t, s \geqq 0$.

(3) $Eh_k(x + \rho(t), -t) = x^k$, $t \geqq 0$.

In the following we develop an operator calculus associated with these h_k and study various properties and extensions based upon the above preliminaries.

III. Notations.

(1) d/dx will be denoted by D.

(2) $L(D)$ will be "any" function of D that is thought of as the generator of a process with independent increments. L is assumed to have constant coefficients (independent of x and t); and generally $L(0) = 0$.

(3) EX or $\langle X \rangle$ will be used to denote expected value.

(4) t, the "time", is independent of all x and D variables. Generally $t \geqq 0$.

Other notations are standard or will be explained as they arise. The functional calculus for D will be implemented by Fourier transform as needed to facilitate and clarify computations.

IV. Algebraic structure. Assume given an operator C such that

$$[D, C] = DC - CD = 1 .$$

For example, $C = x$ as a multiplication operator. By induction it follows readily that

$$[D^r, C] = rD^{r-1}$$

and

$$[D, C^r] = rC^{r-1} , \quad r > 0 .$$

Thus for any polynomial or analytic function $\phi(D)$,

$$[\phi(D), C] = \phi'(D) ,$$

ϕ' denoting the derivative of ϕ. So, even for nondifferentiable ϕ, denote $[\phi(D), x]$ by ϕ'.

PROPOSITION. *Set $C = x + t[\phi(D), x]$. Then $[D, C] = 1$.*

Proof.

$$\begin{aligned} DC &= Dx + tD\phi(D)x - tDx\phi(D) \\ &= xD + 1 + t\phi(D)Dx - tx\phi(D)D - t\phi(D) \\ &= xD + 1 + t\phi(D)xD + t\phi(D) - tx\phi(D)D - t\phi(D) \\ &= (x + t[\phi(D), x])D + 1 . \end{aligned}$$

Since $[D, x] = 1$, this checks that $[D, \phi'] = [\phi, \phi'] = 0$. Denote by A the operator CD. Then the following hold (by induction).

$$\begin{aligned} [A, C^k] &= kC^k \\ [A^k, C] &= C((A + 1)^k - A^k) . \end{aligned}$$

Similarly,

$$[D^k, A] = kD^k$$

and

$$[D, A^k] = ((A + 1)^k - A^k)D .$$

Given a function $\phi(D)$, such that $\phi(D)1 = 0$, set $C = x + t\phi'$. Assume $A1 = 0$. Define $h_k(x, t) = C^k 1$. Then the following properties hold:

(1) $Ch_k = h_{k+1}$.

(2) $Dh_k = kh_{k-1}$.

(3) $Ah_k = CDh_k = kh_k$.

(4) $\partial h_k/\partial t = \phi(D)h_k$.

(5) $h_k(x, 0) = x^k$.

The generating function $e(z) = \sum_0^\infty (z^k/k!)h_k$ has these properties:

(6) $\partial e/\partial z = Ce$, $e(0) = 1$.

(7) $C^k e(z) = e^{zC}h_k$.

The above are easy to check using the commutation relations noted previously. It is worthwhile to check #4 explicitly:

For $k = 0$, $h_k \equiv 1$ and $\partial h_k/\partial t = 0 = \phi(D)1$.

For $k = n + 1$, $h_k = Ch_n$ and

$$\begin{aligned}\frac{\partial h_k}{\partial t} &= \frac{\partial C}{\partial t}h_n + C\frac{\partial h_n}{\partial t} \\ &= \phi' h_n + C\phi h_n \\ &= (\phi C - C\phi)h_n + C\phi h_n \\ &= \phi C h_n = \phi h_k ,\end{aligned}$$

since

$$\phi' = [\phi, x] = [\phi, C] .$$

The condition $\phi 1 = 0$ comes only in proving #4; the condition $A1=0$ for #2. #3 follows from #1 and #2. These h_k thus generalize the powers x^k.

REMARK. In case ϕ is analytic, $\phi(D) = \sum_0^\infty a_p D_p/p!$, the h_k are the determinants of a family of matrices of dimension k. This is seen as follows. Set $\bar{C} \equiv x - t\phi'$. $H_k(x, t) \equiv h_k(x, -t) = \bar{C}^k 1$. Then,

$$\begin{aligned}xH_k = t\phi' H_k + H_{k+1} &= t\sum_1^\infty a_p \frac{D^{p-1}}{(p-1)!}H_k + H_{k+1} \\ &= t\sum_1^\infty a_p \frac{k!}{(p-1)!(k-p+1)!}H_{k-p+1} + H_{k+1} \\ &= t\sum_0^k a_{p+1}\begin{bmatrix} k \\ p \end{bmatrix} H_{k-p} + H_{k+1} .\end{aligned}$$

Thus, $H_k = \det(x - A_k)$ where the matrix A_k has entries:

$$A_{k,rs} = \begin{matrix} 1 & s = r+1 \\ 0 & s > r+1 \\ t\begin{bmatrix} r-1 \\ s-1 \end{bmatrix} a_{r-s+1} & s \leqq r+1 . \end{matrix}$$

In the probabilistic case, ϕ is chosen to be the generator L. Then the moment polynomials h_k can be computed as $C^k 1$. The relations $\langle h_k(\rho(t), -t)\rangle = 0$ yield the moments $\mu_k = \int y^k P_t(y)$ and, conversely, the original definition yields the h_k in terms of the μ_k, $h_k = \sum \begin{bmatrix} k \\ r \end{bmatrix} \mu_r x^{k-r}$.

EXAMPLES.

1. $L = D$. $C = x + t$. $A = (x + t)D$. $h_k = (x + t)^k$.

2. Brownian motion. $L = (1/2)D^2$. $C = x + tL' = x + tD$. $A = xD + tD^2$.

$$h_k = \int_R (x + y)^k \frac{e^{-y^2/2t}}{\sqrt{2\pi t}} dy$$

and H_k are the Hermite polynomials.

3. Poisson process. $L = e^D - 1$. $C = x + te^D$. The first few polynomials are:

$$h_0 = 1.\ h_1 = x + t\ .$$
$$h_2 = x^2 + (2x + 1)t + t^2\ .$$
$$h_3 = x^3 + (3x^2 + 3x + 1)t + (3x + 3)t^2 + t^3\ .$$

4. Singular case. $L = \log D$. Then $C = x + tD^{-1}$. $A = xD + t$. The equations $Ah_k = kh_k$ become:

$$xh' + th = kh$$

yielding

$$h_k = ax^{k-t}\ .$$

Choosing $h_0 = x^{-t}$ gives:

$$h_k = \frac{k!\,x^{k-t}}{(1 - t)(2 - t)\cdots(k - t)} = \frac{(-t)!\,k!\,x^{k-t}}{(k - t)!}\ ,$$

defining $q! \equiv \Gamma(q + 1)$.

V. The operator C. As in § IV, given a generator $L(D)$, define $C = x + tL'$. For general functions $L(D)$, $L(0) = 0$, set

$$p_t(x) = \frac{1}{2\pi}\int_R e^{-i\xi x} e^{tL(i\xi)} \text{ whenever this}$$

may be defined; e.g., if necessary, as a distribution.

PROPOSITION. *The solutions to $Cf = \lambda f$ are of the form*

$$f = k(t)p_t(\lambda - x)\ .$$

Proof.

$xf + tL'f = \lambda f$ becomes by Fourier transformation

$$i\frac{d}{d\xi}\hat{f} + tL'(i\xi)\hat{f} = \lambda\hat{f}$$

which has the solution $\hat{f} = k(t)e^{-i\lambda\xi}e^{t\int_0^\xi L'(is)ids} = k(t)e^{-i\lambda\xi}e^{tL(i\xi)}$.

REMARK. Recall from IV that as long as $Ah_0 = 0$, $Ah_k = kh_k$. This indicates that there is a 2*nd series*, besides C^k1, formed as follows. Take $h_0 = D^{-1}p_t(-x)$, D^{-1} denoting indefinite integral. Then by the above proposition with $\lambda = 0$, $Ah_0 = CDh_0 = 0$. Now define $h_k = C^k h_0$. Then h_{k-1} is actually $k!$ times the kth indefinite integral of $p_t(-x)$ and $Ah_k = kh_k$.

Now proceed to study in more detail the operator C acting on functions in general.

DEFINITION. A function $f(x, t)$ is *harmonic* by definition when

$$\frac{\partial f}{\partial t} = Lf .$$

Then it will be seen that:

THEOREM. *If f is harmonic, then $u = e^{zC}f$ is harmonic. Defining $f_k = C^k f$, then*

COROLLARY. *If f is harmonic, then f_k is harmonic for all $k \geqq 0$. In fact, $e^{zC}f$ harmonic means*

$$\frac{\partial}{\partial t}e^{zC}f = Le^{zC}f \text{ and to conclude that } f_k \text{ is}$$

harmonic note that differentiating with respect to z commutes with both $\partial/\partial t$ and L.

First compute $e^{zC}f$.

LEMMA 1. *Set $\Lambda(D) = L(D + z) - L(D)$. Then*

$$e^{zC}f = e^{zx}e^{t\Lambda}f .$$

Proof. Let $\psi = L'$. For $u = e^{zC}f$,

$$\frac{\partial u}{\partial z} = Cu = xu + t\psi u .$$

Apply Fourier transformation to yield, for $v = \hat{u}$:

$$\frac{\partial v}{\partial z} = i\frac{\partial v}{\partial \xi} + t\psi(i\xi)v$$

which has for solution

$$v = \hat{f}(\xi + iz)e^{t\int_0^z \psi(i\xi - s)ds}$$

as is readily seen, say, by applying Feynman-Kac for the deterministic process $\rho(z) = iz$. Thus,

$$
\begin{aligned}
u &= \int e^{i\xi x}\hat{f}(\xi + iz)e^{t\int_0^z \psi(i\xi - s)ds} \\
&= e^{zx}\int e^{i\xi x}\hat{f}(\xi)e^{t\int_0^z \psi(i\xi + s)ds} \\
&= e^{zx}e^{t\int_0^z \psi(D+s)ds}f .
\end{aligned}
$$

Since $\psi = L'$, $\int_0^z \psi(D + s)ds = L(D + z) - L(D)$.

The theorem will require the aid of the following.

LEMMA 2. *For any* $\phi(D)$,

$$\phi(D)e^{zx}f = e^{zx}\phi(D + z)f .$$

Proof. Utilizing Fourier transformation for the left-hand side,

$$
\begin{aligned}
\phi(D)e^{zx}f &= \int e^{i\xi x}\phi(i\xi)\int e^{-i\xi a}e^{za}f(a) \\
&= \int e^{i\xi x}\phi(i\xi)\hat{f}(\xi + iz) \\
&= \int e^{i\xi x}e^{zx}\phi(i\xi + z)\hat{f}(\xi) \\
&= e^{zx}\phi(D + z)f .
\end{aligned}
$$

Finally the proof of the theorem:

Let $\partial f/\partial t = Lf$. Then, for $u = e^{zC}f$,

$$
\begin{aligned}
\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(e^{zx}e^{t\Lambda}f) &= e^{zx}e^{t\Lambda}\Lambda f + e^{zx}e^{t\Lambda}Lf \\
&= e^{zx}e^{t\Lambda}(L(D + z) - L(D) + L(D))f = e^{zx}e^{t\Lambda}L(D + z)f \\
&= Lu \text{ by Lemma 2.}
\end{aligned}
$$

REMARKS. 1. Particularly, when $L1 = 0$, $e(z) = e^{zC}1$ is harmonic. In fact,

$$
\begin{aligned}
e^{zC}1 &= e^{zx}e^{t(L(D+z)-L(D))}1 = e^{zx}e^{tL(D+z)}1 \\
&= e^{tL(D)}e^{zx}1 , \quad \text{by Lemma 2,} \\
&= e^{tL(z)}e^{zx} \text{ which is the required result.}
\end{aligned}
$$

2. Note that by Lemma 2, generally,

$$
\begin{aligned}
e^{zC}f &= e^{zx}e^{tL(D+z)}e^{-tL}f \\
&= e^{tL}e^{zx}e^{-tL}f .
\end{aligned}
$$

Differentiating with respect to t yields an alternative proof of the theorem. See § IX.

3. In the probabilistic case, the theorem implies that if f is harmonic, $u=e^{zC}f$, then $f_k(\rho(t), -t)$ and $u(\rho(t), -t)$ are martingales.

EXAMPLES.

1. Brownian motion.

$$\Lambda = L(D+z) - L(D) = \frac{1}{2}z^2 + zD.$$

So

$$e^{zC}f = e^{zx}e^{z^2t/2}f(x+zt).$$

2. Poisson process.

$$\Lambda = e^D(e^z - 1).$$

So

$$e^{zC}f = e^{zx}\sum_0^\infty \frac{t^n(e^z-1)^n}{n!}f(x+n).$$

3. Singular process.

$$\Lambda = \log(D+z) - \log D = \log(1+zD^{-1}).$$

$$\begin{aligned} e^{zC}f &= e^{zx}(1+zD^{-1})^t f \\ &= e^{zx}\sum \begin{bmatrix} t \\ k \end{bmatrix} z^k D^{-k} f. \end{aligned}$$

And

$$f_k = \sum_r \begin{bmatrix} t \\ r \end{bmatrix} \frac{k!\,x^{k-r}}{(k-r)!} D^{-r} f.$$

4. $L(D) = D^3/6$. $\Lambda = (1/2)zD^2 + (1/2)z^2D + (1/6)z^3$.

$$e^{zC}f = e^{zx}e^{z^3t/6}\int f\left(x + \frac{z^2t}{2} + \sqrt{zt}\,y\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy.$$

VI. **The operator A.** Given C, define $A = CD$.

This section continues the previous one to analyze the eigenvalue equations $Af_k = kf_k$.

PROPOSITION. *$Af = 0$ if and only if $Af_k = kf_k$, for $k \geqq 0$.*

Proof. Set $u = e^{zC}f$.

$$\begin{aligned} Au &= CDe^{zx}e^{tA}f \\ &= C(zu + e^{zC}Df) \\ &= zCu + e^{zC}Af \\ &= zCu \\ &= z\frac{\partial}{\partial z}u \,. \end{aligned}$$

Thus, $z(\partial/\partial z)z^k = kz^k$ implies $AC^kf = kC^kf$. This follows also from the commutation relation $[A, C^k] = kC^k$.

To see how A acts on functions in general, solve the equation

$$\frac{\partial u}{\partial \alpha} = Au$$

thus computing $e^{\alpha A}f$.

PROPOSITION.

$$e^{\alpha A}f = e^{\alpha x D}e^{tG}f$$

where

$$G = L(De^{\alpha}) - L(D) \,.$$

This is analogous to the result for C, replacing translation by z with multiplication, exponential translation.

Proof. Apply Fourier transformation to

$$\frac{\partial u}{\partial \alpha} = x\frac{\partial u}{\partial x} + t\psi\frac{\partial u}{\partial x}$$

yielding, for $v = \hat{u}$,

$$\begin{aligned} \frac{\partial v}{\partial \alpha} &= i\frac{\partial}{\partial \xi}(i\xi v) + t\psi(i\xi)i\xi v \\ &= -\xi\frac{\partial v}{\partial \xi} + (t\psi(i\xi)i\xi - 1)v \,. \end{aligned}$$

The solution is seen to be

$$v = \hat{f}(\xi e^{-\alpha})e^{t\int_0^{\alpha}\psi(i\xi e^{-s})i\xi e^{-s}ds}e^{-\alpha}$$

by applying, say, Feynman-Kac for the deterministic process $\rho(\alpha) = \xi e^{-\alpha}$. Thus,

$$u = \int e^{i\xi x}\hat{f}(\xi e^{-\alpha})e^{t\int_0^\alpha \psi(i\xi e^{-s})i\xi e^{-s}ds}e^{-\alpha}d\xi$$
$$= \int e^{i\xi e^{\alpha}x}\hat{f}(\xi)e^{t\int_0^\alpha \psi(i\xi e^{\alpha-s})i\xi e^{\alpha-s}ds}$$
$$= e^{t\int_0^\alpha \psi(De^s)De^s ds}f(xe^\alpha) .$$

Setting $t = 0$ yields $e^{\alpha x D}f(x) = f(xe^\alpha)$.

Substituting $\psi = L'$,

$$\int_0^\alpha L'(De^s)De^s ds = L(De^\alpha) - L(D) .$$

Combining these two propositions gives

THEOREM. *$Af = 0$ if and only if*

$$e^{\alpha x D}e^{tG}f_k = e^{\alpha k}f_k \quad \textit{for all} \quad k \geqq 0 ,$$

where $G = L(De^\alpha) - L(D)$.

EXAMPLES.

1. $L = D$. $A = (x + t)D$. And

$$e^{\alpha A}f = e^{t(e^\alpha - 1)D}f(xe^\alpha) = f(xe^\alpha + te^\alpha - t) .$$

Eigenfunctions are $(x + t)^k$.

2. Brownian motion. $A = xD + tD^2$.

$$G = \frac{1}{2}D^2(e^{2\alpha} - 1) .$$

And

$$e^{\alpha A}f = \int f(xe^\alpha + y\sqrt{t(e^{2\alpha} - 1)})\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy .$$

Eigenfunctions are $h_k(x, t)$ where $h_k(x, -t)$ are Hermite polynomials. This is essentially the Ornstein-Uhlenbeck process.

3. Poission process. $A = xD + te^D D$.

$$G = e^{De^\alpha} - e^D .$$

And

$$e^{\alpha A}f = \sum_0^\infty \frac{t^k}{k!}\sum_p \begin{bmatrix} k \\ p \end{bmatrix}(-1)^{k-p}f(pe^\alpha + k - p + xe^\alpha) .$$

4. Singular process. $A = xD + t$.

$$G = \log De^\alpha - \log D = \alpha .$$

And

$$e^{\alpha A}f = e^{\alpha t}f(xe^{\alpha}) .$$

Eigenfunctions are x^{k-t}, as seen previously.

5. Cauchy process. $L = -|D|$.

$$G = -|D|(e^{\alpha} - 1) .$$

And

$$e^{\alpha A}f = \frac{1}{\pi}\int f(xe^{\alpha} + t(e^{\alpha} - 1)y)\frac{dy}{1 + y^2} .$$

For example, set

$$h(x) = \frac{1}{\pi}\int \sqrt{x + yt}\,\frac{dy}{1 + y^2} .$$

Then

$$\frac{1}{\pi}\int h(xe^{\alpha} + t(e^{\alpha} - 1)y)\frac{dy}{1 + y^2} = e^{\alpha/2}h(x) , \quad \text{for all} \quad \alpha \geqq 0 .$$

VII. Multidimensional theory. In the case where $x \in \boldsymbol{R}^N$ and $D = (D_1, D_2, \cdots, D_N)$ the exponential martingale is

$$e^{z\cdot\rho - tL(z)} .$$

The natural extension of the one-dimensional operator C is the vector operator

$$C = (C_1, \cdots, C_N)$$

with components

$$C_j = x_j + t[L, x_j] = x_j + t\frac{\partial L}{\partial D_j} \equiv x_j + tL_j .$$

The generating function

$$e^{z\cdot x + tL(Z)} = \sum_k \frac{z^k}{k!}h_k(x, t)$$

where the usual indexing notations, e.g., $z^k = z_1^{k_1} \cdots z_N^{k_N}$, are employed, is again represented as $e^{z\cdot C}1$. The corresponding basic theorem of § V will be the main topic of this section.

THEOREM. *Given $L(D) = L(D_1, \cdots, D_N)$. For C as defined above, f harmonic implies $e^{z\cdot C}f$ is harmonic.*

The proof will follow from the explicit form of $u = e^{z\cdot C}f$.

PROPOSITION. *The operators C_j commute.*

Proof.
Let $C_1 = x + tL_x$ and $C_2 = y + tL_y$ be two C_j's. Then

$$\begin{aligned} C_1C_2 &= (x + tL_x)(y + tL_y) \\ &= xy + txL_y + tL_xy + tL_xL_y \\ &= xy + txL_y + tyL_x + t\frac{\partial L_x}{\partial D_y} + tL_xL_y \\ &= xy + txL_y + tyL_x + tL_{xy} + tL_xL_y \end{aligned}$$

which is symmetric in x and y since $L_{xy} = L_{yx}$.
By the proposition,

$$e^{z\cdot C}f = \prod_j e^{z_jC_j}f \text{ (independently of the order of the product)}$$

and by the results of § V,

$$e^{z\cdot C}f = e^{z_1x_1}e^{t\Lambda_1}e^{z_2x_2}e^{t\Lambda_2}\cdots e^{z_Nx_N}e^{t\Lambda_N}f$$

where

$$\Lambda_j = L(D_1, \cdots, D_j + z_j, \cdots, D_N) - L(D)\ .$$

Similarly, the operators $A_j = x_jD_j + tL_jD_j$ commute and

$$e^{\alpha\cdot A}f = e^{\alpha_1x_1D_1}e^{tG_1}\cdots e^{\alpha_Nx_ND_N}e^{tG_N}f$$

where

$$G_j = L(D_1, \cdots, D_je^{\alpha_j}, \cdots, D_N) - L(D)\ .$$

Proof of the harmonicity theorem. For $u = e^{z\cdot C}f$,

$$\begin{aligned} \frac{\partial u}{\partial t} = \sum_j e^{z_1x_1}e^{t\Lambda_1}\cdots e^{z_jx_j}e^{t\Lambda_j}\Lambda_je^{z_{j+1}x_{j+1}}e^{t\Lambda_{j+1}}\cdots e^{z_Nx_N}e^{t\Lambda_N}f \\ + \prod_j e^{z_jx_j}e^{t\Lambda_j}\frac{\partial f}{\partial t}\ . \end{aligned}$$

While, using Lemma 2 of § V sequentially (i.e., by induction),

$$\begin{aligned} Lu &= Le^{z_1x_1}e^{t\Lambda_1}\cdots e^{z_Nx_N}e^{t\Lambda_N}f \\ &= e^{z_1x_1}e^{t\Lambda_1}(\Lambda_1 + L)e^{z_2x_2}\cdots e^{t\Lambda_N}f \\ &= e^{z_1x_1}e^{t\Lambda_1}\Lambda_1e^{z_2x_2}\cdots f + e^{z_1x_1}e^{t\Lambda_1}Le^{z_2x_2}e^{t\Lambda_2}\cdots e^{t\Lambda_N}f \\ &= \sum_j e^{z_1x_1}e^{t\Lambda_1}\cdots e^{z_jx_j}e^{t\Lambda_j}\Lambda_j\cdots e^{z_Nx_N}e^{t\Lambda_N}f \\ &\quad + \prod_j e^{z_jx_j}e^{t\Lambda_j}Lf \\ &= \frac{\partial u}{\partial t}\ . \end{aligned}$$

REMARKS 1. Using Remark 2 of § V, it is clear that

$$e^{z\cdot C}f = \prod_j e^{tL}e^{z_jx_j}e^{-tL}f = e^{tL}e^{z\cdot x}e^{-tL}f \, .$$

2. As before, check that, for $L1 = 0$, the exponential generating function satisfies

$$e^{z\cdot x+tL(z)} = e^{z\cdot C}1 \, .$$

Note the general formula as for one dimension:

$$e^{tL(D)}e^{z\cdot x} = e^{tL(z)}e^{z\cdot x} \, .$$

By the above remarks,

$$e^{z\cdot C}1 = e^{tL}e^{z\cdot x}e^{-tL}1 = e^{tL}e^{z\cdot x} = e^{tL(z)}e^{z\cdot x}$$

as required.

EXAMPLES. Here $(x_1, x_2) = (x, y)$, $(z_1, z_2) = (z, w)$.
1. $L = D_xD_y$. $\Lambda_1 = zD_y$. $\Lambda_2 = wD_x$

and

$$\begin{aligned} e^{z\cdot C}f &= e^{wy}e^{twD_x}e^{zx}f(x, y + zt) \\ &= e^{wy+zx}e^{ztw}f(x + wt, y + zt) \, . \end{aligned}$$

2. $L = D_x - (1/2)D_y^2$. $\Lambda_1 = z$. $\Lambda_2 = -(w^2/2) - wD_y$

and

$$\begin{aligned} e^{z\cdot C}f &= e^{zx+wy}e^{zt-w^2t/2}f(x, y - wt) \\ &= e^{z(x+t)}e^{wy-w^2t/2}f(x, y - wt) \, . \end{aligned}$$

VIII. Further examples and remarks.

A. *Homogeneity.*

If the generator L has homogeneity such that $aL(D) = L(a^cD)$, then

$$e^{z\cdot C}1 = e^{z\cdot x+tL(z)}$$

becomes, substituting $z \to a^cz$,

$$e^{a^cz\cdot x+tL(a^cz)} = e^{z\cdot a^cx+atL(z)} \, .$$

Thus,

$$h_k(a^cx, at) = a^{kc}h_k(x, t)$$

or equivalently $t^{kc}h_k(x, 1) = h_k(xt^c, t)$, for every k.
On $\boldsymbol{R}^N$, $aL(D) = L(a^cD)$ means

$$L(a^{c_1}D_1, \cdots, a^{c_N}D_N) = aL(D_1, \cdots, D_N)\ ,$$

and $a^{kc} = a^{k \cdot c}$.

B. *Higher Order Operators.* Extensions.

Consider the equation

$$\frac{\partial u}{\partial z} = xu + \frac{t}{2}u''\ .$$

This is of the type

$$\frac{\partial u}{\partial z} = Cu \text{ with } L = \frac{D^3}{6}\ .$$

So the solution is

$$u = e^{zC}f = e^{zx}e^{t\,'}f = e^{zx+z^3t/6}\int f\left(x + \frac{z^2t}{2} + y\sqrt{zt}\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy\ .$$

On the other hand, for convenience setting $t = 1$, the Feynman-Kac formula gives for Brownian motion $b(z)$,

$$u = < f(x + b(z))e^{\int_0^z (x+b(s))ds} >\ .$$

This suggests considering operators C of the form $\psi(x) + t\phi(D)$. In case either ψ or ϕ generates a process, then the solution is of (generalized) Feynman-Kac type. By Fourier duality,

$$\text{for } C = \psi(x) + t\phi(D)\ , \quad \text{set } \hat{C} = \psi\left(i\frac{d}{d\xi}\right) + t\phi(i\xi)\ .$$

Denote by p the process generated by $\phi(D)$ and by q the process generated by $\psi(i(d/d\xi))$—assuming that these are well-defined in some sense.

Set $u = e^{zC}f$ and $v = \hat{u} = e^{z\hat{C}}\hat{f}$.

$$\frac{\partial u}{\partial z} = \psi(x)u + t\phi(D)u \quad \text{and} \quad \frac{\partial v}{\partial z} = \psi\left(i\frac{d}{d\xi}\right)v + t\phi(i\xi)v\ .$$

Then with $\langle\ \ \rangle$ corresponding to expectation with respect to the process indicated,

$$u = \langle f(x + p(zt))e^{\int_0^z \psi(x+p(st))ds}\rangle$$

$$v = \langle \hat{f}(\xi + q(z))e^{t\int_0^z \phi(i\xi+iq(s))ds}\rangle\ .$$

Applying Fourier transformation to v and u yields the dual representations:

$$u = \langle e^{-iq(z)x} e^{t\int_0^z \phi(D+i(q(s))-q(z)))ds} \rangle f$$

$$v = \langle e^{i\xi p(zt)} e^{\int_0^z \psi(iD_\xi + p(st) - p(zt))ds} \rangle \hat{f} .$$

The solution u is represented in terms of either the p or q process.

In particular, then, if processes are defined for generators of, say, polynomial type of degree $< n$, then any process generated by a polynomial of degree n should satisfy the constraints imposed by the above duality requirements. E.g. For deg $\phi = n$, define p implicitly by requiring $u(p) = u(q)$, using ϕ as a potential for all choices of ψ, deg $\psi < n$.

In this way, inductively, a process and corresponding stochastic calculus is built up for all polynomial generators $L(D)$. By appropriate limiting procedures, extensions to operators with continuous generators and variable coefficients would result.

For example, for $\phi = D^3/6$, $\psi = - x^2/2$, $b(z)$ Brownian motion,

$$\langle f(x + p(zt)) e^{-1/2\int_0^z (x+p(st))^2 ds} \rangle = \langle e^{-ib(z)x} e^{1/6\int_0^z (D+ib(s)-ib(z))^3 ds} \rangle f$$

for all suitable functions f.

C. *Relation with field theory.*

In case L is skew-adjoint, C will be self-adjoint and conversely any set-up with Hermitian operators such that $[D, C] = 1$ is essentially of the above type by Stone-Von Neumann. C is the "creation operator" and D the "annihilation operator", A the "number operator"; h_n are the Wick products. See, e.g., Simon [7] and references there, Segal [5, 6] and Miller [3].

IX. Functional calculus for C. Fundamental operators η_p.

From § V,

$$e^{zC} = e^{tL} e^{zx} e^{-tL} .$$

Expanding in powers of z yields

$$C^n = e^{tL} x^n e^{-tL} .$$

In particular

$$C = e^{tL} x e^{-tL} .$$

So $\phi(C)$ is defined for any function ϕ for which

$$\phi(C) = e^{tL}\phi(x)e^{-tL} \text{ makes sense.}$$

Note that in particular, $\phi(C)f$ can be defined for harmonic f by

$$\phi(C)f = e^{tL}\phi(x)f .$$

Defining C directly, as above, yields an operator for any L that generates a group; for general Markov semigroups, with possibly variable-coefficient generators, $\phi(C)f$ is defined for harmonic f at least.

Before defining the fundamental operators, first recall Leibniz' rule for differentiating a product. It yields the following:

LEMMA. *(General Leibniz formula.)*
As operators,

$$\psi(D)\phi(x) = \sum_{p \geqq 0} \frac{1}{p!} \frac{\partial^p \phi}{\partial x^p} \frac{\partial^p \psi}{\partial D^p} .$$

Consequently,

$$[\psi(D),\ \phi(x)] = \sum_{p \geqq 1} \frac{1}{p!} \frac{\partial^p \phi}{\partial x^p} \frac{\partial^p \psi}{\partial D^p} .$$

Note. Assumptions on ψ are ϕ are not stated explicitly, depending on the domains involved. For the proof, it is assumed that ψ can be approximated by polynomials in some reasonable manner. See, e.g., recent notes on pseudo-differential operators, Taylor [10]—also references are there to papers by Kohn, Nirenberg, and Hormander—for technical clarifications concerning the generalized Leibniz formula.

Proof.

$$D^r \phi = \sum \begin{bmatrix} r \\ k \end{bmatrix} \frac{\partial^k \phi}{\partial x^k} D^{r-k} \text{ by basic Leibniz' rule} .$$

Or,

$$D^r \phi = \sum \frac{1}{k!} \frac{\partial^k \phi}{\partial x^k} \frac{\partial^k D^r}{\partial D^k} .$$

For $\psi \sim \sum c_r D^r$,

$$\psi\phi = \sum\sum \frac{1}{k!} \frac{\partial^k \phi}{\partial x^k} \frac{\partial^k}{\partial D^k} c_r D^r = \sum \frac{1}{k!} \frac{\partial^k \phi}{\partial x^k} \frac{\partial^k \psi}{\partial D^k} .$$

Using this formula,

$C = e^{tL} x e^{-tL} = x + [e^{tL},\ x] e^{-tL} = x + tL'$ as previously defined.

And

$$\phi(C) = e^{tL}\phi(x)e^{-tL} = \sum_{p \geqq 0} \frac{1}{p!} \frac{\partial^p \phi}{\partial x^p} \eta_p(D,\ t)$$

defining

$$\eta_p(D, t) = e^{-tL}\frac{\partial^p}{\partial D^p}e^{tL} .$$

These are the *fundamental operators.*

In particular,

$$C^n = \sum \begin{bmatrix} n \\ p \end{bmatrix} x^{n-p}\eta_p .$$

It is now seen that the moments

$$\mu_r = \int y^r p_t(y) = \eta_r 1 .$$

This enables the formulation of the basic theorem as follows.

HARMONICITY THEOREM.

The following are equivalent:

(1) *f is harmonic.*

(2) *$e^{zC}f$ is harmonic.*

(3) *$\phi(C)f$ is harmonic for all suitable ϕ.*

(4) *$\sum \begin{bmatrix} n \\ p \end{bmatrix} x^{n-p}\eta_p f$ is harmonic for $n \geqq 0$.*

Proof.

$1 \Rightarrow 2$ and 3:

For $u = \phi(C)f$,

$\dfrac{\partial u}{\partial t} = Lu + e^{tL}\phi(x)e^{-tL}\left(\dfrac{\partial}{\partial t} - L\right)f = Lu$, as for Remark #2 in §V.

$3 \Rightarrow 4$: follows from the above formula for C^n via the general Leibniz lemma.

$4 \Rightarrow 1$: $\eta_0 =$ identity.

EXAMPLE. The gamma distributions.

Consider the case

$$P_t(y) = \frac{y^{t-1}e^{-y}}{\Gamma(t)}(y \geqq 0)$$

$$\int e^{i\xi y}P_t(y) = \left[\frac{1}{1 - i\xi}\right]^t .$$

So $L = -\log(1 - D)$, $L' = (1 - D)^{-1}$. And

$$\eta_p = (1 - D)^t\frac{\partial^p}{\partial D^p}(1 - D)^{-t} = \frac{(-1)^p(-t)!}{(-t-p)!}(1 - D)^{-p}$$
$$= t(t + 1)\cdots(t + p - 1)(1 - D)^{-p} .$$

$$C^n = (1 - D)^{-t} x^n (1 - D)^t$$

and

$$C^n 1 = (1 - D)^{-t} x^n .$$

Now it is easy to see that setting $t = -n$ yields $q_n(x) = (x - nL')^n 1 = (1 - D)^n x^n$ an orthogonal system on $[0, \infty)$ relative to e^{-x}, i.e., they are essentially the Laguerre polynomials. Recall the definition of $C^n 1$ in terms of moments of P_t. This yields formally

$$q_n(x) = \int_0^\infty (x + y)^n \frac{y^{-n-1} e^{-y}}{\Gamma(-n)} dy$$

with

$$\mu_r = \frac{1}{\Gamma(-n)} \int_0^\infty y^{r-n-1} e^{-y} dy .$$

From the formula for η_n,

$$\mu_r = \eta_r 1 = \frac{(-1)^r n!}{(n-r)!} (1 - D)^{-r} 1 = \frac{(-1)^r n!}{(n-r)!}$$

since $(1 - D)^{-1} 1 = 1$.

If the gamma function were defined for negative integers, the first formula for μ_r would give

$$\mu_r = \frac{\Gamma(r - n)}{\Gamma(-n)} = \frac{(-1)^r n!}{(n - r)!} .$$

This shows that it would be consistent to define

$$\Gamma(-n) = \frac{(-1)^n}{n!} \text{ if only ratios are computed} .$$

That is, define $\Gamma(0) = \zeta$, with $0 \cdot \zeta = 0\Gamma(0) = \Gamma(1) = 1$. Then setting

$$\Gamma(-n) = \frac{(-1)^n}{n!} \zeta$$

also preserves the basic property $x\Gamma(x) = \Gamma(x + 1)$ as indicated by the computation of μ_r. Note that Γ has simple poles at $-n$ for $n \geqq 0$ with residue $(-1)^n / n!$.

Finally, note the difference between the Laguerre and Hermite systems.

$H_n(x, t) = (x - tD)^n 1$ are orthogonal for $t > 0$ for the measure with density $e^{-x^2/2t}$ while $L_n(x) = (x - n(1 - D)^{-1})^n 1$ requires setting $t = -n$ to have an orthogonal system with respect to the single density $e^{-x} (x \geqq 0)$.

Generally, $H_n = (x - tL')^n\, 1$ does satisfy a generalized three-term recurrence relation,

$$xH_n = tL'H_n + H_{n+1}\,.$$

IXa. Appendix to Section IX. Computing η_p. The following formula for computing the η's is convenient in the case L is analytic.

PROPOSITION.

$$\eta_N = \sum_{\substack{\Sigma k p_k = N \\ p_k \geqq 0}} \frac{t^{p_1+\cdots+p_k}}{\prod_k (k!)^{p_k}} \begin{bmatrix} N \\ p_1 \cdots p_N \end{bmatrix} L_1^{p_1} L_2^{p_2} \cdots L_N^{p_N}$$

where

$$L_j = \left[\frac{\partial}{\partial D}\right]^j L\,.$$

Proof. $\eta_N = e^{-tL}[\partial/\partial D]^N e^{tL}$. Multiply by $a^N/N!$ in the formula above and sum. The left-hand side becomes:

$$e^{t(L(D+a)-L(D))} = \exp\left[t \sum_{N=1}^{\infty} \frac{a^N L_N}{N!}\right].$$

Expanding the exponential and collecting terms corresponding to each a^N yields the formula.

REMARK. Observe that, for $t = 1$, the coefficient of $L_1^{p_1} \cdots L_N^{p_N}$ is the number of ways of grouping N elements to have p_k groups of k elements each.

X. The number operator. Vacuum functions. This section continues § VI as a further study of the operator A. Recall that in one dimension

$$A = CD = e^{tL}xe^{-tL}D = e^{tL}xDe^{-tL}\,.$$

In the multi-dimensional case, the vector operator A acts in each coordinate. As seen in § VI, $Af = 0$ implies that $A_j f_k = k_j f_k$ for each multi-index k. A scalar operator can be defined that acts similarly.

DEFINITION. The *number operator* is the scalar operator defined by

$$A = C \cdot D = \sum C_j D_j = \sum A_j\,.$$

In the following, A will denote this scalar operator; A_j will denote components of the original vector operator.

PROPOSITION.
(1) $A_j f = 0$ *for* $1 \leqq j \leqq N$ *implies* $Af_k = |k| f_k$.
(2) *In fact,* $Af = 0$ *implies* $Af_k = |k| f_k$.

Proof.
(1) follows from above remarks.
(2) $$\begin{aligned} Ae^{z \cdot C} f &= e^{tL} x \cdot D e^{z \cdot x} e^{-tL} f \\ &= e^{tL} x \cdot z e^{z \cdot x} e^{-tL} f + e^{tL} e^{z \cdot x} x \cdot D e^{-tL} f \\ &= z \cdot D_z e^{tL} e^{z \cdot x} e^{-tL} f. \end{aligned}$$
Since,

$$e^{tL} e^{z \cdot x} x \cdot D e^{-tL} f = e^{z \cdot C} A f = 0 .$$

The conclusion follows as in the one-dimensional case.

DEFINITION. A *vacuum function* $f(t, x)$ satisfies by definition:
(1) $\partial f / \partial t = Lf$, i.e., f is harmonic.
(2) $Af = 0$.

Thus, f_k are harmonic functions and eigenfunctions of A as well. The usual choice for the vacuum, as seen previously, is the constant function 1. However, in singular cases, e.g., Example 4 of § IV, other choices may be necessary.

In the case f is independent of t, the vacuum conditions simplify.

PROPOSITION. *When* $\partial f / \partial t = 0$, f *is a vacuum function if and only if*
(1) $Lf = 0$.
(2) $x \cdot Df = 0$.

Proof. To verify (2). Note that since f is independent of t, setting $t = 0$ in the equation $Af = 0$ reduces to (2).

Conversely, assuming (1) and (2) yields:

$$Af = e^{tL} x \cdot D e^{-tL} f = e^{tL} x \cdot Df = 0 .$$

Regarding equation (2) of the above proposition, note the following.

PROPOSITION. $x \cdot Df = 0$ *if and only if* f *is homogeneous of degree zero*, i.e.,

$$f(\lambda x) = f(x) \textit{ for scalar } \lambda .$$

Proof. $x \cdot Df = 0$ implies $e^{\alpha x \cdot D} f = f(x) = f(e^{\alpha} x)$, for scalar α.

Differentiating $f(\lambda x) = f(x)$ with respect to λ yields the converse.

A "general solution" is thus of the form

$$f(x) = \sum_{a \in \boldsymbol{R}^N} Q_a(x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N})$$

with a's such that $\sum a_j = 0$ and the Q_a functions of one variable. Note that any analytic f homogeneous of degree zero must be constant.

EXAMPLES.

1. Brownian notion. $L = 1/2 \sum D_j^2$. $C_j = x_j + tD_j$. $A = C \cdot D = \sum (x_j D_j + tD_j^2)$. Since any harmonic f is analytic, the only vacuums are constant functions.

2. $L = D_x D_y$. $C_x = x + tD_y$. $C_y = y + tD_x$. $A = xD_x + 2tD_x D_y + yD_y$. A harmonic function is of the form

$$f = F(x) + G(y) .$$

And $x \cdot Df = 0$ yields

$$xF'(x) + yG'(y) = 0$$

or

$$xF'(x) = \text{constant} = - yG'(y) .$$

Choosing the particular solutions

$$F = \log x \, , \quad G = - \log y$$

yields

$$f = \log \frac{x}{y} .$$

So

$$u = e^{zC_x + wC_y} f = e^{wy + zx + twz} \log \frac{x + wt}{y + zt}$$

is harmonic and $Au = (zD_z + wD_w)u$.

$$\text{For } h_{nm} = C_x^n C_y^m f, \ Ah_{nm} = (n + m)h_{nm} .$$

E.g.,

$$h_{10} = x(\log x - \log y) - t/y \, , \quad Ah_{10} = h_{10} .$$
$$h_{11} = (xy + t)(\log x - \log y) \, , \quad Ah_{11} = 2h_{11} .$$

3. $L = D_x - D_y^2$. $C_x = x + t$. $C_y = y - 2tD_y$.
$A = xD_x + tD_x + yD_y - 2tD_y^2$.

For f, homogeneous of degree zero, of the form

$$f = Q\left[\frac{x}{y}\right] ,$$

to be harmonic yields the equation

$$yQ' + 2\left[\frac{x}{y}\right]Q' + \left[\frac{x}{y}\right]^2 Q'' = 0\,.$$

Thus, $yQ' = 0$ and Q must be constant.

In case f depends on t, the remark in § V regarding the 2nd series indicates the canonical choice

$$f = D^{-1}p_t(-x) = \int_{-\infty}^{x} p_t(-y)dy = \int_{-x}^{\infty} p_t(y)dy\,.$$

Harmonicity follows immediately from

$$p_t(-x) = \frac{1}{2\pi}\int e^{i\xi x}e^{tL(i\xi)}$$

and the commutativity of the operators $\partial/\partial t$, L, and D^{-1}.

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