ON THE DEGREE OF THE SPLITTING FIELD OF AN IRREDUCIBLE BINOMIAL

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Let $x^m-a$ be irreducible over a field $F$. We give a new proof of Darbi's formula for the degree of the splitting field of $x^m-a$ and investigate some of its properties. We give a more explicit formula in case the only roots of unity in $F$ are $±1$.

A formula for the degree of the splitting field of an irreducible binomial over a field $F$ of characteristic 0 was given in 1926 in the following:

THEOREM (Darbi [1]). Let $ζ_m$ denote a primitive $m$th-root of unity and let $x^m-a \in F[x]$ be irreducible with root $α$. Define an integer $k$ as follows:

$$k = \max\{l : l \mid m \text{ and } α^{m/l} \in F(ζ_m)\}.$$  

Then the degree of the splitting field of $x^m - a$ is $mφ_F(m)/k$, where $φ_F(m) = [F(ζ_m): F]$.

In § 1 of this paper we give a new proof of this theorem which, with an appropriate interpretation of the symbols above, will also be valid when char $F > 0$. In § 2, with the aid of a theorem of Schinzel, we obtain some properties of the number $k$, defined as in (1). Finally in § 3, we will express $k$ explicitly as a function of $a$ and $m$ for a field $F$ of characteristic 0 such that the only roots of unity in $F$ are $±1$.

1. Proof of Darbi's theorem for arbitrary characteristic. Let char $F = p > 0$ and let $m$ be a positive integer. Set $m = m_o p^f$, with $(m_o, p) = 1$ and set $ζ_m = ζ_{m_o}$. Thus $φ_F(m) = φ_F(m_o)$.

Our first step is to reduce the proof of the general theorem to a proof of the separable case, that is, to the case where char $F^f m$. Indeed, let char $F = p > 0$ and $x^m - a$ be irreducible over $F$ with root $α$. The splitting field of $x^m - a$ is $F(α, ζ_m) = F(α^{p^f}, α^{m_o}, ζ_{m_o})$, which in turn is the compositum, over $F$, of $F(α^{p^f}, ζ_{m_o})$, a separable extension of $F$, and $F(α^{m_o})$, a purely-inseparable extension. Thus, if Theorem 1 were true for the separable case, $x^{m_o} - a$ (with splitting field $F(α^{p^f}, ζ_{m_o})$), then we would have:

$$[F(α, ζ_{m_o}): F] = p^f(m_oφ_F(m_o)/k) = mφ_F(m)/k.$$
We therefore assume, for the rest of this paper, that \( \text{char } F \neq m \). To complete the proof we will use the following:

**Lemma (Norris and Velez, [5]).** Let \( x^m - a \) be irreducible over \( F \) with root \( \alpha \). Let \( n = \max \{ l : l \mid m \text{ and } \zeta_i \in F(\alpha) \} \) and suppose \( K \) is a field such that \( F(\zeta_n) \subseteq K \subseteq F(\alpha) \). If \( l = [F(\alpha):K] \), then \( K = F(\alpha^l) \).

**Proof.** Let \( f(x) \) denote the irreducible polynomial that \( \alpha \) satisfies over \( K \). Since \( \alpha^m = \alpha \in F \subseteq K \), we have that \( f(x)|x^m - a \). Thus, every root of \( f(x) \) is of the form, \( \zeta_i^a \), for some \( i \). Hence, \( f(x) = \prod_{i=1}^{\frac{m}{l}} (x - \zeta_i^a) \). The constant term of \( f(x) \), \( \prod_{i=1}^{\frac{m}{l}} \zeta_i^a = \zeta_m^a \), is an element of \( K \subseteq F(\alpha) \). Also \( \alpha^l \in F(\alpha) \), thus \( \zeta_m^a \in F(\alpha) \), and by the definition of \( n \), \( \zeta_m^a \in F(\zeta_n) \subseteq K \), thus \( \alpha^l \in K \). Now \( l = [F(\alpha):K] \) and \( [F(\alpha):F(\alpha^l)] \leq l \), since \( \alpha \) satisfies the binomial \( x^l - \alpha^l \) over \( F(\alpha^l) \). Hence we must have that \( F(\alpha^l) = K \) and \( x^l - \alpha^l \) is irreducible over \( K \).

To complete the proof of Darbi's theorem, let \( k' = [F(\zeta_m) \cap F(\alpha) : F] \). It is clear that the order of the splitting field \( x^m - a \) is \( m/\phi(m)/k' \). We must show that \( k = k' \). Now, by the definition of \( n \) in the above lemma, \( F(\zeta_n) \subseteq F(\zeta_m) \cap F(\alpha) = K \subseteq F(\alpha) \), and thus, by the lemma, we have that there is an integer \( l \) such that \( K = F(\alpha^l) \). Clearly, since \( x^m - a \) is irreducible, \( [K:F] = m/l = k' \). This proves the theorem since \( \alpha^l \in F(\zeta_m) \) and \( l = m/k' \).

2. Some properties of the denominator \( k \) and \( x^k - a \). For irreducible \( x^m - a \in F[x] \), let \( k \) be defined as in formula (1). Set

\[(2) \quad h = \max \{ l : l \mid m \text{ and } x^l - a \text{ has abelian Galois group} \} . \]

Then it is easy to see from the proof of Darbi's theorem that there exist positive integers \( t_1, t_2 \) such that

\[(3) \quad h = \phi(p(h))t_1 = k t_2, \text{ where } t_2 | t_1 . \]

We would like to derive some properties of \( h, t_1, \) and \( t_2 \). For an integer \( q \), let \( w_q \) be the number of the \( q \)-th-roots of unity in \( F \) and \( \mathcal{P}(q) \) be the set of primes dividing \( q \). Then we have:

**Theorem (Schinzel).** A binomial \( x^m - a \in F[x] \) has abelian Galois group iff \( a^{w_m} = c^m \), for some \( c \in F \).


From this we obtain

**Proposition 1.** (A) Let \( x^m - a \) be irreducible with abelian
Galois group. Then \( x^m - a \) is normal and, if \( p \) is a prime and \( p \mid m \), then \( \zeta_p \in F \), that is, \( p(m) \subseteq p(w_m) \). Moreover \( \phi_F(m) \mid m \).

(B) Let \( x^m - a \) be irreducible and \( h, t, \) defined as in (2) and (3). Then \( p(h) \subseteq p(w_h) \) and \( t_1 \mid w_h \).

Proof. (A) Suppose \( p \) prime, \( p \mid m \) and \( \zeta_p \in F \). Then \( \zeta_p \in F \), and, if \( p \mid m \), then \( \zeta_p \in F \). Moreover \( \phi_F(m) \mid m \).

To complete the proof, since \( x^m - a \) is irreducible and normal, \( F(\alpha) \) is the splitting field of \( x^m - a \), for any root \( \alpha \) of \( x^m - a \). Thus \( \zeta_m \in F(\alpha) \), so \( F(\zeta_m) \subset F(\alpha) \) and \( \phi_F(m) \mid m \).

(B) In view of (A), all we need to show is that \( t_1 \mid w_h \). To do this, let \( \beta \) be a root of \( x^h - a \). Then \( t_1 = [F(\beta): F(\zeta_h)] \). Thus, \( F(\beta^i) = F(\zeta_h) \) by the lemma. Since \( x^{h_1} - \beta^{h_1} \) is irreducible over \( F(\zeta_h) \), we have that \( \beta^i \in F(\zeta_h) \) iff \( t_1 \mid l \). However, by Schinzel’s theorem we have \( \alpha^{w_h} = c^h \) (for some \( c \in F \)), so that \( \beta = \zeta_h^i \zeta_h^{w_h} \zeta_h^{c^h/2} \), for some \( i, j \). Thus \( \beta^{w_h} = \zeta_h^i \zeta_h^{w_h} \zeta_h^{c^h/2} \in F(\zeta_h) \), and consequently \( t_1 \mid w_h \).

3. Applications. In this section let \( F \) denote a field with the following two properties: (a) \( \text{char } F = 0 \), and (b) if \( \zeta_m \in F \), then \( \zeta_m = \pm 1 \). Clearly real fields satisfy properties (a) and (b). Furthermore, \( w_m = 1 \) if \( m \) is odd and \( w_m = 2 \) if \( m \) is even.

**Proposition 2.** (A) The irreducible, normal binomials in \( F[x] \) with abelian Galois groups are:

(i) \( x - c \)

(ii) \( x^2 - c, \sqrt{c} \in F \)

(iii) \( x^4 + c^2, c^2 \neq 4d^4, d \in F \)

(iv) \( x^h + c^{h-1}, h \geq 3, \sqrt{2} \in F, c \neq 0 \).

(B) Relative to the irreducible binomial \( x^m - a \in F[x] \),

(i) \( h = \max \{2^s: 2^s \mid m \text{ and } -a = c^{2^{s-1}}, c \in F\} \).

(ii) \( t_1 = \begin{cases} 1, & \text{if } h = 1. \\ 2, & \text{if } h > 1. \end{cases} \)

(iii) \( k = \begin{cases} h, & \text{if } h = 1 \text{ or } h = 2^s, -a = c^{2^{s-1}} \text{ and } \zeta_{w_h} \sqrt{c} \in F(\zeta_m). \\ h/2, & \text{otherwise.} \end{cases} \)

In particular, \( k \) is a power of 2. If \( \sqrt{2} \in F \), then any power of 2 is possible. If \( \sqrt{2} \in F \), then \( k = 1, 2, \text{ or } 4 \).

Proof. (A) If \( x^m - a \) is irreducible, normal, and abelian, then by Proposition 2, we have that \( m = 2^q \), for some \( q \geq 0 \). Schinzel’s theorem then implies \( a^2 = c^{2^q} \), for some \( c \in F \). Thus, if \( q \geq 1 \), \( a = \pm c^{2^q-1} \). The rest follows by Cappelli’s theorem for irreducible
binomials ([4], p. 62).

Conversely, it is easy to check that the binomials (i)—(iv) are irreducible, normal, with abelian Galois group.

(B) Statement (i) follows from (A).

To prove (ii), note first that by Proposition 2, \( t_1 \mid w_{q^2} \). Thus \( t_1 = 1 \) or 2. If \( h = 1 \), then clearly \( t_1 = 1 \). Assume that \( h > 1 \). Recall that \( t_1 = [F(\beta): F(\zeta_{q^2})] \), where \( \beta \) is a root of \( x^{q^2} + a^{q^2} \). If \( h = 2 \), then since \( [F(\zeta_q): F] = 2 \), we must have that \( t_1 = 2 \). If \( q > 2 \), then by (A) we have that \( \sqrt{2} \in F \). Hence \( [F(\zeta_{q^2}): F] = 2^{q^2-1} \), and thus \( t_1 = 2 \).

Finally, to prove (iii), we note that \( t_2 \mid t_1 \) and by (ii), \( t_1 = 1 \) or 2, so \( t_2 = 1 \) or 2. Furthermore, if \( h = 2^r(q \geq 1) \) then \( t_2 = 1 \) iff the splitting field of \( x^{q^2} + a^{q^2} \) is contained in \( F(\zeta_m) \) iff \( \zeta_{q^2+1} \sqrt{c} \in F(\zeta_m) \).

Thus, if the \( h \) of formula (2) has been determined, then

\[
k = \begin{cases} h, & \text{if } h = 1 \text{ or } \sqrt{c} \in F(\zeta_{q^2m}) \\ h/2, & \text{otherwise.} \end{cases}
\]

If \( m = 2^l \cdot p_1^{q_1} \cdots p_k^{q_k} \), with \( l \geq 1 \) and \( p_1, \ldots, p_k \) distinct odd primes, then the condition \( \sqrt{c} \in F(\zeta_{q^2m}) \) is equivalent to the condition \( \sqrt{c} \in F(\zeta_{q^2+1p}) \), where \( P = p_1 \cdots p_k \). For \( F = Q \), the latter is equivalent to \( \sqrt{c} \in Q(\zeta_{q^2p}) \), where \( a = \min\{3, l + 1\} \). For an arbitrary real field however, we cannot do as well. Indeed, given any integer \( q \geq 3 \), there exists an integer \( m \) with \( 2^r \mid m \), a real field \( F \) and \( c \in F \) such that \( \sqrt{c} \in F(\zeta_{q^2m}) \), yet \( \sqrt{c} \in F(\zeta_m) \). (See [2], 5.4.)

Proposition 2 generalizes a theorem of Hooley ([3], pp. 212-214).

REFERENCES


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