SCHUR’S THEOREM AND THE DRAZIN INVERSE

ROBERT E. HARTWIG
SCHUR'S THEOREM AND THE DRAZIN INVERSE

ROBERT E. HARTWIG

It is shown that if $M = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$ is a square $2n \times 2n$ matrix over a ring $R$, such that $AC = CA \in R_{n \times n}$, and with the property that $A$ and $C$ possess Drazin inverses, then $M$ is invertible in $R_{2n \times 2n}$ if and only if $DA - BC$ is invertible in $R_{n \times n}$.

1. Introduction. In a recent paper [7], Herstein and Small extended the classic result of Schur [5, p. 46] to matrices over $E$-rings. These are rings for which every primitive image is artinian. This result states that for a square complex block matrix $M = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$, with $A, B, C, D$ square of the same size such that $AC = CA$, then $M$ is invertible exactly when $\Delta = DA - BC$ is invertible. This is a different but equivalent formulation of the problem as stated in [7].

The purpose of this note is to show that this result by Schur is basically a consequence of the local existence of the Drazin inverse [2] of the matrices $A$ and $C$; that is, the strong-$\pi$-regularity of $A$ and $C$ [1] [4]. The proof of [7] was based on the fact that Schur's result for matrices over $E$-rings is really equivalent to the corresponding result for matrices over simple artinian rings (which may be taken to be division rings). Since artinian rings with unity are noetherian [8], p. 69, it follows that artinian rings with unity are strongly-$\pi$-regular, so that our local result extends the Schur theorem for artinian rings as proven in [7].

The Drazin inverse $a^d$ of a ring element $a$, is the unique solution, if any, to the equations

\[(1) \quad a^dxa = a^d, \quad xax = x, \quad ax = xa,\]

for some $k \geq 0$, while the group inverse $a^g$ of $a$ is the unique solution, if any, of these equations with $k = 0$, or 1. For example, if $a$ is algebraic over some field $F$ and $a^{n+1}b = a^n$, with $ab = ba$, then $a^d = a^nb^{n+1}$. The element $a^d$ exists if and only if $a$ is strongly-$\pi$-regular, that is, when both chains \{a\}$^R$ and \{Ra$^i$\} are ultimately stationary, [5, Theorem 4]. A ring element is called (von Neumann) regular if $aa^{-}a = a$ for some ring element $a^{-}$. If there exists such $a^{-}$ that is invertible, $a$ is called unit-regular.

We shall assume familiarity with the properties of these inverses [4] [2] [6] and in particular with the fact that $ac = ca \Rightarrow a^dc = ca^d$ [4, Theorem 1].

It is known that, unlike regularity and unit regularity, $R_{2 \times 2}$ does
not inherit strong-regularity from $R$ \[9\] \[11\]. It is not known however, whether the strong-$H$-regularity of $R$, or the related concept of finite regularity $(ab = 1 \Rightarrow ba = 1)$ is inherited by $R_{2 \times 2}$ \[10\].

We shall use the notation $^o S$ and $S^o$ to indicate the right and left annihilators of $S$ respectively, e.g.,

$$S^o = \{x \in R; xs = 0, \forall s \in S\}.$$ 

For notational convenience we shall state our results in terms of rings $R$ with unity, with the translation to matrices over $R$, being self evident. In particular $aR + cR = R$ is equivalent to the $1 \times 2$ matrix $[a, c]$ having a right inverse.

2. Preliminaries. The key to our main result are the following two lemmas.

**Lemma 1.** Let $R$ be a ring with unity $1$, and let $e, f$ be commuting idempotents in $R$. If $g = e + f(1 - e)$ then

(i) $g^2 = g$, (ii) $eR + fR = gR$, (iii) $Re + Rf = Rg$, (iv) $e^o \cap f^o = g^o$, (v) $e^o \cap f^o = g^o$,

vi) $eR + fR = R \iff g = 1 \iff Re + Rf = R \iff e^o \cap f^o = 0 \iff (1 - e)(1 - f) = 0.$

**Lemma 2.** Let $R$ be a ring with unity $1$, and let $a, c$ be commuting elements of $R$. Then

(i) $aR + cR = R \iff a^nR + c^nR = R$ for some $m, n \geq 1 \iff a^nR + c^nR = R$ for all $m, n \geq 1$.

(ii) $^oa \cap ^oc = (0) \iff ^o(a^n) \cap ^o(c^n) = 0$ for some $m, n \geq 1 \iff ^o(a^n) \cap ^o(c^n) = 0$ for all $m, n \geq 1$.

(iii) $Ra + Rc = R \iff Ra^m + Rc^n = R$ for some $m, n \geq 1 \iff Ra^m + Rc^n = R$ for all $m, n \geq 1$.

(iv) $a^o \cap c^o = (0) \iff (a^n)^o \cap (c^n)^o = (0)$ for some $m, n \geq 1 \iff (a^n)^o \cap (c^n)^o = (0)$ for all $m, n \geq 1$.

If in addition, the Drazin inverses $a^d$ and $c^d$ exists, these conditions are all equivalent to

(v) $(1 - aa^d)(1 - cc^d) = 0.$

**Proof.** The proof of (i)-(iv) follows by induction. Now suppose that $a^d$ and $c^d$ exist and that index $(a) = k$, index $(c) = l$. Then for all $m \geq k, a^mR = a^kR = a^dR = a^dR$. And so, taking $m \geq k, n \geq l$, we see that (i) is equivalent to

$$R = a^mR + c^nR = a^kR + c^lR = a^dR + c^dR = a^dR + c^dR,$$

which by Lemma 1 is equivalent to

$$(3) \quad (1 - aa^d)(1 - cc^d) = 0.$$
Left-right symmetry now shows that (iii) is also equivalent to (v). Lastly, since for $m \geq k$, $(a^m)^0 = (a^k)^0 = (a^d)^0 = (a^d a)^0$, it follows that with $m \geq k$, $n \geq l$, (iv) is equivalent to $(a^d a)^0 \cap (a^d c)^0 = (0)$, which again by Lemma 1 is equivalent to (v). Symmetry again yields the remaining equivalence.

Before proceeding with our theorem we remark that:
1. It is not necessary for $a^d$ and $c^d$ to exist in order for

$$Ra + Rc = R \iff a^o \cap c^o = (0)$$

to be valid. It would suffice if $a$, $c$ and $c(1 - a^{-1}a)$ were regular.

2. The equivalence of (iv) and (v) has uses in the theory of differential equations, [2] Lemma 1. The above furnishes a short and purely algebraic proof of this useful result.

3. Main results.

**THEOREM 1.** Let $R$ be a ring with unity $1$ and let $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R_{2 \times 2}$ with $ac = ca$. Suppose further that $a^d$ and $[(1 - aa^d)c]^d$ exist. If $\Delta = da - bc$, then:

(i) $\Delta$ is left invertible $\iff M$ is left invertible.

(ii) $M$ is right invertible $\iff \Delta$ is right invertible.

(iii) $M$ is invertible $\iff \Delta$ is invertible.

**Proof.** Consider the matrix

$$(4) \quad N = \begin{bmatrix} a & u \\ b & z \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 1 & -a^d c \\ 0 & 1 \end{bmatrix},$$

where $u = (1 - aa^d)c$ and $z = d - ba^d c$. Since $a$, $a^d$ and $c$ commute it follows that

$$(5) \quad za - bu = (d - ba^d c)a - b(1 - aa^d)c = da - bc = \Delta.$$  

Now because $a^d u = 0 = ua^d = a^d u^d = u^d a^d$, we may construct the matrices:

$$(6) \quad \begin{bmatrix} a & u \\ b & z \end{bmatrix} \begin{bmatrix} a^d & -u \\ u^d & a \end{bmatrix} = \begin{bmatrix} aa^d + uu^d & 0 \\ t & \Delta \end{bmatrix} = T$$

and

$$(7) \quad \begin{bmatrix} a & u \\ -u^d & a^d \end{bmatrix} \begin{bmatrix} a^d & -u \\ u^d & a \end{bmatrix} = \begin{bmatrix} aa^d + uu^d & 0 \\ 0 & aa^d + uu^d \end{bmatrix}$$

$$= \begin{bmatrix} a^d & -u \\ u^d & a \end{bmatrix} \begin{bmatrix} a & u \\ -u^d & a^d \end{bmatrix}.$$
In general however, $aw^d \neq 0$ unless index $(a) \leq 1$. Suppose now that $A$ has a left inverse $A^-$, then by $(5)$,

\[(8)\]

$$R = Ra + Rc = Ra + Ru.$$ 

By Lemma 2, applied to $a$ and $u$, we see that

$$(1 - aa^d)(1 - uu^d) = 0$$

or equivalently

\[(9)\]

$$aa^d + uu^d = 1.$$ 

Hence, by $(7)$, it follows that the matrix $P = \begin{bmatrix} a^d & -u \\ u^d & a \end{bmatrix}$ is invertible.

Now since

$$\begin{bmatrix} 1 & 0 \\ -A^t & A^- \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & A \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

it follows that $M$ has a left inverse $M^-$ and that

$$R = Ra + Rb = Rc + Rd.$$ 

If in addition $AA^- = I$, then

$$T = \begin{bmatrix} 1 & 0 \\ t & A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -A^t & A^- \end{bmatrix} = I$$

and consequently $M$ is also invertible.

Conversely, suppose that $MM^- = I$. Then because $N$ also has a right inverse, it follows that

$$aR + uR = R.$$ 

Again by Lemma 2, applied to $a$ and $u$, we may conclude that $(9)$ holds so that $P$ is invertible. Hence $T = \begin{bmatrix} 1 & 0 \\ t & A \end{bmatrix}$ has a right inverse $T^- = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$. Now $TT^- = I \Rightarrow \gamma = 0 \Rightarrow \delta = 1$, and so $A$ has a right inverse.

If in addition, $M^-M = I$, then $T^-T = I$ and hence again as $\gamma = 0$, $\delta A = 1$, completing the proof.

**Corollary 1.** If $R$ is a ring with unity and $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R_{2 \times 2}$ with $ac = ca$ such that $a^d$ and $c^d$ exist, then $M$ is invertible if and only if $A = da - bc$ is invertible.

**Proof.** Note that $ac = ca$ implies that $aa^d c = c a a^d$, so that $u^d = (1 - aa^d) c^d$. Again because square matrices over artinian ring with unity possess Drazin inverses, this result includes the second part of Theorem 2 of [7].
COROLLARY 2. Let $R$ be a ring with unity 1, and let $a, c \in R$ such that $ac = ca$ and $a^d, [(1 - aa^-)c]^d$ exist. Then if $R = Ra + Rc$ there exists $d \in R$ so that $\begin{bmatrix} a & c \\ c & a + c^a \end{bmatrix}$ is invertible.

Proof. From Theorem 1, it suffices to select $d \in R$ such that $\Delta = da - c^a$ is invertible. One such choice is given by $d = a^d + c^a d$, because then $\Delta = aa^d - w^d$ which has inverse $aa^d - w^d u^d$. Indeed, if $R = Ra + Rc = Ra + Ru$, then $aa^d + uu^d = 1$ which coupled with the fact $a^d u^d = 0$, yields the desired result.

We conclude this note with several remarks.
1. If $a^e$ exists we could also select $d = a + c^a a$ in the last corollary, for then $\Delta = a^2 - u^2$ has as inverse $(a^e)^2 - uu^d$ since now $aa = 0$. Moreover, in this case

$$
\begin{bmatrix} a & c \\ c & a + c^a \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a^2 c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & u^d \\ u^d & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a - (a^e)^2 c^2 & a^d - c(a^e)^2 \\ u^d - c(a^e)^2 & a^e \end{bmatrix}.
$$

2. The fact that: "If $ac = ca$, and $a^d, w^d$ exist, then $R = Ra + Rc$ ensures that $a^d a + w^d u = 1$", should be compared with the corresponding results for Moore-Penrose inverses [6]. Namely, if $a^f$ and $v^f = [c(1 - a^a)]^f$ exists, then

$$
R = Ra + Rc \implies 1 = a^f a + v^f v.
$$

3. If $a$ is unit-regular, that is $aa^a a = a$ for some unit $a^a$, then under suitable conditions $aR + cR = R = Ra + Rc = R$. Indeed if $u = (1 - aa^-)c$ is regular and $c^a$ exists, then

$$
aR + cR = R \implies aa^a + (1 - a^a)cu^- (1 - aa^-) = 1.
$$

Thus $aa^a[(1 - aa^-)cu^- (1 - aa^-)] + cu^- (1 - aa^-) = 1$, which on multiplying through by

$$
p = [1 + aa^- cu^- (1 - aa^-)][a^n]^{-1}
$$

yields:

$$
a + ct = p = \text{unit, where } t = u^- (1 - aa^-)(a^n)^{-1}.
$$

Now if in addition, $ac = ca$ and $a^a c = ca^a$ then we may take $u^- = c^a$. Hence

$$
a + (1 - aa^-)(a^n)^{-1}c^a c = p
$$

implying that $Ra + Rc = R$. 

4. It is now clear how to extend this to the following: If $a^k$ is unit regular for some $k \geq 1$, say $a^k(a^k)^{-1}a^k = a^k$, where $(a^k)^{-1}$ is a unit, and if $c^d$ exist, such that $cc^d$ commutes with $a^k(a^k)^{-1}$ and $(a^k)^{-1}$ then

\[ R = aR + cR \implies R = Ra + Rc. \]

The case where $a^d$ exists and $ac = ca$ easily follows from this example because then $(a^k)^d$ exist, for some $k \geq 1$ and one may then take $(a^k)^{-1} = (a^k)^d + (1 - aa^d)$.

References


Received June 2, 1977 and in revised form December 7, 1977.

North Carolina State University
Raleigh, NC 27607
Simeon M. Berman, *A class of isotropic distributions in $\mathbb{R}^n$ and their characteristic functions* .......................................................... 1

Ezra Brown and Charles John Parry, *The 2-class group of biquadratic fields. II* .......................................................... 11

Thomas E. Cecil and Patrick J. Ryan, *Focal sets of submanifolds* ............ 27

Joseph A. Cima and James Warren Roberts, *Denting points in $B^p$* ........... 41

Thomas W. Cusick, *Integer multiples of periodic continued fractions* .......... 47

Robert D. Davis, *The factors of the ramification sequence of a class of wildly ramified $v$-rings* .................................................. 61

Robert Martin Ephraim, *Multiplicative linear functionals of Stein algebras* .......................................................... 89

Philip Joel Feinsilver, *Operator calculus* .......................................................... 95

David Andrew Gay and William Yslas Vélez, *On the degree of the splitting field of an irreducible binomial* ............................................. 117

Robert William Gilmer, Jr. and William James Heinzer, *On the divisors of monic polynomials over a commutative ring* .......................... 121

Robert E. Hartwig, *Schur’s theorem and the Drazin inverse* ...................... 133

Hugh M. Hilden, *Embeddings and branched covering spaces for three and four dimensional manifolds* .................................................. 139

Carlos Moreno, *The Petersson inner product and the residue of an Euler product* ................................................................................. 149

Christopher Lloyd Morgan, *On relations for representations of finite groups* .................................................................................. 157

Ira J. Papick, *Finite type extensions and coherence* ...................................... 161

R. Michael Range, *The Carathéodory metric and holomorphic maps on a class of weakly pseudoconvex domains* ......................... 173

Donald Michael Redmond, *Mean value theorems for a class of Dirichlet series* ................................................................................. 191

Daniel Reich, *Partitioning integers using a finitely generated semigroup* ........ 233

Georg Johann Rieger, *Remark on a paper of Stux concerning squarefree numbers in non-linear sequences* .......................................... 241

Gerhard Rosenberger, *Alternierende Produkte in freien Gruppen* .............. 243

Ryōtarō Satō, *Contraction semigroups in Lebesgue space* .......................... 251

Tord Sjödin, *Capacities of compact sets in linear subspaces of $\mathbb{R}^n$* .... 261

Robert Jeffrey Zimmer, *Uniform subgroups and ergodic actions of exponential Lie groups* .................................................. 267