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**SCHUR'S THEOREM AND THE DRAZIN INVERSE**

ROBERT E. HARTWIG

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**It is shown that if  $M = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$  is a square  $2n \times 2n$  matrix over a ring  $R$ , such that  $AC = CA \in R_{n \times n}$ , and with the property that  $A$  and  $C$  possess Drazin inverses, then  $M$  is invertible in  $R_{2n \times 2n}$  if and only if  $DA - BC$  is invertible in  $R_{n \times n}$ .**

1. Introduction. In a recent paper [7], Herstein and Small extended the classic result of Schur [5, p. 46] to matrices over  $E$ -rings. These are rings for which every primitive image is artinian. This result states that for a square complex block matrix  $M = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$ , with  $A, B, C, D$  square of the same size such that  $AC = CA$ , then  $M$  is invertible exactly when  $\Delta = DA - BC$  is invertible. This is a different but equivalent formulation of the problem as stated in [7].

The purpose of this note is to show that this result by Schur is basically a consequence of the local existence of the Drazin inverse [2] of the matrices  $A$  and  $C$ ; that is, the strong- $\pi$ -regularity of  $A$  and  $C$  [1] [4]. The proof of [7] was based on the fact that Schur's result for matrices over  $E$ -rings is really equivalent to the corresponding result for matrices over simple artinian rings (which may be taken to be division rings). Since artinian rings with unity are noetherian [8], p. 69, it follows that artinian rings with unity are strongly- $\pi$ -regular, so that our local result extends the Schur theorem for artinian rings as proven in [7].

The Drazin inverse  $a^d$  of a ring element  $a$ , is the unique solution, if any, to the equations

$$(1) \quad a^k x a = a^k, x a x = x, a x = x a,$$

for some  $k \geq 0$ , while the group inverse  $a^\#$  of  $a$  is the unique solution, if any, of these equations with  $k = 0$ , or 1. For example, if  $a$  is algebraic over some field  $\mathcal{F}$  and  $a^{n+1}b = a^n$ , with  $ab = ba$ , then  $a^d = a^n b^{n+1}$ . The element  $a^d$  exists if and only if  $a$  is strongly- $\pi$ -regular, that is, when both chains  $\{a^i R\}$  and  $\{R a^i\}$  are ultimately stationary, [5, Theorem 4]. A ring element is called (von Neumann) *regular* if  $aa^-a = a$  for some ring element  $a^-$ . If there exists such  $a^-$  that is invertible,  $a$  is called *unit-regular*.

We shall assume familiarity with the properties of these inverses [4] [2] [6] and in particular with the fact that  $ac = ca \Rightarrow a^d c = c a^d$  [4, Theorem 1].

It is known that, unlike regularity and unit regularity,  $R_{2 \times 2}$  does

not inherit strong-regularity from  $R$  [9] [11]. It is not known however, whether the strong- $H$ -regularity of  $R$ , or the related concept of finite regularity ( $ab = 1 \Rightarrow ba = 1$ ) is inherited by  $R_{2 \times 2}$  [10].

We shall use the notation  ${}^{\circ}S$  and  $S^{\circ}$  to indicate the right and left annihilators of  $S$  respectively, e.g.,

$$S^{\circ} = \{x \in R; xs = 0, \forall s \in S\}.$$

For notational convenience we shall state our results in terms of rings  $R$  with unity, with the translation to matrices over  $R$ , being self evident. In particular  $aR + cR = R$  is equivalent to the  $1 \times 2$  matrix  $[a, c]$  having a right inverse.

2. Preliminaries. The key to our main result are the following two lemmas.

LEMMA 1. Let  $R$  be a ring with unity 1, and let  $e, f$  be commuting idempotents in  $R$ . If  $g = e + f(1 - e)$  then

- (i)  $g^2 = g$ , (ii)  $eR + fR = gR$ , (iii)  $Re + Rf = Rg$ , (iv)  $e^{\circ} \cap f^{\circ} = g^{\circ}$ , (v)  ${}^{\circ}e \cap {}^{\circ}f = {}^{\circ}g$ , (vi)  $eR + fR = R \Leftrightarrow g = 1 \Leftrightarrow Re + Rf = R \Leftrightarrow e^{\circ} \cap f^{\circ} = (0) \Leftrightarrow {}^{\circ}e \cap {}^{\circ}f = (0) \Leftrightarrow (1 - e)(1 - f) = 0$ .

LEMMA 2. Let  $R$  be a ring with unity 1, and let  $a, c$  be commuting elements of  $R$ . Then

- (i)  $aR + cR = R \Leftrightarrow a^mR + c^nR = R$  for some  $m, n \geq 1 \Leftrightarrow a^mR + c^nR = R$  for all  $m, n \geq 1$ .
- (ii)  ${}^{\circ}a \cap {}^{\circ}c = (0) \Leftrightarrow {}^{\circ}(a^m) \cap {}^{\circ}(c^n) = (0)$  for some  $m, n \geq 1 \Leftrightarrow {}^{\circ}(a^m) \cap {}^{\circ}(c^n) = (0)$  for all  $m, n \geq 1$ .
- (iii)  $Ra + Rc = R \Leftrightarrow Ra^m + Rc^n = R$  for some  $m, n \geq 1 \Leftrightarrow Ra^m + Rc^n = R$  for all  $m, n \geq 1$ .
- (iv)  $a^{\circ} \cap c^{\circ} = (0) \Leftrightarrow (a^m)^{\circ} \cap (c^n)^{\circ} = (0)$  for some  $m, n \geq 1 \Leftrightarrow (a^m)^{\circ} \cap (c^n)^{\circ} = (0)$  for all  $m, n \geq 1$ .

If in addition, the Drazin inverses  $a^d$  and  $c^d$  exists, these conditions are all equivalent to

$$(v) (1 - aa^d)(1 - cc^d) = 0.$$

*Proof.* The proof of (i)-(iv) follows by induction. Now suppose that  $a^d$  and  $c^d$  exist and that  $\text{index}(a) = k, \text{index}(c) = l$ . Then for all  $m \geq k, a^mR = a^kR = a^dR = a^d aR$ . And so, taking  $m \geq k, n \geq l$ , we see that (i) is equivalent to

$$R = a^mR + c^nR = a^kR + c^lR = a^dR + c^dR = a^d aR + c^d cR,$$

which by Lemma 1 is equivalent to

$$(3) (1 - aa^d)(1 - cc^d) = 0.$$

Left-right symmetry now shows that (iii) is also equivalent to (v). Lastly, since for  $m \geq k$ ,  $(a^m)^0 = (a^k)^0 = (a^d)^0 = (a^d a)^0$ , it follows that with  $m \geq k$ ,  $n \geq l$ , (iv) is equivalent to  $(a^d a)^0 \cap (c^d c)^0 = (0)$ , which again by Lemma 1 is equivalent to (v). Symmetry again yields the remaining equivalence.

Before proceeding with our theorem we remark that:

1. It is not necessary for  $a^d$  and  $c^d$  to exist in order for

$$Ra + Rc = R \iff a^0 \cap c^0 = (0)$$

to be valid. It would suffice if  $a, c$  and  $c(1 - a^{-1}a)$  were regular.

2. The equivalence of (iv) and (v) has uses in the theory of differential equations, [2] Lemma 1. The above furnishes a short and purely algebraic proof of this useful result.

### 3. Main results.

**THEOREM 1.** *Let  $R$  be a ring with unity 1 and let  $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R_{2 \times 2}$  with  $ac = ca$ . Suppose further that  $a^d$  and  $[(1 - aa^d)c]^d$  exist. If  $\Delta = da - bc$ , then:*

- (i)  $\Delta$  is left invertible  $\iff M$  is left invertible.
- (ii)  $M$  is right invertible  $\iff \Delta$  is right invertible.
- (iii)  $M$  is invertible  $\iff \Delta$  is invertible.

*Proof.* Consider the matrix

$$(4) \quad N = \begin{bmatrix} a & u \\ b & z \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 1 & -a^d c \\ 0 & 1 \end{bmatrix},$$

where  $u = (1 - aa^d)c$  and  $z = d - ba^d c$ . Since  $a, a^d$  and  $c$  commute it follows that

$$(5) \quad za - bu = (d - ba^d c)a - b(1 - aa^d)c = da - bc = \Delta.$$

Now because  $a^d u = 0 = ua^d = a^d u^d = u^d a^d$ , we may construct the matrices:

$$(6) \quad \begin{bmatrix} a & u \\ b & z \end{bmatrix} \begin{bmatrix} a^d & -u \\ u^d & a \end{bmatrix} = \begin{bmatrix} aa^d + uu^d & 0 \\ t & \Delta \end{bmatrix} = T$$

and

$$(7) \quad \begin{bmatrix} a & u \\ -u^d & a^d \end{bmatrix} \begin{bmatrix} a^d & -u \\ u^d & a \end{bmatrix} = \begin{bmatrix} aa^d + uu^d & 0 \\ 0 & aa^d + uu^d \end{bmatrix} \\ = \begin{bmatrix} a^d & -u \\ u^d & a \end{bmatrix} \begin{bmatrix} a & u \\ -u^d & a^d \end{bmatrix}.$$

In general however,  $au^d \neq 0$  unless  $\text{index}(a) \leq 1$ . Suppose now that  $\Delta$  has a left inverse  $\Delta^-$ , then by (5),

$$(8) \quad R = Ra + Rc = Ra + Ru .$$

By Lemma 2, applied to  $a$  and  $u$ , we see that

$$(1 - aa^d)(1 - uu^d) = 0$$

or equivalently

$$(9) \quad aa^d + uu^d = 1 .$$

Hence, by (7), it follows that the matrix  $P = \begin{bmatrix} a^d & -u \\ u^d & a \end{bmatrix}$  is invertible. Now since

$$\begin{bmatrix} 1 & 0 \\ -\Delta^-t & \Delta^- \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & \Delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ,$$

it follows that  $M$  has a left inverse  $M^-$  and that

$$R = Ra + Rb = Rc + Rd .$$

If in addition  $\Delta\Delta^- = 1$ , then

$$\begin{bmatrix} 1 & 0 \\ t & \Delta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\Delta^-t & \Delta^- \end{bmatrix} = I_2$$

and consequently  $M$  is also invertible.

Conversely, suppose that  $MM^- = I$ . Then because  $N$  also has a right inverse, it follows that

$$aR + uR = R .$$

Again by Lemma 2, applied to  $a$  and  $u$ , we may conclude that (9) holds so that  $P$  is invertible. Hence  $T = \begin{bmatrix} 1 & 0 \\ t & \Delta \end{bmatrix}$  has a right inverse  $T^- = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$ . Now  $TT^- = I \Rightarrow \gamma = 0 \Rightarrow \Delta\delta = 1$ , and so  $\Delta$  has a right inverse. If in addition,  $M^-M = I$ , then  $T^-T = I$  and hence again as  $\gamma = 0$ ,  $\delta\Delta = 1$ , completing the proof.

**COROLLARY 1.** *If  $R$  is a ring with unity and  $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R_{2 \times 2}$  with  $ac = ca$  such that  $a^d$  and  $c^d$  exist, then  $M$  is invertible if and only if  $\Delta = da - bc$  is invertible.*

*Proof.* Note that  $ac = ca$  implies that  $aa^dc = caa^d$ , so that  $u^d = (1 - aa^d)c^d$ . Again because square matrices over artinian ring with unity possess Drazin inverses, this result includes the second part of Theorem 2 of [7].

**COROLLARY 2.** *Let  $R$  be a ring with unity 1, and let  $a, c \in R$  such that  $ac = ca$  and  $a^d, [(1 - aa^d)c]^d$  exist. Then if  $R = Ra + Rc$  there exists  $d \in R$  so that  $\begin{bmatrix} a & c \\ c & d \end{bmatrix}$  is invertible.*

*Proof.* From Theorem 1, it suffices to select  $d \in R$  such that  $\Delta = da - c^2$  is invertible. One such choice is given by  $d = a^d + c^2a^d$ , because then  $\Delta = aa^d - u^2$  which has inverse  $aa^d - u^d u^d$ . Indeed, if  $R = Ra + Rc = Ra + Ru$ , then  $aa^d + uu^d = 1$  which coupled with the fact  $a^d u^d = 0$ , yields the desired result.

We conclude this note with several remarks.

1. If  $a^\#$  exists we could also select  $d = a + c^2a^\#$  in the last corollary, for then  $\Delta = a^2 - u^2$  has as inverse  $(a^\#)^2 - uu^d$  since now  $au = 0$ . Moreover, in this case

$$\begin{aligned} \begin{bmatrix} a & c \\ c & a + c^2a \end{bmatrix}^{-1} &= \begin{bmatrix} 1 & -a^\#c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & u^d \\ u^d & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -ca & 1 \end{bmatrix} \\ &= \begin{bmatrix} a - (a^\#)^3c^2 & u^d - c(a^\#)^2 \\ u^d - c(a^\#)^2 & a^\# \end{bmatrix}. \end{aligned}$$

2. The fact that: "If  $ac = ca$ , and  $a^d, u^d$  exist, then  $R = Ra + Rc$  ensures that  $a^d a + u^d u = 1$ ", should be compared with the corresponding results for Moore-Penrose inverses [6]. Namely, if  $a^\dagger$  and  $v^\dagger = [c(1 - a^\dagger a)]^\dagger$  exists, then

$$R = Ra + Rc \implies 1 = a^\dagger a + v^\dagger v.$$

3. If  $a$  is *unit-regular*, that is  $aa^-a = a$  for some unit  $a^-$ , then under suitable conditions  $aR + cR = R \implies Ra + Rc = R$ . Indeed if  $u = (1 - aa^-)c$  is regular and  $c^\#$  exists, then

$$aR + cR = R \implies aa^- + (1 - a^-)cu^-(1 - aa^-) = 1.$$

Thus  $aa^-[(1 - aa^-)cu^-(1 - aa^-) + cu^-(1 - aa^-)] = 1$ , which on multiplying through by

$$p = [1 + aa^-cu^-(1 - aa^-)](a^-)^{-1}$$

yields:

$$a + ct = p = \text{unit, where } t = u^-(1 - aa^-)(a^-)^{-1}.$$

Now if in addition,  $ac = ca$  and  $a^-c = ca^-$  then we may take  $u^- = c^\#$ . Hence

$$a + (1 - aa^-)(a^-)^{-1}c^\#c = p$$

implying that  $Ra + Rc = R$ .

4. It is now clear how to extend this to the following: If  $a^k$  is unit regular for some  $k \geq 1$ , say  $a^k(a^k)^{\#} = a^k = a^k(a^k)^{\#}$ , where  $(a^k)^{\#}$  is a unit, and if  $c^d$  exist, such that  $cc^d$  commutes with  $a^k(a^k)^{\#}$  and  $(a^k)^{\#}$  then

$$R = aR + cR \implies R = Ra + Rc .$$

The case where  $a^d$  exists and  $ac = ca$  easily follows from this example because then  $(a^k)^{\#}$  exist, for some  $k \geq 1$  and one may then take  $(a^k)^{\#} = (a^k)^{\#} + (1 - aa^d)$ .

#### REFERENCES

- [1] G. Azumaya, *Strongly-II-regular rings*, J. Fac. Science Hokkaido Univ. Series 1, Math., **XIII** (1954-57), 34-39.
- [2] A. Ben Israel and T. N. E. Greville, *Generalized Inverses, Theory and Applications*, Wiley, New York, 1974.
- [3] S. L. Campbell, C. D. Meyer and N. J. Rose, *Applications of the Drazin inverse to linear systems of differential equations*, SIAM J. Appl. Math., **31** (1976), 411-425.
- [4] M. P. Drazin, *Pseudo-inverses in associate rings and semigroups*, Amer. Math. Monthly, **65** (1958), 506-514.
- [5] F. R. Gantmacher, *The Theory of Matrices*, Vol. 1, Chelsea, New York, 1960.
- [6] R. E. Hartwig, *Block generalized inverses*, Arch. Rat. Mech., **61**, (1976), 197-251.
- [7] R. E. Harwig and J. Schoaf, *Group inverses and Drazin inverses of bidiagonal and triangular Toeplitz matrices*, J. Austral. Math. Soc. (A), **XXIV** (1977), 10-34.
- [8] R. E. Hartwig and J. Luh, *On finite regular rings*, Pacific J. Math., **69** (1977), 73-95.
- [9] M. Henriksen, *On a class of regular rings that are elementary divisor rings*, Arch. Math., **24** (1973), 133-141.
- [10] I. N. Herstein and L. W. Small, *An extension of a theorem of Schur*, Lin. Mult. Algebra, **3** (1975), 41-43.
- [11] J. Lambek, *Lectures on Rings and Modules*, Blaisdell, Waltham, 1966.

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