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ON RELATIONS FOR REPRESENTATIONS OF FINITE GROUPS

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Let G be a finite group, and suppose that

$$A: G \longrightarrow \text{GL}(n, \mathbf{C})$$

is a (complex) representation of G with character χ . A (complex) linear relation for A is a formal complex linear combination $\sum_{g \in G} a_g g$ such that $\sum_{g \in G} a_g A(g) = 0$.

We prove the following theorem, which determines the linear relations in terms of the character χ .

THEOREM. Let A be a representation for a finite group G , let χ be the character of A , and let $\{g_1, \dots, g_k\}$ be a subset of G . Then $\sum_{j=1}^k a_j g_j$ is a relation for A if and only if $\sum_{j=1}^k \chi(g_i g_j^{-1}) a_j = 0$, for all $i = 1, \dots, k$.

NOTE 1. If C is the $k \times k$ matrix whose ij -entry is $\chi(g_i g_j^{-1})$ and a is the column vector whose j th entry is a_j , then the above conclusion can be rephrased as follows:

$$\sum_{j=1}^k a_j g_j \text{ is a relation for } A \iff Ca = 0.$$

NOTE 2. The above theorem is a generalization of a result by Russell Merris [3]. His result may be stated in the following way. Let χ be an irreducible character of G , let M be the matrix obtained by applying χ to the entries of the multiplication table of G , let A be any representation of G affording χ , and let S be a subset of G . Then $\{A(g) \mid g \in S\}$ is linearly independent if and only if the rows of M corresponding to S are linearly independent. Our result strengthens Merris' result in three ways: (1) the condition about irreducibility is removed, (2) a way to determine the coefficients of any relation is given, and (3) smaller matrices are involved.

Proof of Theorem. Let χ_1, \dots, χ_r be the irreducible characters of G , and let CG denote the complex group algebra of G . For each $k = 1, \dots, r$, let

$$c_k = (\chi_k(e)/|G|) \sum_{g \in G} \chi_k(g) g.$$

Then c_k is a central idempotent of CG and corresponds to a representation of G with character χ_k in the following way:

Let R_k denote the principal ideal of CG generated by c_k , and let Z_k be any minimal left ideal of CG contained in R_k . Then

$R_k \approx \text{Hom}(Z_k, Z_k)$ and the irreducible group representation

$$A_k: G \longrightarrow \text{GL}(Z_k)$$

given by left multiplication has character χ_k . Furthermore, $\{c_1, \dots, c_k\}$ is a set of mutually annihilating central idempotents of G such that

$$c_1 + c_2 + \dots + c_r = e.$$

See [1, pp. 233-236] and [2, p. 257].

We can write A in terms of these representations as follows:

$$A \approx n_1 A_1 \oplus n_2 A_2 \oplus \dots \oplus n_r A_r$$

where n_k is a nonnegative number given by $n_k = (\chi, \chi_k)$.

Now let

$$L = \sum_{i=1}^n a_i g_i.$$

Then L is a relation for A if it is a relation for those irreducible representations A_k such that $(\chi, \chi_k) \neq 0$. It follows that L is a relation for A iff $c_k L = 0$, for those k such that $(\chi, \chi_k) \neq 0$.

Define

$$c = (\chi(e)/|G|) \sum_{g \in G} \chi(g)g.$$

A straightforward calculation shows that

$$c = \sum_{i=1}^r ((\chi, \chi_i) \chi(e) / \chi_i(e)) c_i.$$

Because of the mutual annihilation property,

$$cL = 0 \iff c_k L = 0 \quad \text{for all } k \text{ such that } (\chi, \chi_k) \neq 0.$$

Thus L is a relation for A if and only if $cL = 0$.

Left multiplication by c is a linear transformation on the complex vector space CG , and thus c has a matrix N with respect to the basis G for CG . The g, h -entry of N is $(\chi(e)/|G|) \chi(gh^{-1})$. Also with respect to this basis, L corresponds to the column vector with a_j as the g_j th entry, and zero otherwise. With a slight abuse of notation, this becomes $a_{g_i} = a_i$ for $i = 1, \dots, k$ and $a_g = 0$ if $g \neq g_i$ for all i .

Since G is finite, we have $\chi(g^{-1}) = \bar{\chi}(g)$, and thus N is hermitian. Since c is a positive linear combination of mutually annihilating idempotents, all eigenvalues of c and hence of N are nonnegative. Thus N is hermitian positive semidefinite, and so there exists a set of vectors $\{v_g \in C^{|\mathcal{G}|} | g \in G\}$ such that the g, h -entry of N is $\langle v_g, v_h \rangle$, where \langle, \rangle is the ordinary hermitian inner product on $C^{|\mathcal{G}|}$. Thus

we can write:

$$cL = \sum_{g, h \in G} \langle v_g, v_h \rangle a_h g = \sum_{g \in G} \langle v_g, v \rangle g$$

where $v = \sum_{h \in G} a_h v_h = \sum_{i=1}^k a_i v_{g_i}$.

If $cL = 0$, then $\langle v_g, v \rangle = 0$, for all $g \in G$. This implies that $\langle v_{g_j}, v \rangle = 0$, for $i = 1, \dots, k$.

Conversely, if $\langle v_{g_i}, v \rangle = 0$, for $i = 1, \dots, k$, then $\langle v, v \rangle = 0$. Since \langle, \rangle is the usual hermitian inner product on $C^{|G|}$, this implies that $v = 0$. But then $\langle v_g, v \rangle = 0$, for all $g \in G$. Thus $cL = 0$.

Thus $cL = 0 \Leftrightarrow \langle v_{g_i}, v \rangle = 0$, for all $i = 1, \dots, k$.

$$\begin{aligned} \text{But } \langle v_{g_i}, v \rangle = 0 &\iff \sum_{j=1}^m \langle v_{g_i}, v_{g_j} \rangle a_j = 0 \\ &\iff \langle \chi(e) / |G| \rangle \sum_{j=1}^k \chi(g_i g_j^{-1}) a_j = 0 \\ &\iff \sum_{j=1}^k \chi(g_i g_j^{-1}) a_j = 0. \end{aligned}$$

Thus L is a relation for A iff

$$\sum_{j=1}^k \chi(g_i g_j^{-1}) a_j = 0, \quad \text{for all } i = 1, \dots, k.$$

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