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Stux studied squarefree numbers of the form $[f(n)]$; his most interesting application is $f(n)=n^c$ for real c with $1<c<4/3$. We would like to point out that a stronger result follows immediately from estimates of Deshouillers.

Let $1 < c < 2$, $x \geqq 1$; denote by $N_c(x; k, l)$ the number of natural numbers $n \leqq x$ with $[n^c] \equiv 1 \bmod k$. According to [1], we have

$$(1) \qquad N_c(x; k, l) = \frac{x}{k} + O_c((x^{1+c}k^{-1})^{1/3}) \quad \text{for} \quad x^{c-5/4} \leqq k < x^{c-1/2} ,$$

$$(2) \qquad N_c(x; k, l) = \frac{x}{k} + O_c((x^{4+c}k^{-1})^{1/7}) \quad \text{for} \quad k < x^{c-5/4} .$$

Denote by $S_c(x)$ the number of squarefree numbers of the form $[n^c]$ with natural $n \leqq x$; the inclusion-exclusion principle in the form $|\mu(n)| = \sum_{d^2|n, d>0} \mu(d)$ gives

$$(3) \qquad S_c(x) = \sum_{d^2 \leqq x^c} \mu(d) N_c(x; d^2, 0) \qquad (x \geqq 1) .$$

For $d^2 \geqq x^{c-1/2}$ we use the trivial estimate $N_c(x; d^2, 0) = O(x^c d^{-2})$; using

$$(4) \qquad \sum_{d>t} d^{-2} = O(t^{-1}) \qquad (t \geqq 1) ,$$

we obtain

$$(5) \qquad S_c(x) = \sum_{d^2 < x^{c-1/2}} \mu(d) N_c(x; d^2, 0) + O(x^{(2c+1)/4}) .$$

In case $c \leqq 5/4$, we use (1) and

$$(6) \qquad \sum_{0<d\leqq t} d^{-2/3} = O(t^{1/3}) \qquad (t \geqq 1)$$

in (5); this gives

$$(7) \qquad S_c(x) = \sum_{d^2 < x^{c-1/2}} \mu(d) d^{-2} x + O_c(x^{(2x+1)/4}) .$$

In case $c > 5/4$, we split the sum in (5) according to $d^2 <$ or $\geqq x^{c-5/4}$ and apply (2) and (1); using $\sum_{0<d\leqq t} d^{-2/7} = O(t^{5/7})$ $(t \geqq 1)$ and (6), we obtain again (7). But (7), $\sum_{d>0} \mu(d) d^{-2} = 6\pi^{-2}$, and (4) give immediately

THEOREM 1. *For real c with $1 < c < 3/2$, we have*

$$S_c(x) = 6\pi^{-2}x + O_c(x^{(2c+1)/4}) \qquad (x \geqq 1) .$$

Looking at $m - [n^c]$ instead of $[n^c]$ we obtain similarly

THEOREM 2. *For real c with $1 < c < 3/2$, the number of representations of the natural number m as $m = q + [n^c]$ with squarefree q and natural n equals*

$$6\pi^{-2}m^{1/c} + O_c(m^{(2c+1)/4c}) .$$

This can easily be generalized to r-free instead of squarefree. It should not be difficult to extend the method of [1] to cover the function class studied in [2].

REFERENCES

1. Jean-Marc Deshouillers, *Sur la répartition des nombres $[n^c]$ dans les progressions arithmétiques*, C.R. Acad. Sci. Paris, **277** (1973), Ser. A., 647–650.
2. Ivan F. Stux, *Distribution of squarefree integers in non-linear sequences*, Pacific J. Math., **59** (1975), 577–584.

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