REMARK ON A PAPER OF STUX CONCERNING SQUAREFREE NUMBERS IN NON-LINEAR SEQUENCES

Georg Johann Rieger
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Stux studied squarefree numbers of the form \([f(n)]\); his most interesting application is \(f(n) = n^c\) for real \(c\) with \(1 < c < 4/3\). We would like to point out that a stronger result follows immediately from estimates of Deshouillers.

Let \(1 < c < 2\), \(x \geq 1\); denote by \(N_c(x; k, l)\) the number of natural numbers \(n \leq x\) with \([n^c] \equiv 1 \mod k\). According to [1], we have

\[
N_c(x; k, l) = \frac{x}{k} + O_c((x^{c+1}k^{-1})^{1/3}) \quad \text{for} \quad x^{c-5/4} \leq k < x^{c-1/2},
\]

\[
N_c(x; k, l) = \frac{x}{k} + O_c((x^{4+c}k^{-1})^{1/7}) \quad \text{for} \quad k < x^{c-5/4}.
\]

Denote by \(S_c(x)\) the number of squarefree numbers of the form \([n^c]\) with natural \(n \leq x\); the inclusion-exclusion principle in the form \(|\mu(n)| = \sum_{d \mid n, d > 0} \mu(d)\) gives

\[
S_c(x) = \sum_{d^2 \leq x^c} \mu(d)N_c(x; d^2, 0) \quad (x \geq 1).
\]

For \(d^2 \geq x^{c-1/2}\) we use the trivial estimate \(N_c(x; d^2, 0) = O(x^c d^{-2})\); using

\[
\sum_{d \geq t} d^{-2} = O(t^{-1}) \quad (t \geq 1),
\]

we obtain

\[
S_c(x) = \sum_{d^2 < x^{c-1/2}} \mu(d)N_c(x; d^2, 0) + O(x^{(2c+1)/4}).
\]

In case \(c \leq 5/4\), we use (1) and

\[
\sum_{0 < d \leq t} d^{-2/3} = O(t^{1/3}) \quad (t \geq 1)
\]

in (5); this gives

\[
S_c(x) = \sum_{d^2 < x^{c-1/2}} \mu(d)d^{-2}x + O_c(x^{(2c+1)/4}).
\]

In case \(c > 5/4\), we split the sum in (5) according to \(d^2 < \text{or} \geq x^{c-5/4}\) and apply (2) and (1); using \(\sum_{0 < d \leq t} d^{-2/7} = O(t^{1/7}) \quad (t \geq 1)\) and (6), we obtain again (7). But (7), \(\sum_{d > 0} \mu(d)d^{-2} = 6\pi^{-2}\), and (4) give immediately
Theorem 1. For real $c$ with $1 < c < 3/2$, we have
\[ S_\beta(x) = 6\pi^{-2}x + O_c(x^{(2c+1)/4}) \quad (x \geq 1). \]

Looking at $m - [n^c]$ instead of $[n^c]$ we obtain similarly

Theorem 2. For real $c$ with $1 < c < 3/2$, the number of representations of the natural number $m$ as $m = q + [n^c]$ with squarefree $q$ and natural $n$ equals
\[ 6\pi^{-3}m^{1/c} + O_c(m^{(2c+1)/4c}). \]

This can easily be generalized to $r$-free instead of squarefree. It should not be difficult to extend the method of [1] to cover the function class studied in [2].

References


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