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**CAPACITIES OF COMPACT SETS IN LINEAR SUBSPACES OF  
 $R^n$**

TORD SJÖDIN

## CAPACITIES OF COMPACT SETS IN LINEAR SUBSPACES OF $R^n$

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**We consider two types of spaces, the Bessel potential spaces  $L_\alpha^p(R^n)$  and the Besov spaces  $A_\alpha^p(R^n)$ ,  $\alpha > 0$ ,  $1 < p < \infty$ . Associated in a natural way with these spaces are classes of exceptional sets. We characterize the exceptional sets for  $A_\alpha^p(R^n)$  by an extension property for continuous functions and prove an inequality between Bessel and Besov capacities.**

The classes of exceptional sets for the spaces  $L_\alpha^p(R^n)$  have been studied by the concept of capacity [5]. Capacity definitions of these classes are given in § 2.

Bessel potential spaces and Besov spaces in  $R^n$  and  $R^{n+1}$  are connected by restriction theorems. A short statement of these results is the following:

$$(1.1) \quad L_\beta^p(R^{n+1})|_{R^n} = A_\alpha^p(R^n)$$

$$(1.2) \quad A_\beta^p(R^{n+1})|_{R^n} = A_\alpha^p(R^n),$$

where  $\alpha > 0$ ,  $1 < p < \infty$ , and  $\beta = \alpha + 1/p$ . (O. V. Besov [4] and E.M. Stein [7].)

The restriction theorem above gives relations between exceptional classes of different spaces  $L_\alpha^p$  and  $A_\alpha^p$  in  $R^n$  and  $R^{n+1}$ .

This enables us to prove an extension theorem for continuous functions on a compact set  $K \subset R^n$  into  $A_\alpha^p(R^n)$  (Theorem 1) analogous to the  $L_\alpha^p(R^n)$  – case contained in [6, Theorem 1]. Finally we prove an inequality between the capacities defining the classes of exceptional sets for  $A_\alpha^p(R^n)$  and  $L_\alpha^p(R^n)$  (Theorem 2).

**2. Preliminaries and statements of the theorems.** We consider the  $n$ -dimensional space  $R^n$  of  $n$ -tuples  $x = (x_1, x_2, \dots, x_n)$ . Points in  $R^{n+1}$  are written  $(x, x_{n+1})$ , where  $x \in R^n$  and  $x_{n+1} \in R^1$ . Then  $R^n$  is identified as the subspace  $\{(x, 0); x \in R^n\}$  of  $R^{n+1}$ . Compact sets are denoted by  $K$ . If  $K \subset R^n$  then  $K$  is a compact subset of  $R^{n+1}$  as well. As usual, the space of  $p$ -summable functions is denoted by  $L^p(R^n)$  with norm  $\|\cdot\|_p$ . The Bessel kernel  $G_\alpha^n$  in  $R^n$  is the  $L^1(R^n)$ -function whose Fourier transform equals  $(1 + |x|^2)^{-\alpha/2}$ ,  $\alpha > 0$ .

The space of convolutions  $U = G_\alpha^n * f$ , where  $f \in L^p(R^n)$ , with the norm  $\|U\|_{\alpha,p} = \|f\|_p$ , is denoted by  $L_\alpha^p(R^n)$ ,  $\alpha > 0$ ,  $1 \leq p < \infty$ . A function  $U \in L_\alpha^p(R^n)$ ,  $1 \leq p \leq \infty$ ,  $0 < \alpha < 1$  if

$$\|U\|_{\alpha,p} = \|U\|_p + \left( \iint \frac{|U(x) - U(y)|^p}{|x - y|^{p(\alpha+n)}} dx dy \right)^{1/p}$$

is finite. (When no limits of integration are indicated it is understood that the integration is over the whole space.)

When  $1 \leq \alpha < 2$  we replace the first difference by the second difference. Finally, for  $\alpha \geq 2$ ,  $U \in A_\alpha^p(R^n)$  if and only if  $U \in L^p$  and  $\partial U / \partial x_i \in A_{\alpha-1}^p(R^n)$ ,  $1 \leq i \leq n$ , with the norm

$$\|U\|_{\alpha,p} = \|U\|_p + \sum_{i=1}^n \left\| \frac{\partial U}{\partial x_i} \right\|_{\alpha-1,p}.$$

We consider the following two capacities for compact sets  $K \subset R^n$ ,  $\alpha > 0$ ,  $1 < p < \infty$ .

$$A_{\alpha,p}^n(K) = \inf |\varphi|_{\alpha,p}^p,$$

$$B_{\alpha,p}^n(K) = \inf \|\varphi\|_{\alpha,p}^p,$$

where, in both cases, the infimum is taken over all  $\varphi \in C_0^\infty(R^n)$  such that  $\varphi(x) \geq 1$  for every  $x \in K$ .  $C_0^\infty(R^n)$  is the infinitely differentiable functions on  $R^n$  with compact support.

The  $B_{\alpha,p}^n$ -capacity has several equivalent definitions [2, 5]. We mention that

$$B_{\alpha,p}^n(K) = \inf \|f\|_q^p$$

where infimum is over  $f \in L_+^p$  such that  $G_\alpha^+ * f(x) \geq 1$  on  $K$ . (A lower superscript + indicates positive elements.) The sign  $\sim$  means that the ratio is bounded from below and above by positive real numbers. Further,  $B_{\alpha,p}^n(K) = (\sup \|\mu\|_1)^p$  where supremum is over positive Borel measures  $\mu$  concentrated on  $K$  with total variation  $\|\mu\|_1 < \infty$  and  $\|G_\alpha^+ * \mu\|_q \leq 1$ .

Here  $q = p/p - 1$ . See [5] where this capacity is denoted by  $b_{\alpha,p}$ . Let  $K$  be a compact subset of  $R^n$ . We have proved that  $B_{\alpha,p}^n(K) = 0$  if and only if every continuous function on  $K$  is the restriction to  $K$  of a continuous function in  $L_\alpha^p(R^n)$  [6, Theorem 1]. We prove here the analogue for  $A_\alpha^p(R^n)$ . Let  $C(E)$  denote the space of continuous functions on a set  $E$  in  $R^n$ .

**THEOREM 1.** *Let  $1 < p < \infty$ ,  $0 < \alpha \cdot p \leq n$  and let  $K$  be a compact subset of  $R^n$ . Then  $A_{\alpha,p}^n(K) = 0$  if and only if every function  $f_0 \in C(K)$  has an extension  $f \in A_\alpha^p(R^n) \cap C(R^n)$ .*

When  $\alpha p > n$ , the capacities  $A_{\alpha,p}^n$  and  $B_{\alpha,p}^n$  are positive unless  $K$  is empty [3].

We denote the exceptional classes for  $L_\alpha^p(R^n)$  and  $A_\alpha^p(R^n)$ ,  $1 < p < \infty$ ,  $\alpha \cdot p \leq n$ , by  $\mathfrak{B}_{\alpha,p}^n$  and  $\mathfrak{U}_{\alpha,p}^n$  respectively [3]. It is well known that for  $K \subset R^n$ :

$$K \in \mathfrak{U}_{\alpha,p}^n \quad \text{if and only if} \quad A_{\alpha,p}^n(K) = 0$$

$$K \in \mathfrak{B}_{\alpha,p}^n \quad \text{if and only if} \quad B_{\alpha,p}^n(K) = 0 .$$

See [3].

It is interesting to note that  $\mathfrak{U}_{\alpha,p}^n$  and  $\mathfrak{B}_{\alpha,p}^n$  can be proved to be identical for  $2 - \alpha/n < p < \infty$  [1, Theorem 1] inspite of the fact that  $L_\alpha^n(R^n) \neq A_\alpha^n(R^n)$  when  $\alpha > 0$  and  $p \neq 2$  [3].

**THEOREM 2.** *Let  $\alpha > 0$ ,  $1 < p < \infty$ , and let  $K$  be a compact subset of  $R^n$ . Then*

$$B_{\alpha,p}^n(K) \leq c \cdot A_{\alpha,p}^n(K) .$$

Constants depending on  $n, p$ , and  $\alpha$  only, not necessarily the same at each occurrence, are denoted by  $c$ .

**REMARK.** David R. Adams [1, p. 3] has proved that  $A_{\alpha,p}^n(K) = 0$  implies  $B_{\alpha,p}^n(K) = 0$  for  $\alpha > 0$ ,  $1 < p < \infty$ . Theorem 2 makes it possible to compare the capacities  $B_{\alpha,p}^n$  and  $A_{\alpha,p}^n$  for all sets.

It will become clear from the proofs of Theorem 1 and Theorem 2 that the restriction theorem described in (1, 1) and (1, 2) is an essential tool. (An exact formulation is given in Theorem A in §3.)

At this point we just note that Theorem 2 has an alternative formulation. Under the assumptions of Theorem 2,

$$B_{\alpha,p}^n(K) \leq c \cdot B_{\beta,p}^{n+1}(K) , \quad K \subset R^n , \quad \beta = \alpha + \frac{1}{p} .$$

The inclusions  $L_\alpha^p(R^n) \subset A_\alpha^n(R^n)$  for  $2 \leq p < \infty$  and  $A_\alpha^n(R^n) \subset L_\alpha^p(R^n)$ , for  $1 < p \leq 2$ , are well known [3]. They give immediately the inequalities

$$B_{\alpha,p}^n(K) \leq c \cdot A_{\alpha,p}^n(K) , \quad 1 < p \leq 2 ,$$

and

$$(2.1) \quad A_{\alpha,p}^n(K) \leq c \cdot B_{\alpha,p}^n(K) , \quad 2 \leq p < \infty .$$

Combining Theorem 2 with (2.1) gives,

$$A_{\alpha,p}^n(K) \sim B_{\alpha,p}^n(K) , \quad 2 \leq p < \infty .$$

**3. Proof of Theorem 1.** We first define two operators  $E$  and  $R$  in the following way. Let  $\varphi \in C_0^\infty(R^{n+1})$ , then

$$R\varphi(x) = \varphi(x, 0) , \quad x \in R^n .$$

Let  $f \in C_0^\infty(R^1)$  and  $g \in C_0^\infty(R^n)$  be such that  $f(0) = 1$  and  $\int g(x) dx = 1$ .

When  $\psi \in C_0^\infty(R^n)$  we put

$$E\psi(x, x_{n+1}) = f(x_{n+1}) \cdot \int \psi(x - x_{n+1} \cdot y) \cdot g(y) dy,$$

$x \in R^n, x_{n+1} \in R^1$ . See for example, E. M. Stein [7].

**THEOREM A.** *Let  $\alpha > 0, 1 < p < \infty$ , and  $\beta = \alpha + 1/p$ . Then*

(a) *the map  $R$  is a continuous map from  $L_\beta^p(R^{n+1})(A_\beta^p(R^{n+1}))$  to  $A_\alpha^p(R^n)$ ;*

(b) *the map  $E$  is a continuous map from  $A_\alpha^p(R^n)$  to  $L_\beta^p(R^{n+1})(A_\beta^p(R^{n+1}))$ .*

This theorem is due to E.M. Stein [7] and O.V. Besov [4]. Let  $K \subset R^n, \alpha > 0, 1 < p < \infty$ , then

$$(3.1) \quad B_{\beta,p}^{n+1}(K) \sim A_{\alpha,p}^n \sim A_{\beta,p}^{n+1}(K)$$

where  $\beta = \alpha + 1/p$ .

This is an immediate consequence of Theorem A and the definitions of the capacities.

*Proof of Theorem 1.* Let  $K$  be a compact subset of  $R^n$  such that  $A_{\alpha,p}^n(K) = 0$ . Let  $f_0 \in C(K)$ . Since  $B_{\beta,p}^{n+1}(K) = 0, \beta = \alpha + 1/p$ , by (3.1), there is a function  $f \in L_\beta^p(R^{n+1}) \cap C(R^{n+1})$  such that  $f(x) = f_0(x)$  when  $x \in K$  [6, Theorem 1]. Taking the restriction  $Rf$  we have  $Rf \in A_\alpha^p(R^n) \cap C(R^n)$  by Theorem A.

Conversely suppose that every  $f_0 \in C(K)$  has an extension  $f \in A_\alpha^p(R^n) \cap C(R^n)$ . Let  $f_0 \in C(K)$  then  $Ef \in L_\beta^p(R^{n+1}) \cap C(R^{n+1}), \beta = \alpha + 1/p$ . By [6, Theorem 1] we must have  $B_{\beta,p}^{n+1}(K) = 0$ , which implies  $A_{\alpha,p}^n(K) = 0$ . The proof is complete.

**4. Proof of Theorem 2.** We begin with a lemma. Let  $f \in L_+^p(R^{n+1})$  then we define  $g(y) = \left( \int f(y, t)^p dt \right)^{1/p}, y \in R^n$ .

The function  $g$  belongs to  $L_+^p(R^n)$  and

$$\|g\|_p = \|f\|_p.$$

(The notation  $\|\cdot\|_p$  means that the integral defining the norm is taken over all the variables and over the whole space.)

**LEMMA 1.** *Let  $\alpha > 0, 1 < p < \infty, \beta = \alpha + 1/p$ . Then for  $f \in L_+^p(R^{n+1})$ ,*

$$G_\beta^{n+1} * f(x, 0) \leq c \cdot G_\alpha^n * g(x), \quad x \in R^n.$$

In the proof of Lemma 1 we use some well known properties of the Bessel kernel  $G_\alpha^n(r)$  (see for example [3]):

$$\begin{aligned} G_\alpha^n(r) &\sim r^{\alpha-n}, \quad r \longrightarrow 0, \quad \text{for } 0 < \alpha < n \\ G_\alpha^n(r) &\sim r^{(\alpha-n-1)/2} \cdot e^{-r}, \quad r \longrightarrow \infty, \quad \text{for } \alpha > 0. \end{aligned}$$

*Proof of Lemma 1.* Suppose  $\alpha \cdot p \leq n$  and let  $f \in L_+^p(R^{n+1})$  and  $g(y) = \left( \int f(y, t)^p dt \right)^{1/p}$ . We have

$$G_\beta^{n+1} * f(x, 0) = \iint G_\beta^{n+1}(\sqrt{|x-y|^2 + t^2}) \cdot f(y, t) dy dt.$$

For  $|y-x| \leq 1$  we get the estimate:

$$\begin{aligned} \int G_\beta^{n+1}(\sqrt{|x-y|^2 + t^2}) \cdot f(y, t) dt &\leq c \cdot \int (\sqrt{|x-y|^2 + t^2})^{\beta-n-1} \\ &\cdot f(y, t) dt \\ &= c \cdot |x-y|^{\beta-n} \cdot \int (\sqrt{1+t^2})^{\beta-n-1} \cdot f(y, |x-y| \cdot t) dt \\ &\leq c \cdot |x-y|^{\beta-n} \cdot \left( \int f(y, |x-y| \cdot t)^p dt \right)^{1/p} \\ &= c \cdot |x-y|^{\beta-n-1/p} \cdot \left( \int f(y, t)^p dt \right)^{1/p} \\ &\leq c \cdot G_\alpha^n(x-y) \cdot g(y). \end{aligned}$$

For  $|y-x| \geq 1$  we get

$$\begin{aligned} \int G_\beta^{n+1}(\sqrt{|x-y|^2 + t^2}) f(y, t) dt &\leq c \cdot \int (\sqrt{|x-y|^2 + t^2})^{(\beta-n-2)/2} \\ &\cdot e^{-\sqrt{|x-y|^2 + t^2}} \cdot f(y, t) dt \\ &= c \cdot |x-y|^{(\beta-n)/2} \cdot \int (\sqrt{1+t^2})^{(\beta-n-2)/2} \cdot e^{-|x-y| \cdot \sqrt{1+t^2}} \\ &\cdot f(y, |x-y| \cdot t) dt. \end{aligned}$$

We divide the last integral in two parts

$$I = \int_{-1}^1 \quad \text{and} \quad II = \int_{|t| \geq 1}.$$

Then using the inequality  $\sqrt{1+x} \geq 1 + x/3$ ,  $0 \leq x \leq 1$  we get

$$\begin{aligned} I &\leq e^{-|x-y|} \cdot \int_{-1}^1 e^{-|x-y| \cdot t^2/3} \cdot f(y, |x-y| \cdot t) dt \\ &\leq |x-y|^{-1/2} \cdot e^{-|x-y|} \cdot \int e^{-t^2/3} \cdot f(y, \sqrt{|x-y|} |t|) dt \\ &\leq c \cdot |x-y|^{(-1-1/p) \cdot /2} \cdot e^{-|x-y|} \cdot g(y). \end{aligned}$$

Further we have

$$\begin{aligned} II &\leq e^{-\sqrt{2} \cdot |x-y|} \cdot \int (\sqrt{1+t^2})^{(\beta-n-2)/2} \cdot f(y, |x-y| \cdot t) dt \\ &\leq c \cdot e^{-\sqrt{2} \cdot |x-y|} \cdot |x-y|^{-1/p} \cdot g(y). \end{aligned}$$

Collecting our results we have

$$\int G_\beta^{\alpha+1}(\sqrt{|x-y|^2+t^2}) \cdot f(y, t) dt \leq c \cdot G_\alpha^n(x-y) \cdot g(y)$$

which gives

$$G_\beta^{\alpha+1} * f(x, 0) \leq c \cdot G_\alpha^n * g(x),$$

where  $\beta = \alpha + 1/p$ .

The case  $\alpha \cdot p > n$  is much simpler and the proof is omitted.

*Proof of Theorem 2.* According to the relation (3.1) it suffices to prove that for every  $f \in L_+^p(\mathbb{R}^{n+1})$  such that  $G_\beta^{\alpha+1} * f(x, 0) \geq 1$  for  $x \in K$ , there exists  $g \in L_+^p(\mathbb{R}^n)$  such that  $G_\alpha^n * g(x) \geq 1$  for  $x \in K$  and

$$\|g\|_p \leq c \cdot \|f\|_p.$$

But this follows immediately from Lemma 1. This proves the theorem.

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