

Pacific Journal of Mathematics

**ON REPRESENTING ANALYTIC GROUPS WITH THEIR
AUTOMORPHISMS**

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A real or complex Lie group is said to be *faithfully representable* if it has a faithful finite-dimensional analytic representation. Let G be a real or complex analytic group, and let A denote the group of all analytic automorphisms of G , endowed with its natural structure of a real or complex Lie group. The natural semidirect product $G \rtimes A$ is a real or complex Lie group, sometimes called the *holomorph* of G . We show that if G is faithfully representable and if the maximum nilpotent normal analytic subgroup of G is simply connected then $G \rtimes A$ is faithfully representable.

This result follows quite easily from well-known representation-theoretical results and techniques. What we use is contained in [1, Ch. XVIII], and all the references given below are to this. Nominally, these references cover only the real case. However, as explained *loc. cit.*, both the results and their proofs are almost identical in the complex case.

Thanks are due to Martin Moskowitz who drew my attention to this question and who obtained a number of special results that are consequences of the theorem below and contain suggestions for its proof.

PROPOSITION. *Let G be a faithfully representable real or complex analytic group, and let N be the maximum nilpotent normal analytic subgroup of G . If N is simply connected there is a faithful finite-dimensional analytic representation of G whose restriction to N is unipotent.*

Proof. By Theorem 4.3 (or 4.7), G is a semidirect product $B \rtimes H$, where B is solvable and simply connected, and H is reductive. The construction of B is carried out in the proof of Theorem 4.2, and this shows that, if N is simply connected, one can arrange to have $N \subset B$ (one begins with a semidirect product decomposition of the radical of G having the form $M \rtimes Q$, where Q is reductive, and M is simply connected and contains N).

By Theorem 3.1, there exists a faithful finite-dimensional analytic representation ρ of B whose restriction to N is unipotent. Now ρ satisfies the conditions of Theorem 2.2, so that (enlarging the representation space of ρ) one can extend ρ to a finite-dimensional

analytic representation of G whose restriction to B is faithful and whose restriction to N is still unipotent. Any given faithful finite-dimensional analytic representation of G , via restriction to H and the canonical homomorphism $G \rightarrow H$, yields a finite-dimensional analytic representation of G whose kernel is precisely B . The direct sum of this and the representation obtained above satisfies the requirements of the proposition.

THEOREM. *Let G be a faithfully representable real or complex analytic group. Let N be the maximum nilpotent normal analytic subgroup of G , and let A be the group of all analytic automorphisms of G . If N is simply connected then $G \rtimes A$ is faithfully representable.*

Proof. Let B denote the group of all Lie algebra automorphisms of the Lie algebra $\mathcal{L}(G)$ of G . This is an algebraic linear group, and we denote its algebraic identity component by B_1 . The Lie algebra of B_1 may be identified with the Lie algebra of all derivations of $\mathcal{L}(G)$. If R is the radical of G then $\mathcal{L}(R)$ is the radical of $\mathcal{L}(G)$, while $\mathcal{L}(N)$ is the maximum nilpotent ideal of $\mathcal{L}(G)$. Therefore, by a well-known result from Lie algebra theory, every derivation of $\mathcal{L}(G)$ sends $\mathcal{L}(R)$ into $\mathcal{L}(N)$. This implies that B_1 acts trivially on $\mathcal{L}(R)/\mathcal{L}(N)$.

Let A° denote the group of all elements of A whose canonical images in B belong to B_1 . Then A° acts trivially on R/N . Since B_1 is normal and of finite index in B , the group A° is normal and of finite index in A .

Let I denote the group of inner automorphisms of G . Clearly, $I \subset A^\circ$. Now let S be a maximal semisimple analytic subgroup of G , and let T be the subgroup of A° consisting of the elements of A° that keep the elements of S fixed. Since every maximal semisimple analytic subgroup of G is a G -conjugate of S , every coset of I in A contains an element that stabilizes S . It follows from this that the group TI is normal in A° . Since the group of inner automorphisms of S is of finite index in the group of all analytic automorphisms of S , it follows also that TI is of finite index in A° .

Clearly, A° contains the identity component A_1 of A . If n is the index of TI in A° then a^n belongs to TI for every a in A_1 . Since A_1 is an analytic group, these elements a^n generate A_1 . Hence $A_1 \subset TI$. We conclude that TI is open and of finite index in A .

Now let ρ be a representation of G with the properties stated in the proposition. Let V denote the space of representative functions on G that are associated with ρ . Then V is finite-dimensional and stable under the right and left translation actions of G . If ρ'

is the semisimple representation of G that is associated with ρ then N is contained in the kernel of ρ' . Let x be an element of the radical of G , and let τ be an element of T . Then $\tau(x)x^{-1} \in N$, so that $\rho'(\tau(x)x^{-1})$ is the identity map on the representation space. Since the elements of T keep the elements of S fixed, it is now clear that Lemma 2.1 applies to our present situation and gives the result that the space $V \circ T$ spanned by the composites of the elements of V with those of T is still finite-dimensional. From the 2-sided G -stability of V , it is clear that $V \circ (TI) = V \circ T$.

Let us write J for TI , and W for $V \circ T$. We define an action of $G \rtimes J$ on W by setting $(x, \alpha) \cdot w = x \cdot (w \circ \alpha^{-1})$, where, for any function f on G , the *translate* $x \cdot f$ is defined by $(x \cdot f)(y) = f(yx)$. Thus, our definition means that the value of $(x, \alpha) \cdot w$ at y is $w(\alpha^{-1}(yx))$. In order to verify that this is indeed a representation of $G \rtimes J$ on W , it suffices to check that, compatibly with $\alpha x = \alpha(x)\alpha$ (in $G \rtimes J$), one has $\alpha \cdot (x \cdot w) = \alpha(x) \cdot (\alpha \cdot w)$. Evidently, this representation is analytic, and its restriction to G is faithful.

In order to obtain a representation of $G \rtimes A$ whose restriction to G is faithful, we use the ordinary group algebra $F[G \rtimes A]$, where F is the field of real numbers or the field of complex numbers. We form the tensor product $F[G \rtimes A] \otimes_{F[G \rtimes J]} W$ and let $G \rtimes A$ act via multiplication on the left factor. As a vector space, this module is the direct sum of $[A:J]$ copies of W , and thus is finite-dimensional. Since $A_1 \subset J$, it is clear that this representation is analytic.

Finally, we make $\mathcal{L}(G)$ into a $G \rtimes A$ -module via the canonical homomorphisms $G \rtimes A \rightarrow A \rightarrow B$, so that the kernel of this representation of $G \rtimes A$ on $\mathcal{L}(G)$ is precisely G . The direct sum of the two representations we have constructed satisfies the requirements of our theorem.

The simplest example of a faithfully representable analytic group G such that $G \rtimes A$ is not faithfully representable is the direct product $\mathbf{R} \times T$ of the additive group \mathbf{R} of real numbers and the multiplicative group T of complex numbers of absolute value 1. For every real number a , this group has an analytic automorphism a^* , where $a^*(r, u) = (r, \exp(iar)u)$. If S is the space of representative functions associated with a representation of G that is not trivial on T , it is easy to see that the space spanned by the functions $f \circ a^*$ with f in S and a in \mathbf{R} is not finite-dimensional.

We may summarize our results as follows.

Summary. Let K be a real or complex Lie group having a semidirect product decomposition $G \rtimes H$, where G is connected and faithfully representable, and the maximum nilpotent normal analy-

tic subgroup N of G is simply connected. There is a finite-dimensional analytic representation of K whose restriction to G is faithful and whose restriction to N is unipotent. If H is faithfully representable there is a faithful such representation.

Proof. Let A denote the group of all analytic automorphisms of G . The semidirect product decomposition of K defines an analytic homomorphism $\eta: H \rightarrow A$ and hence an analytic homomorphism $\eta^*: K \rightarrow G \rtimes A$. If ρ is a representation of $G \times A$ as obtained in the theorem, then $\rho \circ \eta^*$ clearly satisfies the requirements of the first statement of the summary. The second statement then follows immediately.

REFERENCE

1. G. Hochschild, *The structure of Lie groups*, Holden-Day, San Francisco, 1965.

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