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RADII OF CONVEXITY FOR CERTAIN CLASSES OF UNIVALENT ANALYTIC FUNCTIONS

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Let $P(\alpha, \beta)$ denote the class of functions $p(z)=1+b_1z+\cdots$ which are analytic and satisfy the inequality |(p(z)-1)/ $\{2\beta(p(z)-\alpha)-(p(z)-1)\}|<1$ for some α, β $(0\leq \alpha<1, 0<\beta\leq 1)$ and all $z\in E=\{z:|z|<1\}$. Also, let $P_b(\alpha, \beta)=\{p\in P(\alpha, \beta): p'(0)=2b\beta(1-\alpha), 0\leq b\leq 1\}$. In the present paper, we determine sharp estimates for the radii of convexity for functions in the classes $R_a(\alpha, \beta)$ and $S_a^*(\alpha, \beta)$ where $R_a(\alpha, \beta)=\{f(z)=z+a\beta(1-\alpha)z^2+\cdots: f'\in P_a(\alpha, \beta), 0\leq a\leq 1\}$. The results thus obtained not only sharpen and generalize the various known results but also give rise to several new results.

1. Introduction. Let P denote the class of functions

$$(1.1) p(z) = 1 + b_1 z + b_2 z^2 + \cdots$$

which are analytic and satisfy $\operatorname{Re}(p(z)) > 0$ for $z \in E \equiv \{z: |z| < 1\}$. Considerable work has been done to study the various aspects of the above mentioned class (see e.g., [11], [12] and others). Some of these results have also been extended to the class $P(\alpha)$ of functions p(z) which are analytic and satisfy $\operatorname{Re}(p(z)) > \alpha, 0 \leq \alpha < 1$ for $z \in E$. If $p \in P(\alpha)$, it is easily seen that $|b_1| \leq 2(1 - \alpha)$. Further, we note that if $\tau = \exp\{-i \arg b_i\}$ then $p(\tau z) = 1 + |b_1|z + \cdots$ and so while studying $P(\alpha)$, there is no loss of generality if one takes the first coefficient b_1 in (1.1) to be nonnegative.

McCarty in [8] determined a lower bound on $\operatorname{Re} zp'(z)/p(z)$ for functions p(z) in the class $P_b(\alpha) \equiv \{p \in P(\alpha): p'(0) = 2b(1-\alpha), 0 \leq b \leq 1\}$. He also applied the results obtained to determine the sharp estimates for the radii of convexity of the two classes $R_a(\alpha)$ and $S_a^*(\alpha)$ for each $a \in [0, 1]$ and $\alpha \in [0, 1)$ where

$$R_a(lpha) = \{f(z) = z + a(1-lpha)z^2 + \cdots : f' \in P_a(lpha)\}$$

and

$$S^*_{a}(lpha)=\{g(z)=z+2a(1-lpha)z^2+\cdots : zg'/g\in P_{a}(lpha)\}$$
 .

For still another class $R'_a(\alpha)$ defined by $R'_a(\alpha) = \{f(z) = z + a(1 - \alpha)z^2 + \cdots : |f'(z) - 1| < \alpha, 1/2 < \alpha \leq 1, z \in E\}$ Goel [4] determined the radius of convexity.

In the present paper, we propose an approach by which it is not only possible to have a unified study of the above mentioned classes but of various other classes as well. For this purpose we introduce the following classes:

$$egin{aligned} P(lpha,\,eta) &= \{p(z) = 1 + b_1 z + \cdots : |(p(z) - 1)/\{2eta(p(z) - lpha) \ &- (p(z) - 1)\}| < 1, & ext{for } lpha \in [0,\,1), \ eta \in (0,\,1] ext{ and } z \in E\} \ P_b(lpha,\,eta) &= \{p \in P(lpha,\,eta) : p'(0) = 2beta(1 - lpha), \ 0 &\leq b \leq 1\} \ R_a(lpha,\,eta) &= \{f(z) = z + aeta(1 - lpha)z^2 + \cdots : f' \in P_a(lpha,\,eta), \ 0 \leq a \leq 1\} \ S_a^*(lpha,\,eta) &= \{g(z) = z + 2aeta(1 - lpha)z^2 + \cdots : zg'/g \in P_a(lpha,\,eta), \ 0 \leq a \leq 1\} \end{aligned}$$

and determine sharp estimates for the radii of convexity for functions in $R_a(\alpha, \beta)$ and $S_a^*(\alpha, \beta)$.

2. Preliminary lemmas. Let B denote the class of analytic functions w(z) in E which satisfy the conditions w(0)=0 and |w(z)|<1 for $z \in E$. We require the following lemmas:

LEMMA 1 [15]. If $w \in B$, then for $z \in E$

(2.1)
$$|zw'(z) - w(z)| \leq \frac{|z|^2 - |w(z)|^2}{1 - |z|^2}.$$

LEMMA 2. Let $w \in B$. Then we have

$$\begin{array}{l} \text{(2.2)} \ \operatorname{Re}\left\{\frac{zw'(z)}{(1+sw(z))(1+tw(z))}\right\} &\leq -\frac{1}{(s-t)^2}\operatorname{Re}\left\{sp(z) \ + \ \frac{t}{p(z)} - s - t\right\} \\ &+ \frac{1}{(s-t)^2}\frac{r^2|sp(z) - t|^2 - |1 - p(z)|^2}{(1-r^2)|p(z)|} \end{array}$$

where p(z) = (1 + tw(z))/(1 + sw(z)), |z| = r and $-1 \leq t < s \leq 1$.

Using the estimate (2.1), the lemma follows easily. Hence we omit the proof.

LEMMA 3. If p(z) = (1 + tw(z))/(1 + sw(z)), $w \in B$, then for each $b \in [0, 1]$ and s, t satisfying $-1 \leq t < s \leq 1$, p(z) lies in the disc

$$arDelta(z)\equiv \{\zeta\colon |\,\zeta-A_b\,| \leq D_b\}$$
 ,

where

$$egin{aligned} A_b &= rac{(1+br)^2 - str^2(b+r)^2}{(1+br)^2 - s^2r^2(b+r)^2}; D_b &= rac{(s-t)r(b+r)(1+br)}{(1+br)^2 - s^2r^2(b+r)^2} \ and \ r &= |m{z}| < 1. \end{aligned}$$

Proof. Since p(z) = (1 + tw(z))/(1 + sw(z)), we have

(2.3)
$$w(z) = \frac{1 - p(z)}{sp(z) - t} = -[bz + \cdots] = -z\phi(z)$$

where ϕ is analytic and $|\phi(z)| \leq 1$ for $z \in E$ with $\phi'(0) = b$. Now, since $(\phi(z) - b)/(1 - b\phi(z))$ is subordinate to z, it follows that $\phi(z)$ is subordinate to (z + b)/(1 + bz) and so

(2.4)
$$\left|\frac{1-p(z)}{sp(z)-t}\right| \leq |z|\frac{(|z|+b)}{(1+b|z|)}$$

Putting $p(z) = \xi + i\eta$, (2.4) gives

$$\left|\hat{arsigma}+i\eta\,-rac{(1+br)^2-str^2(b+r)^2}{(1+br)^2-s^2r^2(b+r)^2}
ight| \leq rac{(s-t)r(b+r)(1+br)}{(1+br)^2-s^2r^2(b+r)^2}$$

Hence the lemma.

LEMMA 4. If p(z) = (1 + tw(z))/(1 + sw(z)), $w \in B$, then for |z| = r, $0 \leq r < 1$, we have

$$\operatorname{Re}\left\{kp(z) + \frac{t}{p(z)}\right\} - \frac{r^{2}|sp(z) - t|^{2} - |1 - p(z)|^{2}}{(1 - r^{2})|p(z)|}$$

$$(2.5) \geq \begin{bmatrix} \frac{2}{1 - r^{2}} [\sqrt{(1 + t)(1 - tr^{2})(k(1 - r^{2}) + 1 - s^{2}r^{2})} - (1 - str^{2})] \\ if \quad R_{b} \leq R^{*} \\ W/W^{*} \qquad if \quad R_{b} \geq R^{*} \end{bmatrix}$$

where

$$W = t(kt + s^2)r^4 + 2bt\{(k + s) + (kt + s^2)\}r^3 + [b^2(1 + t)\{(k + t) + (kt + s^2)\} + 2t(k + s) - (s - t)^2]r + 2b\{(k + t) + t(k + s)\}r + (k + t),$$

(2.5; b)
$$W^* = \{1 + rb(1 + t) + tr^2\}\{1 + rb(1 + s) + sr^2\}$$

and $R^{*2} = (1 + t)(1 - tr^2)/(k(1 - r^2) + 1 - s^2r^2)$, $R_b = A_b - D_b$ where A_b , D_b are defined as in Lemma 3 and $k \ge s$, $-1 \le t < s \le 1$.

Proof. Let |z| = r, and $p(z) = A_b + \xi + i\eta \equiv \operatorname{Re}^{i\psi}$, then $-\pi/2 < \psi < \pi/2$. Denoting the left hand side of (2.5) by $U_b(\xi, \eta)$, we get

$$(2.6) U_b(\xi,\eta) = k(A_b + \xi) + t(A_b + \xi)R^{-2} + \frac{1 - s^2r^2}{1 - r^2}[((A_b + \xi) - A_1)^2 + \eta^2 - D_1^2]R^{-1}$$

and

(2.7)
$$\frac{\partial U_b}{\partial \eta} = \eta R^{-4} V_b(\xi, \eta)$$

where

$$egin{aligned} V_b(\hat{\xi},\eta) &= -2t(A_b+\hat{\xi}) + (D_1^2+2A_1(A_b+\hat{\xi})-A_1^2) \Big(rac{1-s^2r^2}{1-r^2}\Big) R \ &+ \Big(rac{1-s^2r^2}{1-r^2}\Big) R^3 \end{aligned}$$

(2.8)
$$= -2tR\cos\psi + (D_1^2 - A_1^2 + 2A_1R\cos\psi)\Big(\frac{1-s^2r^2}{1-r^2}\Big)R \\ + \Big(\frac{1-s^2r^2}{1-r^2}\Big)R^3 \equiv M_b(R,\psi)(\text{say}) .$$

Since for fixed r, $0 \leq r < 1$, $A_b - D_b$ decreases as b increases over the interval [0, 1], it follows that $R \geq R \cos \psi \geq A_b - D_b \geq A_1 - D_1$. Thus, for all b, $0 \leq b \leq 1$,

$$egin{aligned} M_b(R, \ \psi) &\geq R \cos \psi \Big[-2t + (D_1^2 - A_1^2 + 2A_1R \cos \psi + R^2) \left(rac{1-s^2r^2}{1-r^2}
ight) \Big] \ &\geq 2R \cos \psi \Big[\Big(rac{1-s^2r^2}{1-r^2} \Big) (A_1 - D_1)^2 - t \Big] &> 0, \end{aligned}$$

for all s, t satisfying $-1 \leq t < s \leq 1$. Thus $V_b(\xi, \eta) \equiv M_b(R, \psi)$ is positive for all points in the disc $\Delta(z)$. Now, (2.7) gives that, for every fixed ξ , $U_b(\xi, \eta)$ is increasing function of η for positive η and is a decreasing function of η for negative η . Thus, the minimum of $U_b(\xi, \eta)$ inside the disc Δ is attained on the diameter forming part of the real axis. Setting $\eta = 0$ in (2.6), we obtain

(2.9)
$$\min_{-1 \le \eta \le 1} U_b(\xi, \eta) \equiv N_b(R) = \left(k + \frac{1 - s^2 r^2}{1 - r^2}\right)R + \frac{(1 + t)(1 - tr^2)}{(1 - r^2)}R^{-1} - 2A_1\left(\frac{1 - r^2 s^2}{1 - r^2}\right)$$

where $R = A_b + \xi \in [A_b - D_b, A_b + D_b]$. Thus the absolute minimum of $N_b(R)$ in $(0, \infty)$ is attained at

(2.10)
$$R^* = \left(\frac{(1+t)(1-tr^2)}{k(1-r^2)+1-s^2r^2}\right)^{1/2}$$

and the value of this minimum is equal to

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(2.11)
$$N_{b}(R^{*}) = \frac{1}{1-r^{2}} \left[\sqrt{(k(1-r^{2})+1-s^{2}r^{2})(1+t)(1-tr^{2})} - (1-str^{2}) \right].$$

Since it is easily seen that $R^* < A_1 + D_1$ and that $A_b + D_b$ is a decreasing function of b for $0 \leq b \leq 1$, it follows that $R^* < A_b + D_b$ for $b \in [0, 1]$; but R^* is not always greater than $A_b - D_b$. In case $R^* \notin [A_b - D_b, A_b + D_b]$, it can be easily verified that $N_b(R)$ increases with R in $[A_b - D_b, A_b + D_b]$. Thus the minimum of $N_b(R)$ on the segment $[A_b - D_b, A_b + D_b]$ is attained at $R_b = A_b - D_b$. The value of this minimum equals

$$N_b(R_b)\equiv N_b(A_b-D_b)=W/W^*$$
 ,

where W and W^{*} are given by (2.5; a) and (2.5; b). Moreover $N_b(R^*) = N_b(R_b)$ for those values of k, s, and t for which $R_b = R^*$. Hence the lemma.

3. The class $R_a(\alpha, \beta)$. Let $R(\alpha, \beta)$ be the class of functions $f(z) = z + a_2 z^2 + \cdots$ which are analytic and satisfy the inequality $|(f'(z) - 1)/\{2\beta(f'(z) - \alpha) - (f'(z) - 1)\}| < 1$ for some $\alpha, \beta(0 \le \alpha < 1, 0 < \beta \le 1)$ and $z \in E$. One of the authors [9] has shown that for $f \in R(\alpha, \beta), |a_2| \le \beta(1 - \alpha)$. Define

$$R_a(\alpha, \beta) = \{f(z) = z + a\beta(1-\alpha)z^2 + \cdots : f' \in P_a(\alpha, \beta), \ 0 \leq a \leq 1\}.$$

Now, we determine a sharp estimate for the radii of convexity for functions in $R_{\alpha}(\alpha, \beta)$.

THEOREM 1. Let $f \in R_a(\alpha, \beta)$, then f is convex in $|z| < r_0$ where r_0 is the smallest positive root of the equation

if $R_a \geq R^*$ and

$$r_{\scriptscriptstyle 0} = [\{-lphaeta + \sqrt{lpha(1-2lphaeta + lphaeta^2)}\}/(1-2lphaeta)]^{_{1/2}}$$

if $R_a \leq R^*$ where

$$R_{a}=rac{1+2lphaeta ar+(2lphaeta-1)r^{2}}{1+2eta ar+(2eta-1)r^{2}}$$
 , $R^{*}=\Big(rac{lpha(1-(2lphaeta-1)r^{2})}{1-(2eta-1)r^{2}}\Big)^{^{1/2}}$

and r = |z| < 1. The result is sharp for each α , $\beta(0 \leq \alpha < 1, 0 < \beta \leq 1)$ and $0 \leq a \leq 1$.

Proof. Since $f \in R_a(\alpha, \beta)$, an application of Schwarz's lemma gives

(3.1)
$$f'(z) = \frac{1 + (2\alpha\beta - 1)w(z)}{1 + (2\beta - 1)w(z)}$$

where $w \in B$. Logarithmic differentiation of (3.1) gives

$$(3.2) \quad 1 + z \frac{f''(z)}{f'(z)} = 1 - 2\beta(1-\alpha) \left\{ \frac{zw'(z)}{(1 + (2\beta - 1)w(z))(1 + (2\alpha\beta - 1)w(z))} \right\} .$$

Applying (2.2) with $s = 2\beta - 1$, $t = 2\alpha\beta - 1$ to (3.2), we get

(3.3) Re
$$\left\{1 + z \frac{f''(z)}{f'(z)}\right\} \ge \frac{1}{2\beta(1-\alpha)} \left[\operatorname{Re} \left\{ (2\beta - 1)p(z) + \frac{2\alpha\beta - 1}{p(z)} \right\} - \frac{r^2 |(2\beta - 1)p(z) + 1 - 2\alpha\beta|^2 - |1 - p(z)|^2}{(1 - r^2)|p(z)|} \right] + \frac{1 - 2\alpha\beta}{\beta(1 - \alpha)}$$

where $p(z) = (1 + (2\alpha\beta - 1)w(z))/(1 + (2\beta - 1)w(z))$. An application of Lemma 4 with $k = s = 2\beta - 1$, $t = 2\alpha\beta - 1$ to (3.3) gives

$$(3.4) \operatorname{Re} \left\{ 1 + z \frac{f''(z)}{f'(z)} \right\}$$

$$\geq \begin{bmatrix} \frac{1}{\beta(1-\alpha)(1-r^2)} [\sqrt{4\alpha\beta^2(1-(2\beta-1)r^2)(1+(1-2\alpha\beta)r^2)} \\ -(1+(1-2\alpha\beta)(2\beta-1)r^2) + (1-2\alpha\beta)(1-r^2)] \\ \text{if } R_a \leq R^* , \\ 1+4\alpha\beta ar + (4\alpha\beta^2a^2 - 2(1+\beta-3\alpha\beta))r^2 \\ +4\beta \times (2\alpha\beta-1)ar^3 + (2\beta-1)(2\alpha\beta-1)r^4 \\ \hline (1+2\beta ar + (2\beta-1)r^2)(1+2\alpha\beta ar + (2\alpha\beta-1)r^2) \\ \text{if } R_a \geq R^* \end{bmatrix}$$

where

Now the theorem follows easily from (3.4).

The function given by

$$f'(z)=rac{1-2lphaeta az+(2lphaeta-1)z^2}{1-2eta az+(2eta-1)z^2} ext{ if } R_a \geqq R^*$$
 ,

and

$$f'(z)=rac{1-2lphaeta cz+(2lphaeta-1)z^2}{1-2eta cz+(2eta-1)z^2}\, ext{ if }R_a \leqq R^*$$

where c is determined by the relation

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$$rac{1-2lphaeta cr+(2lphaeta-1)r^2}{1-2eta cr+(2eta-1)r^2}=R^*=~\sqrt{rac{lpha(1+(1-2lphaeta)r^2)}{(1-(2eta-1)r^2)}}$$

show that the results obtained in the theorem are sharp.

Putting $\beta = 1$, in Theorem 1, we get the following result due to McCarty [8].

COROLLARY 1(a). Each $f \in R_a(\alpha)$ maps $|z| < r_0$ onto a convex region where r_0 is the smallest positive root of the equation

 $1+4lpha ar+(6lpha-4+4lpha a^2)r^2+4(2lpha-1)ar^3+(2lpha-1)r^4=0$ if $R_a\geqq R^*$ and

$$r_{\scriptscriptstyle 0} = [\{-lpha + \sqrt{lpha(1-lpha)}\}/(1-2lpha)]^{\scriptscriptstyle 1/2}$$

if $R_a \leq R^*$, where

$$R_{a}=rac{1+2lpha ar+(2lpha-1)r^{2}}{1+2ar+r^{2}}$$
 , $R^{*}=\Bigl(rac{lpha(1-(2lpha-1)r^{2})}{1-r^{2}}\Bigr)^{1/2}$

and r = |z| < 1. The result is sharp for each α $(0 \leq \alpha < 1)$ and $0 \leq a \leq 1$.

COROLLARY 1(b). Let $f \in R'_a(\alpha)$, then f is convex in $|z| < r_o$ where r_o is the smallest positive root of the equation

$$1 + 2(1 - \alpha)ar + ((1 - \alpha)a^2 - 3\alpha)r^2 - 2\alpha ar^3 = 0$$

if $R_a \geq R^*$ and

$$r_{\scriptscriptstyle 0} = [\{-(1-lpha) + \sqrt{(1-lpha)(1+3lpha)}\}/2lpha]^{1/2}$$

if $R_{a} \leq R^{*}$, where

$$R_a = rac{1+(1-lpha)ar-lpha r^2}{1+ar} \,, \quad R^* = [(1-lpha)(1+lpha r^2)]^{1/2}$$

and r = |z| < 1. The result is sharp for each α $(0 \leq \alpha < 1)$ and $0 \leq \alpha \leq 1$.

The result is obtained by replacing α by $1 - \alpha$ and β by 1/2 in Theorem 1. It may be noted that this result was obtained by Goel [4] under the additional restriction $1/2 \leq \alpha \leq 1$.

REMARK. Replacing (α, β) by (0, 1), or by $(0, 1 - \delta)$, $0 \leq \delta < 1$ or by $(0, (2\delta - 1)/2\delta)$, $1/2 < \delta \leq 1$, or by $((1 - \gamma)/1 + \gamma, (1 + \gamma(/2), 0 < \gamma \leq 1)$, or by $((1 - \delta + 2\gamma\delta)/(1 + \delta), (1 + \delta)/2)$, $0 \leq \gamma < 1$, $0 < \delta \leq 1$, we get the estimates for the radii of convexity for functions with fixed second coefficient of the classes introduced and studied by MacGregor [7], Shaffer [13], Goel [3], Caplinger and Causey [1] and the authors [6] respectively.

4. The class $S_a^*(\alpha, \beta)$. Let $S^*(\alpha, \beta)$ be the class of functions $g(z) = z + a_2 z^2 + \cdots$ which are analytic and satisfy the inequality $|(zg'(z)/g(z)-1)/\{2\beta(zg'(z)/g(z)-\alpha)-(zg'(z)/g(z)-1)\}| < 1$, for some $\alpha, \beta(0 \leq \alpha < 1, 0 < \beta \leq 1)$ and $z \in E$. The authors [5] have shown that for $g \in S^*(\alpha, \beta)$, $|a_2| \leq 2\beta(1-\alpha)$. Define

$$S_a^*(lpha,\,eta) \!=\! \{g(z) \!=\! z \!+\! 2aeta(1\!-\!lpha) z^2 \!+\! \cdots \!: zg' / g \in P_a(lpha,\,eta), \ 0 \!\leq\! a \!\leq\! 1\}$$
 .

Now, we determine a sharp estimate for the radii of convexity for functions in $S^*_{\alpha}(\alpha, \beta)$.

THEOREM 2. Let $g \in S_a^*(\alpha, \beta)$, then g is convex in $|z| < r_0$ where r_0 is the smallest positive root of the equation

if $R_a \geq R^*$ and

$$r_{_0} = [(5lpha - 1)/\{(1 - lpha + 4eta lpha^2) + 4lpha \sqrt{(1 + eta - 3lpha eta + lpha^2 eta^2)}]^{_{1/2}}$$

if $R_a \leq R^*$, where

$$R_{a}=rac{1+2lphaeta ar+(2lphaeta-1)r^2}{1+2eta ar+(2eta-1)r^2}$$
 , $R^{*}=\Big(rac{lpha(1+(1-2lphaeta)r^2)}{(2-lpha)-(2eta-lpha)r^2}\Big)^{1/2}$

and r = |z| < 1. The result is sharp for each α, β ($0 \leq \alpha < 1, 0 < \beta \leq 1$) and $0 \leq a \leq 1$.

Proof. Since $g \in S^*_a(\alpha, \beta)$, an application of Schwarz's lemma gives

(4.1)
$$z \frac{g'(z)}{g(z)} = \frac{1 + (2\alpha\beta - 1)w(z)}{1 + (2\beta - 1)w(z)}$$

where $w \in B$. Logarithmic differentiation of (4.1) gives

(4.2)
$$1 + z \frac{g''(z)}{g'(z)} = \frac{1 + (2\alpha\beta - 1)w(z)}{1 + (2\beta - 1)w(z)} \\ - 2\beta(1 - \alpha) \left\{ \frac{zw'(z)}{(1 + (2\beta - 1)w(z))(1 + (2\alpha\beta - 1)w(z))} \right\}.$$

Applying (2.2) with $s = 2\beta - 1$, $t = 2\alpha\beta - 1$ to (4.2), we get

$$(4.3) \qquad \operatorname{Re}\left\{1 + z \frac{g''(z)}{g'(z)}\right\} \geq \frac{1}{2\beta(1-\alpha)} \left[\operatorname{Re}\left\{(4\beta - 1 - 2\alpha\beta)p(z) + \frac{2\alpha\beta - 1}{p(z)}\right\} - \frac{r^2|(2\beta - 1)p(z) + 1 - 2\alpha\beta|^2 - |1 - p(z)|^2}{(1 - r^2)|p(z)|}\right] \\ + \frac{\alpha + \alpha\beta - 1}{\beta(1-\alpha)}$$

where $p(z) = (1 + (2\alpha\beta - 1)w(z))/(1 + (2\beta - 1)w(z))$. Now, an application of Lemma 4 with $k = 4\beta - 1 - 2\alpha\beta$, $s = 2\beta - 1$ and $t = 2\alpha\beta - 1$ to (4.3) gives the required results easily.

The functions given by

$$zrac{g'(z)}{g(z)}=rac{1-2lphaeta az+(2lphaeta-1)z^2}{1-2eta az+(2eta-1)z^2} ext{ if } R_a \geqq R^*$$

and

$$zrac{g'(z)}{g(z)}=rac{1-2lphaeta cz+(2lphaeta-1)z^2}{1-2eta cz+(2eta-1)z^2} ext{ if } R_a \leq R^*$$

where c is determined by the relation

$$rac{1-2lphaeta cr+(2lphaeta-1)r^2}{1-2eta cr+(2eta-1)r^2}=R^*=\Bigl(rac{lpha(1-(2lphaeta-1))r^2}{(2-lpha)-(2eta-lpha)r^2}\Bigr)^{1/2}$$

show that the results obtained in the theorem are sharp.

Taking $\beta = 1$, in Theorem 2, we get the following result due to McCarty [8] which also includes the result obtained by Tepper [16].

COROLLARY 2(a). Each $g \in S_a^*(\alpha)$ maps $|z| < r_0$ onto a convex region where r_0 is the smallest positive root of the equation

if $R_a \ge R^*$ and

$$r_{\scriptscriptstyle 0} = [(5lpha - 1)/\{(4lpha^2 - lpha + 1) + 4lpha
u/(\overline{lpha^2 - 3lpha + 2})\}]^{_{1/2}}$$

if $R_a \leq R^*$ where

$$R_{a}=rac{1+2lpha ar+(2lpha-1)r^{2}}{1+2ar+r^{2}}$$
 , $R^{*}=\Bigl(rac{lpha(1-(2lpha-1))r^{2}}{(2-lpha)(1-r^{2})}\Bigr)^{1/2}$

and r = |z| < 1. The result is sharp for each α $(0 \le \alpha < 1)$ and $0 \le \alpha \le 1$.

REMARKS. (i) Replacing (α, β) by (0, 1/2), or by $(0, (2\delta - 1)/2\delta)$, $1/2 < \delta \leq 1$, or by $((1 - \gamma)/1 + \gamma, (1 + \gamma)/2)$, $0 < \gamma \leq 1$, we may obtain the estimates for the radii of convexity for functions with fixed second coefficient of the classes introduced and studied by Eenigenburg [2], Ram Singh [14] and Padmanabhan [10] respectively.

(ii) Setting a = 1 in Theorem 1 and Theorem 2 we get the sharp estimates for the radii of convexity for functions in $R(\alpha, \beta)$ and $S^*(\alpha, \beta)$. These were obtained by the authors in [9] and [5] and thus also include the results obtained in [1], [2], [13] etc.

(iii) By setting a = 0 in Theorem 1 and Theorem 2, we may get the results for functions in $R(\alpha, \beta)$ and $S^*(\alpha, \beta)$ with missing second coefficient and in particular for odd functions in these classes.

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