RADII OF CONVEXITY FOR CERTAIN CLASSES OF UNIVALENT ANALYTIC FUNCTIONS

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Let \( P(\alpha, \beta) \) denote the class of functions \( p(z) = 1 + b_1 z + \cdots \) which are analytic and satisfy the inequality \( \frac{2\beta (p(z) - \alpha) - (p(z) - 1)}{2\beta (p(z) - \alpha)} \leq 1 \) for some \( \alpha, \beta \) \((0 \leq \alpha < 1, 0 < \beta \leq 1)\) and all \( z \in E = \{z : |z| < 1\} \). Also, let \( P_b(\alpha, \beta) = \{p \in P(\alpha, \beta) : p'(0) = 2b\beta(1 - \alpha), 0 \leq b \leq 1\} \). In the present paper, we determine sharp estimates for the radii of convexity for functions in the classes \( R_\alpha(\alpha, \beta) \) and \( S^*_\alpha(\alpha, \beta) \) where \( R_\alpha(\alpha, \beta) = \{f(z) = z + a\beta(1 - \alpha)z^2 + \cdots : f' \in P_\alpha(\alpha, \beta), 0 \leq a \leq 1\} \), \( S^*_\alpha(\alpha, \beta) = \{g(z) = z + 2a\beta(1 - \alpha)z^2 + \cdots : zg'/g \in P_\alpha(\alpha, \beta), 0 \leq a \leq 1\} \). The results thus obtained not only sharpen and generalize the various known results but also give rise to several new results.

1. Introduction. Let \( P \) denote the class of functions

\[
p(z) = 1 + b_1 z + b_2 z^2 + \cdots
\]

which are analytic and satisfy \( \text{Re}(p(z)) > 0 \) for \( z \in E = \{z : |z| < 1\} \). Considerable work has been done to study the various aspects of the above mentioned class (see e.g., [11], [12] and others). Some of these results have also been extended to the class \( P(\alpha) \) of functions \( p(z) \) which are analytic and satisfy \( \text{Re}(p(z)) > \alpha \), \( 0 \leq \alpha < 1 \) for \( z \in E \). If \( p \in P(\alpha) \), it is easily seen that \( |b_1| \leq 2(1 - \alpha) \). Further, we note that if \( \tau = \exp\{-i \text{arg } b_1\} \) then \( p(\tau z) = 1 + |b_1| z + \cdots \) and so while studying \( P(\alpha) \), there is no loss of generality if one takes the first coefficient \( b_1 \) in (1.1) to be nonnegative.

McCarty in [8] determined a lower bound on \( \text{Re}(zp'(z)/p(z)) \) for functions \( p(z) \) in the class \( P_b(\alpha) = \{p \in P(\alpha) : p'(0) = 2b(1 - \alpha), 0 \leq b \leq 1\} \). He also applied the results obtained to determine the sharp estimates for the radii of convexity of the two classes \( R_\alpha(\alpha) \) and \( S^*_\alpha(\alpha) \) for each \( \alpha \in [0, 1] \) and \( \alpha \in [0, 1] \) where

\[
R_\alpha(\alpha) = \{f(z) = z + a(1 - \alpha)z^2 + \cdots : f' \in P_\alpha(\alpha)\}
\]

and

\[
S^*_\alpha(\alpha) = \{g(z) = z + 2a(1 - \alpha)z^2 + \cdots : zg'/g \in P_\alpha(\alpha)\}.
\]

For still another class \( R'_\alpha(\alpha) \) defined by \( R'_\alpha(\alpha) = \{f(z) = z + a(1 - \alpha)z^2 + \cdots : |f'(z) - 1| < \alpha, 1/2 < \alpha \leq 1, z \in E\} \) Goel [4] determined the radius of convexity.

In the present paper, we propose an approach by which it is not only possible to have a unified study of the above mentioned
classes but of various other classes as well. For this purpose we introduce the following classes:

\[ P(\alpha, \beta) = \{ p(z) = 1 + b(z) + \cdots : |(p(z) - 1)/(2\beta(p(z) - \alpha) - (p(z) - 1))| < 1, \text{ for } \alpha \in [0, 1), \beta \in (0, 1] \text{ and } z \in E \} \]

\[ P_b(\alpha, \beta) = \{ p \in P(\alpha, \beta): p'(0) = 2b\beta(1 - \alpha), \ 0 \leq b \leq 1 \} \]

\[ R_a(\alpha, \beta) = \{ f(z) = z + a\beta(1 - \alpha)z^2 + \cdots : f' \in P_a(\alpha, \beta), \ 0 \leq a \leq 1 \} \]

\[ S^*_a(\alpha, \beta) = \{ g(z) = z + 2a\beta(1 - \alpha)z^2 + \cdots : zg' \in P_a(\alpha, \beta), \ 0 \leq a \leq 1 \} \]

and determine sharp estimates for the radii of convexity for functions in \( R_a(\alpha, \beta) \) and \( S^*_a(\alpha, \beta) \).

2. Preliminary lemmas. Let \( B \) denote the class of analytic functions \( w(z) \) in \( E \) which satisfy the conditions \( w(0) = 0 \) and \( |w(z)| < 1 \) for \( z \in E \). We require the following lemmas:

**Lemma 1** [15]. If \( w \in B \), then for \( z \in E \)

\[(2.1) \quad |zw'(z) - w(z)| \leq \frac{|z|^2 - |w(z)|^2}{1 - |z|^2} . \]

**Lemma 2.** Let \( w \in B \). Then we have

\[(2.2) \quad \text{Re} \left\{ \frac{zw'(z)}{(1 + sw(z))(1 + tw(z))} \right\} \leq -\frac{1}{(s-t)^2} \text{Re} \left\{ \frac{sp(z) + t}{p(z)} - s-t \right\} \]

\[ + \frac{1}{(s-t)^2} \frac{r^2|sp(z) - t|^2 - |1 - p(z)|^2}{(1 - r^2)|p(z)|} \]

where \( p(z) = (1 + tw(z))/(1 + sw(z)), \ |z| = r \text{ and } -1 \leq t < s \leq 1. \)

Using the estimate (2.1), the lemma follows easily. Hence we omit the proof.

**Lemma 3.** If \( p(z) = (1 + tw(z))/(1 + sw(z)), \ w \in B \), then for each \( b \in [0, 1] \) and \( s, t \) satisfying \( -1 \leq t < s \leq 1 \), \( p(z) \) lies in the disc

\[ A(z) \equiv \{ \zeta: |\zeta - A_b| \leq D_b \} , \]

where

\[ A_b = \frac{(1 + br)^2 - str^2(b + r)^2}{(1 + br)^2 - s^2r^2(b + r)^2} , \quad D_b = \frac{(s - t)r(b + r)(1 + br)}{(1 + br)^2 - s^2r^2(b + r)^2} \]

and \( r = |z| < 1. \)
Proof. Since \( p(z) = (1 + tw(z))/(1 + sw(z)) \), we have

\[
(2.3) \quad w(z) = \frac{1 - p(z)}{sp(z) - t} = -[bz + \cdots] = -z\phi(z)
\]

where \( \phi \) is analytic and \( |\phi(z)| \leq 1 \) for \( z \in \mathbb{E} \) with \( \phi'(0) = b \). Now, since \( (\phi(z) - b)/(1 - b\phi(z)) \) is subordinate to \( z \), it follows that \( \phi(z) \) is subordinate to \( (z + b)/(1 + bz) \) and so

\[
(2.4) \quad \left| \frac{1 - p(z)}{sp(z) - t} \right| \leq |z| \frac{(|z| + b)}{(1 + b|z|)}.
\]

Putting \( p(z) = \xi + i\eta \), (2.4) gives

\[
\left| \xi + i\eta - \frac{(1 + br)^2 - str^2(b + r)^2}{(1 + br)^2 - s^2r^2(b + r)^2} \right| \leq \frac{(s - t)r(b + r)(1 + br)}{(1 + br)^2 - s^2r^2(b + r)^2}.
\]

Hence the lemma.

**Lemma 4.** If \( p(z) = (1 + tw(z))/(1 + sw(z)) \), \( w \in B \), then for \( |z| = r \), \( 0 \leq r < 1 \), we have

\[
\Re \left\{ k p(z) + \frac{t}{p(z)} \right\} - r^2 |sp(z) - t|^2 - |1 - p(z)|^2
\]

\[
(2.5) \geq \begin{cases} 
\frac{2}{1 - r^2} [V(1 + t)(1 - tr^2)(k(1 - r^2) + 1 - s^2r^2) - (1 - str^2)] & \text{if } R_b \leq R^* \\
W/W^* & \text{if } R_b \geq R^*
\end{cases}
\]

where

\[
W = t(kt + s^2)r^4 + 2bt((k + s) + (kt + s^2))r^3
\]

\[
(2.5; a) \quad + [b(1 + t)((k + t) + (kt + s^2))] + 2t(k + s) - (s - t)^2 |r
\]

\[
+ 2b((k + t) + t(k + s))r + (k + t)
\]

\[
(2.5; b) \quad W^* = (1 + rb(1 + t) + tr^2)[1 + rb(1 + s) + sr^2]
\]

and \( R^{*^2} = (1 + t)(1 - tr^2)(k(1 - r^2) + 1 - s^2r^2) \), \( R_b = A_b - D_b \) where \( A_b, D_b \) are defined as in Lemma 3 and \( k \leq s, -1 \leq t < s \leq 1 \).

Proof. Let \( |z| = r \), and \( p(z) = A_b + \xi + i\eta \equiv \Re^{\psi} \), then \( -\pi/2 < \psi < \pi/2 \). Denoting the left hand side of (2.5) by

\[
U_b(\xi, \eta), \text{ we get}
\]

\[
(2.6) \quad U_b(\xi, \eta) = k(A_b + \xi) + t(A_b + \xi)R^{-2} + \frac{1 - s^2r^2}{1 - r^2} [(A_b + \xi) - A_b]^2 + \eta^2 - D_b^2]R^{-1}
\]
and

\begin{equation}
\frac{\partial U_b}{\partial \eta} = \eta R^{-4} V_b(\xi, \eta)
\end{equation}

where

\begin{equation}
V_b(\xi, \eta) = -2t(A_b + \xi) + (D_i^2 + 2A_i(A_b + \xi) + A_i^3) \left( \frac{1 - s^2 \gamma^2}{1 - \gamma^2} \right) R
+ \left( \frac{1 - s^2 \gamma^2}{1 - \gamma^2} \right) R^3
\end{equation}

\begin{equation}
= -2tR \cos \psi + (D_i^2 - A_i^3 + 2A_iR \cos \psi) \left( \frac{1 - s^2 \gamma^2}{1 - \gamma^2} \right) R
+ \left( \frac{1 - s^2 \gamma^2}{1 - \gamma^2} \right) R^3 \equiv M_b(R, \psi)(\text{say}).
\end{equation}

Since for fixed \( r \), \( 0 \leq r < 1 \), \( A_b - D_b \) decreases as \( b \) increases over the interval \([0, 1]\), it follows that \( R \geq R \cos \psi \geq A_b - D_b \geq A_i - D_i \).

Thus, for all \( b \), \( 0 \leq b \leq 1 \),

\[ M_b(R, \psi) \geq R \cos \psi \left[ -2t + (D_i^2 - A_i^3 + 2A_iR \cos \psi + R^3) \left( \frac{1 - s^2 \gamma^2}{1 - \gamma^2} \right) \right] \]

\[ \geq 2R \cos \psi \left[ \left( \frac{1 - s^2 \gamma^2}{1 - \gamma^2} \right)(A_i - D_i)^3 - t \right] > 0, \]

for all \( s, t \) satisfying \(-1 \leq t < s \leq 1\). Thus \( V_b(\xi, \eta) = M_b(R, \psi) \) is positive for all points in the disc \( \Delta(z) \). Now, (2.7) gives that, for every fixed \( \xi \), \( U_b(\xi, \eta) \) is increasing function of \( \eta \) for positive \( \eta \) and is a decreasing function of \( \eta \) for negative \( \eta \). Thus, the minimum of \( U_b(\xi, \eta) \) inside the disc \( \Delta \) is attained on the diameter forming part of the real axis. Setting \( \eta = 0 \) in (2.6), we obtain

\begin{equation}
\min_{-1 \leq t \leq 1} U_b(\xi, \eta) = N_b(R) = \left( k + \frac{1 - s^2 \gamma^2}{1 - \gamma^2} \right) R + \frac{(1 + t)(1 - t \gamma^2)}{(1 - \gamma^2)} R^{-1} - 2A_i \left( \frac{1 - \gamma^2 s^2}{1 - \gamma^2} \right)
\end{equation}

where \( R = A_b + \xi \in [A_b - D_b, A_b + D_b] \). Thus the absolute minimum of \( N_b(R) \) in \((0, \infty)\) is attained at

\begin{equation}
R^* = \left( \frac{(1 + t)(1 - t \gamma^2)}{k(1 - \gamma^2) + 1 - s^2 \gamma^2} \right)^{1/2}
\end{equation}

and the value of this minimum is equal to
(2.11) \[ N_b(R^*) = \frac{1}{1 - r^2} \left[ \sqrt{(k(1 - r^2) + 1 - s^2r^2)(1 + t)(1 - tr^2)} \right. \\
\left. - (1 - str^2) \right]. \]

Since it is easily seen that \( R^* < A_b + D_b \) and that \( A_b + D_b \) is a decreasing function of \( b \) for \( 0 \leq b \leq 1 \), it follows that \( R^* < A_b + D_b \) for \( b \in [0, 1] \); but \( R^* \) is not always greater than \( A_b - D_b \). In case \( R^* \in [A_b - D_b, A_b + D_b] \), it can be easily verified that \( N_b(R) \) increases with \( R \) in \([A_b - D_b, A_b + D_b]\). Thus the minimum of \( N_b(R^*) \) on the segment \([A_b - D_b, A_b + D_b]\) is attained at \( R_b = A_b - D_b \). The value of this minimum equals

\[ N_b(R_b) = N_b(A_b - D_b) = W/W^* , \]

where \( W \) and \( W^* \) are given by (2.5; a) and (2.5; b). Moreover \( N_b(R^*) = N_b(R_b) \) for those values of \( k, s, \) and \( t \) for which \( R_b = R^* \). Hence the lemma.

3. The class \( R_a(\alpha, \beta) \). Let \( R(\alpha, \beta) \) be the class of functions \( f(z) = z + a_2z^2 + \cdots \) which are analytic and satisfy the inequality 
\[ |(f'(z) - 1)/[2\beta(f(z) - \alpha) - (f(z) - 1)]| < 1 \]
for some \( \alpha, \beta(0 \leq \alpha < 1, \ 0 < \beta \leq 1) \) and \( z \in E \). One of the authors [9] has shown that for
\( f \in R(\alpha, \beta), \ |a_z| \leq \beta(1 - \alpha) \). Define

\[ R_a(\alpha, \beta) = \{ f(z) = z + a\beta(1 - \alpha)z^2 + \cdots : f' \in P_a(\alpha, \beta), \ 0 \leq a \leq 1 \} . \]

Now, we determine a sharp estimate for the radii of convexity for functions in \( R_a(\alpha, \beta) \).

**Theorem 1.** Let \( f \in R_a(\alpha, \beta) \), then \( f \) is convex in \( |z| < r_0 \) where \( r_0 \) is the smallest positive root of the equation

\[ 1 + 4\alpha\beta ar + (4\alpha\beta^3a^2 - 2(1 + \beta - 3\alpha\beta))r^2 + 4\beta(2\alpha\beta - 1)ar^3 \]
\[ - (2\beta - 1)(2\alpha\beta - 1)r^4 = 0 \]

if \( R_a \geq R^* \) and

\[ r_0 = [(-\alpha\beta + \sqrt{\alpha(1 - 2\alpha\beta + \alpha\beta^2)}/(1 - 2\alpha\beta)]^{1/2} \]

if \( R_a \leq R^* \) where

\[ R_a = \frac{1 + 2\alpha\beta ar + (2\alpha\beta - 1)r^2}{1 + 2\beta ar + (2\beta - 1)r^2}, \ R^* = \left( \frac{\alpha(1 - (2\alpha\beta - 1)r^2)}{1 - (2\beta - 1)r^2} \right)^{1/2} \]

and \( r = |z| < 1 \). The result is sharp for each \( \alpha, \beta(0 \leq \alpha < 1, \ 0 < \beta \leq 1) \) and \( 0 \leq a \leq 1 \).
Proof. Since \( f \in R_\alpha(\alpha, \beta) \), an application of Schwarz's lemma gives

\[
(3.1) \quad f'(z) = \frac{1 + (2\alpha \beta - 1)w(z)}{1 + (2\beta - 1)w(z)}
\]

where \( w \in B \). Logarithmic differentiation of (3.1) gives

\[
(3.2) \quad 1 + z \frac{f''(z)}{f'(z)} = 1 - 2\beta(1 - \alpha) \left\{ \frac{zw'(z)}{(1 + (2\beta - 1)w(z))(1 + (2\alpha \beta - 1)w(z))} \right\}.
\]

Applying (2.2) with \( s = 2\beta - 1, \ t = 2\alpha \beta - 1 \) to (3.2), we get

\[
(3.3) \quad \text{Re} \left\{ 1 + z \frac{f''(z)}{f'(z)} \right\} \geq \frac{1}{2\beta(1 - \alpha)} \left[ \text{Re} \left\{ (2\beta - 1)p(z) + \frac{2\alpha \beta - 1}{p(z)} \right\} - \frac{r^2 |(2\beta - 1)p(z) + 1 - 2\alpha \beta|}{1 - r^2} \right] + \frac{1 - 2\alpha \beta}{\beta(1 - \alpha)}
\]

where \( p(z) = (1 + (2\alpha \beta - 1)w(z))/(1 + (2\beta - 1)w(z)) \). An application of Lemma 4 with \( k = s = 2\beta - 1, \ t = 2\alpha \beta - 1 \) to (3.3) gives

\[
(3.4) \quad \text{Re} \left\{ 1 + z \frac{f''(z)}{f'(z)} \right\} \geq \frac{1}{\beta(1 - \alpha)(1 - r^2)} \left[ \sqrt{4\alpha \beta^2(1 - (2\beta - 1)r^2)(1 + (1 - 2\alpha \beta)r^2)} ight. \\
\left. - (1 + (1 - 2\alpha \beta)(2\beta - 1)r^2) + (1 - 2\alpha \beta)(1 - r^2) \right]
\]

if \( R_\alpha \leq R^* \),

\[
\frac{1 + 4\alpha \beta ar + (4\alpha \beta^2 a^2 - 2(1 + \beta - 3\alpha \beta))r^2 + 4\beta \times (2\alpha \beta - 1)ar^2 + (2\beta - 1)(2\alpha \beta - 1)r^4}{(1 + 2\beta ar + (2\beta - 1)r^2)(1 + 2\alpha \beta ar + (2\alpha \beta - 1)r^2)}
\]

if \( R_\alpha \geq R^* \)

where

\[
R_\alpha = \frac{1 + 2\alpha \beta ar + (2\alpha \beta - 1)r^2}{1 + 2\beta ar + (2\beta - 1)r^2}, \quad R^* = \left( \frac{a(1 - (2\alpha \beta - 1)r^2)}{1 - (2\beta - 1)r^2} \right)^{1/2},
\]

\[0 \leq a \leq 1.\]

Now the theorem follows easily from (3.4).

The function given by

\[
f''(z) = \frac{1 - 2\alpha \beta az + (2\alpha \beta - 1)z^2}{1 - 2\beta az + (2\beta - 1)z^2} \quad \text{if } R_\alpha \geq R^*,
\]

and

\[
f''(z) = \frac{1 - 2\alpha \beta cz + (2\alpha \beta - 1)z^2}{1 - 2\beta cz + (2\beta - 1)z^2} \quad \text{if } R_\alpha \leq R^*
\]

where \( c \) is determined by the relation
show that the results obtained in the theorem are sharp.

Putting \( \beta = 1 \), in Theorem 1, we get the following result due to McCarty [8].

**Corollary 1(a).** Each \( f \in R_\alpha(\alpha) \) maps \( |z| < r_0 \) onto a convex region where \( r_0 \) is the smallest positive root of the equation

\[
1 + 4aar + (6\alpha - 4 + 4a^2)r^2 + 4(2\alpha - 1)ar^3 + (2\alpha - 1)r^4 = 0
\]

if \( R_\alpha \geq R^* \) and

\[
r_0 = \left\{ -\alpha + \sqrt{\alpha(1 - \alpha)/(1 - 2\alpha)} \right\}^{1/2}
\]

if \( R_\alpha \leq R^* \), where

\[
R_\alpha = \frac{1 + 2a2ar + (2\alpha - 1)r^2}{1 + 2ar + r^2}, \quad R^* = \left( \frac{\alpha(1 - (2\alpha - 1)r^2)}{1 - r^2} \right)^{1/2}
\]

and \( r = |z| < 1 \). The result is sharp for each \( \alpha \ (0 \leq \alpha < 1) \) and \( 0 \leq a \leq 1 \).

**Corollary 1(b).** Let \( f \in R'_\alpha(\alpha) \), then \( f \) is convex in \( |z| < r_0 \) where \( r_0 \) is the smallest positive root of the equation

\[
1 + 2(1 - \alpha)ar + ((1 - \alpha)a^2 - 3\alpha)r^2 - 2aar^2 = 0
\]

if \( R_\alpha \geq R^* \) and

\[
r_0 = \left\{ - (1 - \alpha) + \sqrt{(1 - \alpha)(1 + 3\alpha)}/2\alpha \right\}^{1/2}
\]

if \( R_\alpha \leq R^* \), where

\[
R_\alpha = \frac{1 + (1 - \alpha)ar - ar^2}{1 + ar}, \quad R^* = \left( (1 - \alpha)(1 + ar^2) \right)^{1/2}
\]

and \( r = |z| < 1 \). The result is sharp for each \( \alpha \ (0 \leq \alpha < 1) \) and \( 0 \leq a \leq 1 \).

The result is obtained by replacing \( \alpha \) by \( 1 - \alpha \) and \( \beta \) by \( 1/2 \) in Theorem 1. It may be noted that this result was obtained by Goel [4] under the additional restriction \( 1/2 \leq \alpha \leq 1 \).

**Remark.** Replacing \((\alpha, \beta)\) by \((0, 1)\), or by \((0, 1 - \delta)\), \(0 \leq \delta < 1\) or by \((0, (2\delta - 1)/2\delta)\), \(1/2 < \delta \leq 1\), or by \(((1 - \gamma)/1 + \gamma, (1 + \gamma)/(2))\), \(0 < \gamma \leq 1\), or by \(((1 - \delta + 2\gamma\delta)/(1 + \delta), (1 + \delta)/2)\), \(0 \leq \gamma < 1, 0 < \delta \leq 1\), we get the estimates for the radii of convexity for functions with
fixed second coefficient of the classes introduced and studied by MacGregor [7], Shaffer [13], Goel [3], Caplinger and Causey [1] and the authors [6] respectively.

4. The class $S_\alpha^*(\alpha, \beta)$. Let $S_\alpha^*(\alpha, \beta)$ be the class of functions $g(z) = z + a_2z^2 + \cdots$ which are analytic and satisfy the inequality $|(zg'(z)/g(z) - 1)/(2\beta(zg'(z)/g(z) - \alpha) - (zg'(z)/g(z) - 1))| < 1$, for some $\alpha, \beta(0 \leq \alpha < 1, 0 < \beta \leq 1)$ and $z \in E$. The authors [5] have shown that for $g \in S^*(\alpha, \beta)$, $|a_2| \leq 2\beta(1 - \alpha)$. Define

$$S_\alpha^*(\alpha, \beta) = \{g(z) = z + 2\alpha\beta(1 - \alpha)z^3 + \cdots : zg'/g \in P_\alpha(\alpha, \beta), 0 \leq \alpha \leq 1\}.$$

Now, we determine a sharp estimate for the radii of convexity for functions in $S_\alpha^*(\alpha, \beta)$.

**Theorem 2.** Let $g \in S_\alpha^*(\alpha, \beta)$, then $g$ is convex in $|z| < r_0$ where $r_0$ is the smallest positive root of the equation

$$1 + 2\beta(3\alpha - 1)ar + (4\alpha^2\beta^2a^3 + 8\alpha\beta - 2 - 4\beta)r^2
- 2\beta(1 + \alpha - 4\alpha\beta^2)ar^3 + (1 - 2\alpha\beta)^2r^4 = 0$$

if $R_a \geq R^*$ and

$$r_0 = [(5\alpha - 1)/((1 - \alpha + 4\beta\alpha^2) + 4\alpha\sqrt{(1 + \beta - 3\alpha\beta + \alpha^2\beta^2)})]'^3$$

if $R_a \leq R^*$, where

$$R_a = \frac{1 + 2\alpha\beta ar + (2\alpha\beta - 1)r^2}{1 + 2\beta ar + (2\beta - 1)r^2}, \quad R^* = \left(\frac{\alpha(1 + (1 - 2\alpha\beta)r^2)}{(2 - \alpha - (2\beta - \alpha)r^2}\right)^{1/2}$$

and $r = |z| < 1$. The result is sharp for each $\alpha, \beta (0 \leq \alpha < 1, 0 < \beta \leq 1)$ and $0 \leq \alpha \leq 1$.

**Proof.** Since $g \in S_\alpha^*(\alpha, \beta)$, an application of Schwarz's lemma gives

$$z \frac{g'(z)}{g(z)} = \frac{1 + (2\alpha\beta - 1)w(z)}{1 + (2\beta - 1)w(z)}$$

where $w \in B$. Logarithmic differentiation of (4.1) gives

$$1 + z \frac{g''(z)}{g'(z)} = \frac{1 + (2\alpha\beta - 1)w(z)}{1 + (2\beta - 1)w(z)}
- 2\beta(1 - \alpha)\left\{\frac{zw'(z)}{(1 + (2\beta - 1)w(z))(1 + (2\alpha\beta - 1)w(z))}\right\}.$$

Applying (2.2) with $s = 2\beta - 1$, $t = 2\alpha\beta - 1$ to (4.2), we get
(4.3) \[ \operatorname{Re}\left\{ 1 + z \frac{g''(z)}{g'(z)} \right\} \geq \frac{1}{2\beta(1 - \alpha)} \operatorname{Re}\left\{ (4\beta - 1 - 2\alpha\beta)p(z) \right\} \]
\[+ \frac{2\alpha\beta - 1}{p(z)} \right\} \right\} \right\} - \frac{r^2 |(2\beta - 1)p(z) + 1 - 2\alpha\beta| - |1 - p(z)|^2}{(1 - r^2)|p(z)|} \]
\[+ \frac{\alpha + \alpha\beta - 1}{\beta(1 - \alpha)} \]

where \( p(z) = (1 + (2\alpha\beta - 1)w(z))/(1 + (2\beta - 1)w(z)) \). Now, an application of Lemma 4 with \( k = 4\beta - 1 - 2\alpha\beta \), \( s = 2\beta - 1 \) and \( t = 2\alpha\beta - 1 \) to (4.3) gives the required results easily.

The functions given by
\[ z \frac{g'(z)}{g(z)} = \frac{1 - 2\alpha\beta az + (2\alpha\beta - 1)z^2}{1 - 2\beta az + (2\beta - 1)z^2} \text{ if } R_a \geq R^* \]
and
\[ z \frac{g'(z)}{g(z)} = \frac{1 - 2\alpha\beta cz + (2\alpha\beta - 1)z^2}{1 - 2\beta cz + (2\beta - 1)z^2} \text{ if } R_a \leq R^* \]
where \( c \) is determined by the relation
\[ \frac{1 - 2\alpha\beta cr + (2\alpha\beta - 1)r^2}{1 - 2\beta cr + (2\beta - 1)r^2} = R^* = \left( \frac{\alpha(1 - (2\alpha\beta - 1)r^2)}{(2 - \alpha - (2\beta - \alpha)r^2)} \right)^{1/2} \]
show that the results obtained in the theorem are sharp.

Taking \( \beta = 1 \), in Theorem 2, we get the following result due to McCarty [8] which also includes the result obtained by Tepper [16].

**Corollary 2(a).** Each \( g \in S_*(\alpha) \) maps \( |z| < r_o \) onto a convex region where \( r_o \) is the smallest positive root of the equation
\[ 1 + (6\alpha - 2)ar + (4\alpha^2a^2 + 8\alpha - 6)r^2 + (8\alpha^2 - 2\alpha - 2)ar^3 \]
\[+ (2\alpha - 1)r^4 = 0 \]
if \( R_a \geq R^* \) and
\[ r_o = \left[ (5\alpha - 1)/((4\alpha^2 - \alpha + 1) + 4\alpha\sqrt{(\alpha^2 - 3\alpha + 2)}) \right]^{1/2} \]
if \( R_a \leq R^* \) where
\[ R_a = \frac{1 + 2\alpha ar + (2\alpha - 1)r^2}{1 + 2ar + r^2} \]
\[ R^* = \frac{(\alpha(1 - (2\alpha - 1)r^2)^2}{(2 - \alpha)(1 - r^2)} \]
and \( r = |z| < 1 \). The result is sharp for each \( \alpha \) (0 \( \leq \alpha < 1 \)) and \( 0 \( \leq a \leq 1 \).
REMARKS. (i) Replacing \((\alpha, \beta)\) by \((0,1/2)\), or by \((0,(2\delta - 1)/2\delta), 1/2 < \delta \leq 1\), or by \(((1 - \gamma)/1 + \gamma, (1 + \gamma)/2), 0 < \gamma \leq 1\), we may obtain the estimates for the radii of convexity for functions with fixed second coefficient of the classes introduced and studied by Eenigenburg [2], Ram Singh [14] and Padmanabhan [10] respectively.

(ii) Setting \(a = 1\) in Theorem 1 and Theorem 2 we get the sharp estimates for the radii of convexity for functions in \(R(\alpha, \beta)\) and \(S^*(\alpha, \beta)\). These were obtained by the authors in [9] and [5] and thus also include the results obtained in [1], [2], [13] etc.

(iii) By setting \(a = 0\) in Theorem 1 and Theorem 2, we may get the results for functions in \(R(\alpha, \beta)\) and \(S^*(\alpha, \beta)\) with missing second coefficient and in particular for odd functions in these classes.

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