PEIRCE IDEALS IN JORDAN ALGEBRAS

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In attempting to investigate infinite-dimensional simple Jordan algebras $J$ having rich supplies of idempotents, it would be helpful to know that the Peirce subalgebra $J_1(e)$ relative to an idempotent $e$ in $J$ remains simple. This clearly holds for associative and alternative algebras because any ideal in a Peirce space is the projection of a global ideal. The corresponding result is false for Jordan algebras: there are multiplications of the ambient algebra $J$ which send $J_1$ to itself (therefore leave invariant the projection of a global ideal), but are not expressible as multiplication by elements of $J_1$ (therefore need not leave invariant an arbitrary ideal of $J_1$). We show that an ideal $K_1$ is the projection of a global ideal iff it is invariant under the multiplications $V_{J_1/2}J_1/2$ and $U_{J_1/2}U_{J_1/2}$. This yields an explicit expression for the global ideal generated by a Peirce ideal. We then show that if $J$ is a simple Jordan algebra with idempotent, the Peirce subalgebras $J_1$ and $J_0$ inherit simplicity.

Throughout we consider a quadratic Jordan algebra $J$ over an arbitrary ring of scalars $\Phi$ with product

$$U_x y$$

quadratic in $x$ and linear in $y$. Linearization yields a trilinear product

$$\{xyz\} = U_{x,y} = V_{x,y,z}.$$

(See [1] for basic results on quadratic Jordan algebras.) If $e$ is an idempotent element of $J$, $e^2 = e$, then we have a Peirce decomposition $J = J_1 \oplus J_{1/2} \oplus J_0$ where $J_1$, $J_0$ are subalgebras. We wish to relate the ideals in these Peirce subalgebras $J_1$ to ideals in the ambient algebra $J$.

Analogous results hold for Jordan triple systems. However, in this case $U_e$ is merely an involution on $J_1$ rather than the identity map, and this causes such technical complications in the Peirce identities that the basic argument is lost sight of. We prefer to do the simpler Jordan algebra case first, and treat the triple system case separately [3].

We recall a few basic identities satisfied by Jordan algebras:

1. $U_{U(x)y} = U_x U_y U_x$
2. $V_{U(x,y)z} = U_x U_y U_z + U_z U_y U_x + V_{x,y} U_z V_{y,z} - U_{U(x)U(y)z, z}$
3. $U_{V(x,y)z, z} = V_{x,y} U_z + U_z V_{y,z}$
4. $U_{x,y} U_z = V_{x,y} V_{z, z} - V_{U(x)U(y)z, z}$

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In a Peirce decomposition we have the following identities for \( i = 1, 0 \) and \( j = 1 - i \):

\[
\begin{align*}
(P1) \quad & U_{x_i y_{1/2}} = U_{x_i} U_{y_{1/2}} \text{ on } J_j(x_i \in J_i, y_{1/2} \in J_{j/2}) \\
(P2) \quad & U_{x_i y_{1/2}} = U_{y_{1/2}} U_{x_i} \text{ on } J_i \\
(P3) \quad & x_{1/2} y_{1/2} = x_{1/2} E_i(x_{1/2} y_{1/2}) - y_{1/2} E_j(x_{1/2} y_{1/2}) \\
(P4) \quad & [x_{1/2} a_i y_{1/2}] = E_j(x_{1/2} a_i y_{1/2}) \\
(P5) \quad & [x_{1/2} y_{1/2} a_i] = E_i(x_{1/2} a_i) \\
(P6) \quad & a_i E_i(x_{1/2} b_j) = (a_i E_i x_{1/2}) \circ b_j \\
(P7) \quad & a_i E_i x_{1/2} = a_i (a_i E_i x_{1/2}) \quad (V_{a_i} = V_{a_i} \text{ on } J_{j/2}) \\
(P8) \quad & U_{a_i b_i} x_{1/2} = a_i (b_i (a_i E_i x_{1/2})) \quad (V_{U(a_i) b_i} = V_{a_i} V_{b_i} V_{a_i} \text{ on } J_{j/2}) \\
(P9) \quad & [a_i b_i x_{1/2}] = a_i (b_i E_i x_{1/2}) 
\end{align*}
\]

where \( E_i \) denotes the Peirce projection on the Peirce space \( J_i \).

1. Ideal-building. A subspace \( K \) of a Jordan algebra is an ideal if it is both an outer ideal

\[(1.1) \quad U_j K \subset K \quad (U_j K \subset K, V_j K \subset K) \]

and an inner ideal

\[(1.2) \quad U_K \hat{J} \subset K \quad (U_K J \subset K, K^2 \subset K). \]

Here \( \hat{J} = \Phi 1 + J \) denotes the unital hull of the Jordan algebra \( J \); if \( J \) is itself unital then \( \hat{J} = J \), and the conditions \( V_j K \subset K \) and \( K^2 \subset K \) are superfluous \( (V_x = U_{x,1}, x^2 = U_{x,1}) \). A useful observation is that once \( K \) is known to be an outer ideal it is an inner ideal as soon as

\[(1.3) \quad U_{k,j} K \subset K \text{ for some spanning set } \{k_i\} \text{ of } K.\]

From now on we fix an idempotent \( e \) in \( J \) and consider the corresponding Peirce decomposition

\[ J = J_1 \oplus J_{1/2} \oplus J_0. \]

Then the unital hull \( \hat{J} = \Phi 1 + J = \Phi (1 - e) + J \) can be identified with \( J_1 \oplus J_{1/2} \oplus \hat{J}_e. \) Note that any ideal \( K \subset J \) is invariant under the Peirce projections \( E_i \) since these are multiplication operators, therefore \( K \) is the direct sum of its Peirce components

\[ K = K_1 \oplus K_{1/2} \oplus K_e \quad (K_e = K \cap J_e). \]

Triple products of Peirce elements largely reduce to simpler bilinear products:

\[
U_{x_1 + x_{1/2} + x_0}(y_1 + y_{1/2} + y_0) = U_{x_1} y_1 + U_{x_{1/2}} (y_1 + y_{1/2} + y_0) + U_{x_0} y_0 \\
\quad + \{x_1 y_{1/2} x_0\} + \{x_1 y_1 x_{1/2}\} + \{x_1 y_{1/2} x_{1/2}\} + \{x_0 y_{1/2} x_{1/2}\}
\]
Correspondingly, the ideal conditions (1.1), (1.2) for $K$ reduce to simpler conditions on the Peirce components $K_i$.

**IDEAL CRITERION 1.5.** A subspace $K = K_1 \oplus K_1/2 \oplus K_0$ is an ideal of a Jordan algebra $J = J_1 \oplus J_1/2 \oplus J_0$ iff for $i = 1, 0, j = 1 - i$

(C1) $K_i$ is an ideal in $J_i$

(C2) $E_i(J_1/2 \circ K_1/2) \subset K_i$

(C3) $J_i \circ K_1/2 \subset K_1/2$

(C4) $K_i \circ J_1/2 \subset K_1/2$

(C5) $U_{j,1/2} K_i \subset K_j$

(C6) $U_{1/2} \hat{J}_i \subset K_i$ for some spanning set $\{k_{1/2}\}$ of $K_{1/2}$.

If $1/2 \in \Phi$ the conditions (C5), (C6) are superfluous.

**Proof.** Clearly these inclusions are all necessary by the Peirce relations and the fact that any product involving a factor from an ideal falls back in that ideal.

A routine calculation shows (C1)-(C5) suffice to establish outerness: $U_{3/2} K \subset K$ follows from (1.4) since $U_{3/2} K_i \subset K_i$ by (C1); $U_{3/2} K_i \subset K_j$ by (C5); $J_{1/2} \circ (J_1/2 \circ K_1/2) \subset K_1/2$ by (C2), (C4); $K_i \circ J_1/2 \subset K_1/2$ by (C3); $J_i \circ (J_1/2 \circ K_1/2) \subset K_1/2$ by (C3), (C4); $E_i(J_1/2 \circ (J_1/2 \circ K_1/2)) \subset K_i$ by (C3), (C2).

Once we have outerness, innerness (1.3) follows for the spanning set of elements $k_i \in K_i$ ($i = 1, 0$) and the given $k_{1/2} \in K_{1/2}$ since $U_{k_{1/2}} \hat{J} = U_{k} \hat{J}_i \subset K_i$ by (C1), $U_{k_{1/2}} \hat{J} \subset K_j$ by (C6), and $U_{k_{1/2}} \hat{J}_i = k_{1/2} \circ E_i(k_{1/2} \circ J_1/2) - J_{1/2} \circ E_0(k_{1/2}^2) \subset K_1/2$ by (C3), (C4), and $E_0(k_{1/2}^2) = U_{k_{1/2}} \subset K_0$ by (C6).

Since $2U_{x} = U_{x,x}$ and always $U_{J_1/2,J_{1/2}} K_i = E_3(J_1/2 \circ (K_1 \circ J_{1/2})) \subset K_j$ by (C4), (C2), $U_{J_{1/2},K_1} \hat{J}_i = E_3(J_{1/2} \circ (J_1 \circ K_{1/2})) \subset K_j$ by (C3), (C2), we see that (C5), (C6) are consequences of (C2)-(C4) when $1/2 \in \Phi$.

**REMARK 1.6.** In characteristic 2 situations we cannot dispense with (C5) and (C6)—they really are necessary in addition to the other conditions. For example, if $J$ is the special Jordan algebra $\Phi e_{11} + \Phi(e_{12} + e_{21}) + \Phi e_{22}$ of symmetric $2 \times 2$ matrices over $\Phi$ of characteristic 2, then relative to $e = e_{11}$ we have $J_1 = \Phi e_{11}$, $J_{1/2} = \Phi(e_{12} + e_{21})$, $J_0 = \Phi e_{22}$ so $J_{1/2} \circ J_{1/2} = 2\Phi(e_{12} + e_{21})^2 = 0$, and thus (C2) is automatic for any $K$.

If we take $K_i = K_0 = 0, K_{1/2} = J_{1/2}$ then (C1)-(C5) hold trivially, but not (C6) since $U_{J_{1/2}, J_i} = \Phi U_{e_{12} + e_{21}, e_{11}} = \Phi e_{ii} = J_i \neq 0$. Thus (C6) is not a consequence of the other conditions. If we take $K = \lambda \Phi e_{11}$, $K_{1/2} = \lambda \Phi(e_{12} + e_{21})$, $K_0 = \lambda \Phi e_{22}$ for noninvertible $\lambda$ in a domain $\Phi$ of charac-
teristic 2, then (C1), (C2)-(C4) hold trivially, as does (C6) by
\[ U_{\lambda(e_1+e_2)}(\Phi e_i) = \lambda^2 \Phi e_{ij}, \]
but (C5) is not a consequence since \( U_{e_1+e_2}(\lambda \Phi e_1) = \lambda \Phi e_2 \not\subset \lambda^2 \Phi e_2 = K_i. \)

Next we introduce the key notions of invariance. An ideal \( K_i \) in a Peirce space \( J_i \) \((i=1,0)\) is \textit{invariant} if it is both \( U \)-invariant
\[ (1.7) \quad U_{J_{1/2}} U_{J_{1/2}} K_i \subset K_i \]
and \( V \)-invariant
\[ (1.8) \quad V_{J_{1/2},J_{1/2}} K_i = E_i(J_{1/2} \circ (J_{1/2} \circ K_{1/2})) \subset K_i. \]
By the Peirce relations and (P5) the maps \( U_{x_{1/2}} U_{y_{1/2}} \) and \( V_{x_{1/2},y_{1/2}} \) map \( J_i \) into itself, though in general they cannot be compressed into a multiplication from \( J_i \).

\( V \)-invariance is the more fundamental notion, and goes a long way towards ensuring \( U \)-invariance. For example, the special case \( z = y \) in (0.4) shows
\[ (1.9) \quad 2 U_x U_y = V_{x,y} V_{x,y} - V_{U(x)y,y}, \]
so whenever we can divide by 2 \( V \)-invariance implies \( U \)-invariance.

We can flip an invariant ideal from one diagonal Peirce space to the other.

**Flipping Lemma 1.10.** If \( K_i \) is an ideal in a Peirce space \( J_i \) \((i=1,0)\) then \( K_j = U_{J_{1/2}} K_i \) is an ideal in \( J_j \). If \( K_i \) is \( V \)-invariant or \( U \)-invariant, so is the flipped ideal \( K_j. \)

\textit{Proof.} \( K_j \) is outer since \( U_{J_{1/2}} K_i = U_{J_{1/2}} U_{I_{1/2}} K_i = U_{J_{1/2}} J_{1/2} K_i \) (by (P1)) \( \subset U_{J_{1/2}} K_i = K_j \) as in (1.1), and for the spanning set of elements \( k_j = U_{y_{1/2}} k_i \) we have by (0.1) \( U_{J_{1/2}} J_{1/2} U_{y_{1/2}} k_i = U_{J_{1/2}} U_{J_{1/2}} K_i \subset U_{J_{1/2}} K_i = K_j \), so by (1.3) \( K_j \) is an ideal. If \( K_i \) is \( V \)-invariant so is \( K_j \), since by (0.3) \( V_{J_{1/2},J_{1/2}} K_i = V_{J_{1/2},J_{1/2}} U_{J_{1/2}} K_i \subset U_{V(J_{1/2},J_{1/2})} K_i = U_{J_{1/2}} J_{1/2} K_i \subset U_{J_{1/2}} K_i \) (by \( V \)-invariance of \( K_i \)) = \( K_j \), and \( K_j \) trivially inherits \( U \)-invariance
\[ U_{J_{1/2}} U_{J_{1/2}} K_i = U_{J_{1/2}} U_{J_{1/2}} U_{J_{1/2}} K_i \subset U_{J_{1/2}} K_i \]
(by \( U \)-invariance) = \( K_j \).

Now we are ready to establish the main result of this section, describing the global ideal generated by an invariant Peirce ideal.

**Projection Theorem 1.11.** An ideal \( K_i \) in a Peirce space
J_i (i = 1, 0) is the Peirce projection of a global ideal K in J iff K_i is invariant. In this case the ideal generated by K_i takes the form

\[ K = K_i \uplus K_i \circ J_{1/2} \uplus U_{J_{1/2}} K_i. \]

If \( 1/2 \in \Phi \) we have \( U_{J_{1/2}} K_i = E_i (J_{1/2} \circ (K_i \circ J_{1/2})). \)

**Proof.** We have already noted that if \( K_i \) is the projection of an ideal \( K \) then by the Peirce relations and invariance of \( K \) under all multiplications from \( J, K_i \) must be invariant. We must establish the converse. Since the ideal generated by \( K_i \) must certainly contain the above products, if we can show the above \( K \) actually is an ideal then we will have exhibited \( K_i \) as the projection of an ideal \( K \) which is thus precisely the ideal generated by \( K_i. \)

We verify the conditions of the Ideal Criterion (1.5). \( K_i \) is an invariant ideal in \( J \) by hypothesis, and \( K_j = U_{J_{1/2}} K_i \) is an invariant ideal in \( J_j \) by the Flipping Lemma 1.10. Thus (C1) holds. For (C2), note \( E_i (J_{1/2} \circ K_{1/2}) = E_i (J_{1/2} \circ (J_{1/2} \circ K_i)) = \{ J_{1/2} \circ J_{1/2} \circ K_i \} = V_{J_{1/2}} J_{1/2} K_i \subset K_i \) by (P5) and \( V \)-invariance, also \( E_i (J_{1/2} \circ K_{1/2}) = \{ J_{1/2} \circ J_{1/2} \} \subset U_{J_{1/2}} K_i = K_j \) by (P4). For (C3), \( J_j \circ K_{1/2} = J_j \circ (K_j \circ J_{1/2}) = K_j \circ (J_j \circ J_{1/2}) \subset K_j \circ J_{1/2} = K_{1/2} \) by (P6), while \( J_i \circ K_{1/2} = J_i \circ (K_i \circ J_{1/2}) = (J_i \circ K_i) \circ J_{1/2} = K_i \circ J_{1/2} = K_{1/2} \subset K_{1/2} \) by (P7) and the fact that \( K_i \subset J_i. \) For (C4) we have \( K_j \circ J_{1/2} = K_{1/2} \) by definition, and \( K_j \circ J_{1/2} = U_{J_{1/2}} K_i \circ J_{1/2} \subset U_{J_{1/2}} J_{1/2} \circ K_i = J_{1/2} \circ K_i = K_{1/2} \) by \( V \)-invariance of \( K_i. \) For (C5), \( U_{J_{1/2}} K_i = K_j \) by definition, while \( U_{J_{1/2}} K_i = U_{J_{1/2}} U_{J_{1/2}} K_i \subset K_i \) by \( U \)-invariance of \( K_i. \) For (C6), the spanning elements \( k_{1/2} = k_i \circ y_{1/2} \) satisfy \( U_{k_i \circ y_{1/2}} \hat{J} = U_{y_{1/2}} U_{k_i \circ y_{1/2}} \subset U_{J_{1/2}} K_i = K_j \) by (P2) and \( K_i \subset J_i, \) similarly \( U_{k_i \circ y_{1/2}} \hat{J} = U_{k_i \circ y_{1/2}} \hat{J} \subset U_{k_i \circ y_{1/2}} K_i \subset K_i \) by (P1) and \( K_i \subset J_i. \) Thus (C1–C6) hold, and \( K \) is an ideal.

**Example 1.12.** The connector ideal generated by an off-diagonal Peirce space \( J_{1/2} \) is

\[ I(J_{1/2}) = U_{J_{1/2}} J_0 \uplus J_{1/2} \uplus U_{J_{1/2}} J_1. \]

**Proof.** It suffices to verify conditions (C1–C6) of (1.5): (C3–C6) are automatic since \( K_{1/2} = J_{1/2}, \) \( K_j = U_{J_{1/2}} \hat{J} \); (C1) follows from the Flipping Lemma 1.10 applied to \( \hat{J} \) in \( J \); (C2) follows from \( E_i (J_{1/2} \circ J_{1/2}) = \{ J_{1/2} \circ J_{1/2} \} \subset U_{J_{1/2}} \hat{J} \subset K_i \) by (P4).

**Example 1.13.** If \( Z_i \) denotes the kernel of the Peirce specialization of \( J_i \) on \( J_{1/2}, \)

\[ Z_i = \{ z_i \in J_i \mid z_i \circ J_{1/2} = 0 \} \]
then $Z = Z_1 \oplus Z_0$ is an ideal in $J$ which annihilates the connector ideal, $U_Z I(J_{1/2}) = 0$.

Proof. Any time $K$ has $K_{1/2} = 0$ the conditions (C2), (C3), (C6) become vacuous and (C4) becomes the condition $K_i \subset Z_i$. If we take $K_i = Z_i$ (C4) is thus satisfied, as is (C1) since the Peirce specialization is a homomorphism of $J_i$ into $\operatorname{End}(J_{1/2})$ by (P7), (P8) and therefore its kernel is an ideal. Moreover, these are interchanged by $U_{J_{1/2}}$ as in (C5) since $U_{x_{1/2}}z_i \circ y_{1/2} = V_{y_{1/2}}U_{x_{1/2}}z_i = \{U_{x_{1/2}}y_{1/2}z_{1/2} - U_{x_{1/2}}V_{y_{1/2}}z_i\}$ (by (0.3) with $x = 1) = \{x_{1/2}z_i E_i(y_{1/2} \circ x_{1/2})\} - U_{x_{1/2}}V_{y_{1/2}}z_i = (x_{1/2}z_i) \circ E_j(y_{1/2} \circ x_{1/2}) - U_{x_{1/2}}(y_{1/2} \circ z_i) = 0$ by (P9) if $z_i \circ x_{1/2} = z_i \circ y_{1/2} = 0$.

Thus $Z$ is an ideal in $J \cdot U_Z I(J_{1/2}) = 0$ since by (1.4) we have $U_{x_{z_1+0}}(k_1 + k_{1/2} + k_0) = U_z k_1 + U_{x_0}k_0 + z_1 \circ (y_{1/2} \circ z_0) = 0$ where $U_z K_i = U_{x_i}U_{J_{1/2}}J_j = U_{x_i}z_{1/2}J_j$ by (P1) and $z_i \circ J_{1/2} = 0$.

**Proposition 1.14.** If $J$ is a prime Jordan algebra and $e \neq 1, 0$ a proper idempotent, then $J_{1/2} \neq 0$ and the Peirce specializations of $J_1$ and $J_0$ on $J_{1/2}$ are faithful (hence $J_1, J_0$ are special Jordan algebras).

Proof. If $J_{1/2} = 0$ then $J = J_1 \oplus J_0$ would be a direct sum of ideals, whereupon primeness would force $J = J_1$ (hence $e = 1$) or $J = J_0$ (hence $e = 0$). Thus $J_{1/2}$ cannot vanish if $e$ is proper. Then $U_Z I(J_{1/2}) = 0$ for $I(J_{1/2}) \neq 0$ forces $Z = 0$ by primeness.

Thus in any prime exceptional Jordan algebra $J$, as soon as we examine a proper piece $J_i(e)$ or $J_0(e)$ it is special (in some sense $J$ has no smaller exceptional pieces), and exceptionality results only from the way $J_i$ and $J_j$ are tied together via $J_{1/2}$.

In §4 we will see that when $J$ is simple the same is true of $J_1$ and $J_0$, so $J$ is built up of pieces which are simple and special.

Note that if $J$ is simple and $e$ proper we have $J_{1/2} \neq 0$ by 1.14, so by simplicity $I(J_{1/2}) = J$ and by (1.12) we have

\[(1.15) \quad U_{J_{1/2}J_0} = J_1, \quad U_{J_1}J_1 = J_0.\]

We can improve on this by removing the hat from $J_0$. To do this we need to look at the ideal generated by $J_0$. Trivially $J_i$ is an invariant ideal in $J_i$, and $J_i \circ J_{1/2} = e \circ J_{1/2} = J_{1/2}$, so by 1.11 we have

**Example 1.16.** The ideal in $J$ generated by a diagonal Peirce space $J_i(e)$ is

\[
\begin{align*}
(i = 1) \quad I(J_i) &= J_1 \oplus J_{1/2} \oplus U_{J_{1/2}}J_1 \\
(i = 0) \quad I(J_0) &= J_0 \oplus J_0 \circ J_{1/2} \oplus U_{J_{1/2}}J_0.
\end{align*}
\]
If \( J \) is simple then \( e \neq 0 \) implies \( J_1 \neq 0 \) and hence \( I(J_1) = J_1 \), once more leading to \( U_{J_1/2} J_1 = J_0 \). If we knew \( e \neq 1 \) implied \( J_0 \neq 0 \) we could similarly deduce \( I(J_0) = J \) by simplicity and hence \( U_{J_1/2} J_0 = J_1 \) (without the hat).

Surprisingly, it takes a bit of arguing to establish \( J_0 \neq 0 \). Suppose in fact \( J_1 = 0 \). Then for \( z_{1/2} \in J_{1/2} \) we would have \( z_{1/2}^2 \in J_{1/2} + J_0 = J_{1/2} \), and hence \( z_{1/2} \) would be trivial since \( U_{z_{1/2}} J = U_{z_{1/2}} J_1 = U_{z_{1/2}} U_{z_{1/2}} J_1 \subset U_{z_{1/2}} J_0 = 0 \). But a simple \( J \) with idempotent is not nil and therefore has no trivial elements, so \( z_{1/2}^2 = 0 \) and \( z_{1/2} \neq 0 \) for all \( z_{1/2} \), \( w_{1/2} \in J_{1/2} \). But then by (1.4) \( U_{z_{1/2}} w_{1/2} = z_{1/2} \circ E_1(z_{1/2} \circ w_{1/2}) - w_{1/2} \circ E_0(z_{1/2}^2) = 0 \), so \( U_{z_{1/2}} J_{1/2} = 0 \), and since already \( U_{z_{1/2}} J_1 \subset J_0 = 0 \) we have \( U_{z_{1/2}} J = 0 \) and \( z_{1/2} \) would be trivial. Again \( J \) has no trivial elements, so \( z_{1/2} = 0 \), \( J_{1/2} = 0 \), contradicting 1.14.

**Proposition 1.17.** If \( J \) is a simple Jordan algebra and \( e \neq 1, 0 \) a proper idempotent, then

\[
U_{J_{1/2}} J_0 = J_1, \quad U_{J_{1/2}} J_1 = J_0.
\]

2. Invariance. To construct global ideals we must begin with invariant Peirce ideals. We now turn to the question of conditions under which an ideal is automatically invariant. Throughout this section we will be concerned with ideals \( K_t \) in a diagonal Peirce space \( J_t (t = 1, 0) \).

While \( V_{J_{1/2}, J_{1/2}} K_t \) and \( U_{J_{1/2}, J_{1/2}} K_t \) are not in general contained in \( K_t \), they are in some sense contained in the “square root” and “fourth root” of \( K_t \); \( V_{J_{1/2}, J_{1/2}} \) maps \( K_t \) into \( K_t \), and \( U_{J_{1/2}, J_{1/2}} \) maps \( K_t \) into \( K_t \). More precisely, we have the following useful technical result.

**Lemma 2.1.** For any ideal \( K_t \subset J_t \) we have

\[
(2.2) \quad V_{J_{1/2}, J_{1/2}}(U_{K_t} J_t) \subset K_t
\]

and

\[
(2.3) \quad U_{J_{1/2}} U_{J_{1/2}}(U_{K_t} J_t) \subset K_t.
\]

In general, for \( x, y \in J_{1/2} \), \( k \in K_t \), \( a \in J_t \) we have

\[
(2.4) \quad V_{x, y} u_k a = U_{V(x, y)k, x} a - U_k V_{y, x} a \in K_t
\]

\[
(2.5) \quad U_x U_y u_k a = U_{(xy)k, x} a - U_k U_x U_y a - V_{x, y} U_k V_{y, x} a \\
\quad + U_{k, U(x)U(y)k} a \subset U_{(xy)k, x} a + K_t
\]

so that whenever \( k \in K_t \) is \( V \)-invariant, \( \{xyk\} = V_{x, y} k \in K_t \), then \( U_k J_t \) is \( U \)-invariant, \( U_x U_y (U_k a) \in K_t \).
Proof. For (2.4) we have by (0.3) $V_{x,y}U_k a U_{(zyk),k} a - U_k V_{y,z} a \in U_j \cdot K_i a - U_k V_{y,z} a \subset K_i$ whenever $K_i \subset J_i$. For (2.5) we use (0.2): $U_{(zyk)} a = [U_x U_y U_k + U_k U_y U_z + V_{z,y} U_k V_{y,z} - U_k V_{y,z}] a \equiv U_x U_y U_k a$ modulo $K_i$ since $U_k U_y U_z a \in U_K U_a \subset K_i$. Applying (2.4) to $k \in K_i$, $a \in \hat{J}_i$, yields (2.2), and applying (2.5) to $k \in U_K J_i$, so $\{xyz\} = 0$ by (2.2)) yields (2.3).

**Example 2.6.** If $B_i, C_i$ are invariant ideals in $J_i$ so is their product $U_{ni} C_i$.

**Proof.** For $V$-invariance apply (2.4), for $U$-invariance apply (2.5).

**Example 2.7.** If $K_i$ is an idempotent ideal in $J_i$, $U_{K_i} J_i = K_i$, then $K_i$ is invariant.

**Example 2.8.** If $B_i$ are invariant ideals in $J_i$ so is their sum $\sum B_i$ and their intersection $\cap B_i$.

**Example 2.9.** For any ideal $K_i \subset J_i$ the infinite Penico derived ideal $P^\infty(K_i) = \bigcap P^n(K_i)$ is an invariant ideal $(P^{n+1}(K_i) = P(P^n(K_i))$ where $P(L_i) = U_{L_i} J_i)$. Similarly for the infinite derived ideal $D^\infty(K_i)$ (where $D(L_i) = U_{L_i} J_i$). Thus either $K_i$ contains a nonzero invariant ideal, or else it is $\infty$-nilpotent: $P^\infty(K_i) = 0$.

**Proof.** $V$-invariance of $P^\infty(K_i)$ follows from (2.2),

$$V_{j/2,j/2} (P^{n+1}(K_i)) \subset P^n(K_i),$$

and $U$-invariance from (2.3), $U_{j/2,j/2} (P^{n+1}(K_i)) \subset P^n(K_i)$. For $D^\infty(K_i)$ we use (2.4) to get $V$-invariance, $V_{j/2,j/2} D^{n+1}(K_i) \subset D^n(K_i)$ and (2.5) to get $U$-invariance, $U_{j/2,j/2} D^{n+1}(K_i) \subset D^n(K_i)$ (note $V_{x,y} U_{d_{n+1}} V_{y,z} D_{n+1} \oplus V_{x,z} D^{n+1} \subset D^n$ by the relation for the $V$’s).

We have seen in the Flipping Lemma 1.10 that one way of obtaining an invariant Peirce ideal is to flip an invariant ideal by $U_{j/2}$. Another way of obtaining an invariant Peirce ideal is to take the kernel of $U_{j/2}$ instead of the image.

**Kernel Lemma 2.10.** $\text{Ker } U_{j/2} = \{z \in J_i | U_{j/2} z = U_{j/2} z \hat{J}_i = 0\}$ is an invariant ideal in $J_i$.

**Proof.** $K_i = \text{Ker } U_{j/2}$ is trivially $U$-invariant $(U_{j/2} U_{j/2} K_i = 0)$, and is $V$-invariant because by (0.3) $U_{j/2}(V_{x/2,y/2} z) \subset U_{(y_1/2,z_1/2,j_1/2)} - V_{y/2,z/2,j/2} U_{j/2} z = 0$, and by (0.2) and (0.3)
\[ U_{J_{1/2}} U_{V(x_1/2, y_1/2) U_{J_i}} = U_{J_{1/2}} U_{z_1/2} U_{y_1/2} U_z + U_z U_{y_1/2} U_{x_1/2} + V_{x_1/2, y_1/2} U_z V_{y_1/2, x_1/2} \]

\[ - U_{V(x_1/2, y_1/2) U_{J_{1/2}} U_{J_i}} \subset U_{J_{1/2}} (U_{J_{1/2}} U_{J_i}) + U_{J_{1/2}} (U_{J_i} J_i) \]

\[ + \left( U_{V_{y_1/2, x_1/2} U_{J_{1/2}} U_{J_i}} - V_{y_1/2, x_1/2} U_{J_{1/2}} U_{J_i} \right) = 0 = 0. \]

\( K_i \) is a linear subspace since for \( z, w \in K_i \) we have \( U_{J_{1/2}} U_{z+w} \tilde{J}_i = U_{J_{1/2}} (U_z + U_w + U_{z+w}) \tilde{J}_i \) where by (0.3) \( U_{J_{1/2}} U_{z+w} \tilde{J}_i = U_{J_{1/2}} V_{z+w} \tilde{J}_i = (U_{J_{1/2}} U_{z+w} \tilde{J}_i - V_{z+w} U_{J_{1/2}} \tilde{J}_i) = 0 \). It is an outer ideal since \( U_{J_{1/2}} (U_{J_i} z) = U_{J_{1/2}} z \subset U_{J_{1/2}} z = 0 \) by (P2), \( U_{J_{1/2}} U_{U_{J_i} z} \tilde{J}_i = U_{J_{1/2}} U_{z} U_{J_i} \tilde{J}_i = U_{J_{1/2}} U_{z} U_{J_i} \tilde{J}_i = 0 \) by (0.1), (P2), and is an inner ideal since \( U_{J_{1/2}} (U_{z} J_i) = 0 \), \( U_{J_{1/2}} (U_{U_{J_i} z}) \tilde{J}_i = U_{J_{1/2}} U_{z} U_{J_i} \tilde{J}_i = U_{J_{1/2}} U_{z} U_{J_i} \tilde{J}_i = 0 \) by (0.1).

We can easily show that a strongly semiprime ideal is invariant. Recall that \( K_i \) is strongly semiprime in \( J_1 \) if \( J_i = J_i/K_i \) is strongly semiprime in the sense of having no trivial elements \( U_{J_i} J_i = 0 \); this is equivalent to \( U_{J_i} J_i \subset K_i \iff z_i \in K_i \).

**Theorem 2.11.** Any strongly semiprime ideal \( K_i \subset J_i \) is invariant.

**Proof.** For \( x, y \in J_{1/2}, k \in K_i \) we have \( \{xyk\} \in K_i \implies U_{\{xyk\}} J_i \subset K_i \) (strong semiprimeness) \( \iff U_z U_y U_k J_i \subset K_i \) (using (2.5)) \( \iff U_z U_y U_k J_i = U_z U_y U_{U_{J_i} J_i} \subset K_i \) (by (0.1)) \( \iff U_z U_y k \in K_i \). This shows \( V \)-invariance implies \( U \)-invariance. Further, since \( \{xy(U_y a)\} \in K_i \) by (2.2) it shows \( U_z U_y (U_y a) \in K_i \), i.e., \( U_y U_k J_i \subset K_i \), hence by the above \( \{xyk\} \in K_i \), establishing \( V \)-invariance.

Since any maximal ideal in a unital algebra is strongly semiprime (the quotient is simple with unit, therefore contains no nil ideals, therefore contains no trivial elements), we have the important

**Corollary 2.12.** Any maximal ideal \( M_1 \) in \( J_1 \) is invariant.

This immediately shows that \( J_i \) is simple if \( J \) is. We return to this in §4, where we use a flipping argument to deduce that \( J_0 \) is simple as well. In the remainder of this section we undertake a more delicate analysis to show \( K_i \) is invariant if it is merely semiprime in \( J_i \) (in the sense that \( \tilde{J}_i \) is semiprime), or even if it has no trivial ideals \( U_{\tilde{J}_i} J_i = 0 \) (this is equivalent to \( U_{\tilde{J}_i} J_i \subset K_i \implies B_i \subset K_i \) for \( B_i \subset J_i \)).

**Lemma 2.13.** If \( K_i \) is an ideal in \( J_i \) then \( H(K_i) = K_i + V_{J_{1/2}} \subset J_i \) is again an ideal in \( J_i \). In fact, for any particular \( x, y \in J_{1/2} \) the subspaces
\[ K_i^{(1)} = K_i + U_x U_y U_{K_i} \hat{J}_i \]
\[ K_i^{(2)} = K_i + V_{x,y} K_i + U_x U_y U_{K_i} \hat{J}_i \]
\[ K_i^{(3)} = K_i + V_{x,y} K_i + U_x U_y K_i \]

are ideals in \( J_i \) with
\[ K_i \subset K_i^{(1)} \subset K_i^{(2)} \subset K_i^{(3)} \]

and with each trivial modulo the preceding:
\[ U_{K_i^{(3)}} \hat{J}_i \subset K_i^{(2)} , \quad U_{K_i^{(2)}} \hat{J}_i \subset K_i^{(1)} , \quad U_{K_i^{(1)}} \hat{J}_i \subset K_i . \]

**Proof.** Since \( H(K_i) \) is just the sum of all \( K_i^{(3)} \) for all possible \( x,y \in J_{i/2} \), it suffices to prove the \( K_i^{(j)} \) are ideals.

We first show each \( K_i^{(j)} \) is an outer ideal: \( U_{\hat{J}_i} K_i^{(j)} \subset K_i^{(j)} \). For \( a \in \hat{J}_i \) and \( k \in L_i \triangleleft J_i \) we have

\[ U_a V_{x,y} k = \{ U_{[yza],n} - V_{y,z} U_a \} k \]  
(by (0.3))

\[ = U_{[yza],n} k - V_{y,z} U_a k - V_{x,y} U_a k \]  
(by (0.6))

\[ \in U_{\hat{J}_i} L_i - V_{\hat{J}_i} U_{\hat{J}_i} L_i - V_{x,y} U_a L_i \subset L_i + V_{x,y} L_i \]

\[ U_a U_x U_y k = \{ U_{(azb)}, a = \} - V_{y,z} U_a V_{x,y} + U_{a, U(y) U(z)} k \]  
(by (0.2))

\[ = \{ U_{(azb)} + (U_a U_y - U_{azb} + V_{x,y} V_{y,z} - V_{U(y) U(z)}) U_a \]  
(by (0.2), (0.6))

\[ - (V_{x,y} - V_{y,z}) U_a V_{x,y} + U_{a, U(y) U(z)} k \]

\[ = \{ U_{(azb)} + U_a U_y U_a - (V_{x,y} + V_{U(y) U(z)}) U_a + V_{x,y} U_{a, [yza]} \]  
(by (0.3))

\[ - V_{x,y} U_a V_{x,y} + U_{a, U(y) U(z)} k \]

\[ \in U_{\hat{J}_i} L_i + U_{x,y} L_i - U_{\hat{J}_i} V_{x,y} L_i + V_{x,y} U_{\hat{J}_i} L_i \]  
(by (2.3))

\[ \subset L_i + U_{x,y} L_i - L_i + V_{x,y} U_{\hat{J}_i} L_i = U_{x,y} U_{\hat{J}_i} L_i \]

(\( \text{using our previous calculation to move } V_{\hat{J}_i}, U_{\hat{J}_i} \text{ past } V_{x,y} \)).

Taking \( L_i = K_i \) shows \( K_i^{(j)} \) is outer, while \( L_i = U_{K_i} \hat{J}_i \subset K_i \) shows \( K_i^{(j)} \) is outer (using (2.2) for \( K_i^{(1)} \)).

Now we show the \( K_i^{(j)} \) are inner, in fact the stronger assertion that each is trivial modulo its predecessor: \( U_{K_i^{(j)}} \hat{J}_i \subset K_i^{(j-1)} \subset K_i^{(j)} \). For \( j = 1 \) we have \( K_i^{(1)} \equiv U_x U_y U_{K_i} \hat{J}_i \) modulo the ideal \( K_i^{(0)} = K_i \), so

\[ U_{K_i^{(1)}} \hat{J}_i = U_{U(y) U(z)} U_{U(K_i)} \hat{J}_i U_x U_y \hat{J}_i \]

\[ \subset U_x U_y U_{U(K_i)} \hat{J}_i J_i \subset K_i \equiv 0 \]  
(by (2.3))

so \( U_{K_i^{(1)}} \hat{J}_i \subset K_i \). In particular, \( K_i^{(1)} \) is inner and thus an ideal. Once \( K_i^{(1)} \) is an ideal we have for \( j = 2 \) that \( K_i^{(2)} \equiv V_{x,y} K_i \) modulo \( K_i^{(1)} \), so

\[ U_{K_i^{(2)}} \hat{J}_i \equiv U_{[xyK_i]} \hat{J}_i \equiv U_x U_y U_{K_i} \hat{J}_i \subset K_i^{(1)} \equiv 0 \]  
(by (2.5))
so $U_{K_t}(\hat{J}_t) \subset K^{(2)}_t$ and $K^{(2)}_t$ too is an ideal. Then we have $K^{(3)}_t = U_x U_y K_t$ modulo the ideal $K^{(2)}_t$, so

$$U_{K^{(3)}_t}(\hat{J}_t) = U_x U_y U_{K^{(3)}_t} U_y U_{K^{(3)}_t} J_t \subset U_x U_y U_{K^{(3)}_t} J_t \subset K^{(4)}_t \equiv 0$$

so $U_{K^{(3)}_t}(\hat{J}_t) \subset K^{(2)}_t$ and $K^{(3)}_t$ is also an ideal trivial modulo its predecessor.

Our calculations show each $U_x U_y K_t$ is an ideal and each $K_t + V_{x,y} K_t$ is an outer ideal; if $1/2 \in \Phi$ outer ideals are ideals, so $U_{J_{1/2}} U_{J_{1/2}} K_t$ and $K_t + V_{J_{1/2}, J_{1/2}} K_t$ are both ideals in this case.

**Remark 2.14.** If invertible elements are dense one can show

$$B(J_{1/2}, J_{1/2}) K_t \triangleleft J_t \quad (B(x, y) = I + V_{x,y} + U_{x,y}) .$$

Indeed, $U_x B(x, U_y) z = B(U_x x, y) U_x z$ shows for invertible $x, y \in J_t$ that

$$U_x B(J_{1/2}, J_{1/2}) K_t = U_{e_0 + x_1} B(J_{1/2}, e_0 + x_1) K_t = U_x B(J_{1/2}, U_y J_{1/2}) K_t = (U_x B(x, y) U_y J_{1/2}) U_x K_t$$

hence if such $x$ are dense $B K_t$ is outer, and it is inner since for the spanning set of $B(x_{1/2}, y_{1/2}) K_t$ we have $U_{B(x_{1/2}, y_{1/2}) K_t} = B(x, y) U_{B(x, y) J_1} \subset B(x, y) U_{B(x, y) K_t}$. It is not known if this holds in general. If $\Phi$ is a field with more than two elements then $B(J_{1/2}, J_{1/2}) K_t$ is just $K_t + V(J_{1/2}, J_{1/2}) K_t + U(J_{1/2}, U(J_{1/2}) K_t$ and thus is certainly an ideal.

Now we can establish invariance of semiprime ideals.

**Theorem 2.15.** Any semiprime ideal $K_t \triangleleft J_t$ is invariant.

**Proof.** Semiprimeness means $J_t/K_t$ contains no trivial ideals. But then $K_t = K^{(2)}_t \subset K^{(1)}_t \subset K^{(2)}_t \subset K^{(3)}_t$ with $K^{(2)}_t/K^{(1)}_t \subset K^{(3)}_t$ and with $x, y \in J_{1/2}$ for any particular $x, y \in J_{1/2}$, and thus $K_t$ is $V$- and $U$-invariant.

**Remark 2.16.** We have established invariance of $K_t$ as long as $\hat{J}_t = J_t/K_t$ contains no ideals $\hat{L}_t$ consisting entirely of trivial elements (i.e., $U_{L_t} \hat{J}_t \subset K_t \Rightarrow L_t \subset K_t$). It is not known whether an algebra without such ideals is necessarily semiprime; this holds whenever $1/2 \in \Phi$ since $\hat{L}_t = 0$ implies $2 U_{\hat{J}_t} J_t = \hat{L}_t \circ (\hat{L}_t \circ J_t) - \hat{L}_t \circ J_t = 0$.

3. The invariant hull. If we have no specific information about a given ideal $K_t \triangleleft J_t$ which allows us to conclude it is invariant, we must enlarge it by applying all possible $V$'s and $U$'s until the result is invariant. The **invariant hull** $\text{Inv}(K_t)$ of the ideal $K_t$ is the smallest invariant ideal containing $K_t$. 

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**PEIRCE IDEALS IN JORDAN ALGEBRAS**

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In (1.9) we saw that $V$-invariance implies $U$-invariance when $1/2 \in \Phi$. More generally,

**Proposition 3.1.** The subalgebra $E(\mathcal{H}, \mathcal{V})$ of $\text{End}(J_\iota)$ generated by the restrictions to $J_\iota$ of $V_{J_{1/2}J_{1/2}}$ and $U_{J_{1/2}U_{J_{1/2}}}$ reduce to $\mathcal{H} + \mathcal{V}$ where $\mathcal{H}$ is the linear span of all operators

$$U_{x_1}U_{y_1} \cdots U_{x_n}U_{y_n}$$
and $\mathcal{V}$ the linear span of all

$$V_{x_1,y_1} \cdots V_{x_n,y_n}$$

where $x_i, y_i$ belong to some spanning set for $J_{1/2}$. Further, $2\mathcal{H} \subset \mathcal{V}$.

**Proof.** The Jordan identities (0.4), (0.5) show that the partially linearized $U$-operators $U_xU_{y,z}$ and $U_{y,z}U_x$ can be replaced by products of $V$-operators: $U_xU_{y,z} \in \mathcal{V}$; $U_{y,z}U_x \in \mathcal{V}$. In particular, for $y = z$ we see as in (1.9)

$$2U_xU_y \in \mathcal{V}.$$ 

These together with the further Jordan identities

$$(0.8) \quad U_xU_yV_{z,w} = U_{\{xyz\}z}U_{w,y} - V_{z,y}U_xU_{w,y} - U_xU_{U(y)z,w} \in \mathcal{V}$$

$$(0.9) \quad V_{w,z}U_yU_x = U_{w,y}U_{\{xyz\}z} - U_{w,y}U_xV_{y,z} - U_{U(y)z,w}U_x \in \mathcal{V}$$

show that any mixed term involving a product of $U$'s with at least one $V$ factors, or 2 times any product of $U$'s can be expressed solely in terms of $V$'s,

$$\mathcal{H} + \mathcal{V} \subset \mathcal{V}, \quad 2\mathcal{H} \subset \mathcal{V}.$$ 

Thus the subalgebra generated by $\mathcal{H}$ and $\mathcal{V}$ reduces to $\mathcal{H} + \mathcal{V}$ with $2U \subset \mathcal{V}$.

Since $V_{x,y}$ is bilinear in $x, y$, if \{u_i\} spans $J_{1/2}$ then the $V_{u_i,u_j}$ span $V_{J_{1/2}J_{1/2}}$, and $U_{J_{1/2}U_{J_{1/2}}}$ is spanned by the $U_{u_i}U_{u_j}$ modulo terms $U_{u_i}U_{u_j}U_{u_k}$, $U_{u_j}U_{u_k}U_{u_i}$, $U_{u_i}U_{u_i}U_{u_k} \in \mathcal{V}$.

**Remark 3.2.** For $x, y \in J_{1/2}$ we have an operator identity on $J_\iota$

$$U_xU_y = U_x^2 = U_{E_\iota(x^2)}; \quad U_xU_y + U_yU_x + U_x^2 = U_{E_\iota(x+y)} + U_{E_\iota(x^2),E_\iota(y^2)}$$

showing $U_{x_1}U_{x_2} \cdots U_{x_{2n}}$ is an alternating function of the variables $x_i \in J_{1/2}$ modulo products with fewer $U$'s and either more $V$'s or more multiplications from $J_\iota$ (which automatically leave any ideal $K_\iota \subset J_\iota$ invariant). Thus $\mathcal{H}$ is spanned modulo $\mathcal{V}$ and $\mathcal{M}(J_\iota)$ by
all \( U_{u_1}U_{u_2} \cdots U_{u_{2n}} \) for \( u_1 < \cdots < u_{2n} \) in some ordered spanning set for \( J_{1/2} \).

**Theorem 3.3.** The invariant hull of a given ideal \( K_i \triangleleft J_i \) is

\[
\text{Inv} (K_i) = \mathcal{U} K_i + \mathcal{V} K_i = \sum_{k=0}^{\infty} V_{J_{1/2}J_{1/2}}^j K_i + \sum_{m=0}^{\infty} U_{J_{1/2}}^{2m} K_i.
\]

If \( 1/2 \in \Phi \) this reduces to \( \sum V_{J_{1/2}J_{1/2}}^j K_i \).

**Proof.** A subspace is \( U \) - and \( V \) -invariant iff it is invariant under the subalgebra generated by all \( U \)'s and \( V \)'s, which by 3.1 is just \( \mathcal{U} + \mathcal{V} \); so \( \mathcal{U} K_i + \mathcal{V} K_i \) is the invariant closure of \( K_i \). To see this remains an ideal in \( J_i \) if \( K_i \) is to begin with, note that this invariant closure can also be represented as \( \text{Inv} (K_i) = \sum_{n=0}^{\infty} H^*(K_i) \) where \( H(L_i) = L_i + V_{J_{1/2}J_{1/2}}^j L_i + U_{J_{1/2}} L_i \), where by Lemma 2.13 each \( H^*(K_i) \) is an ideal and therefore their sum is too.

If \( 1/2 \in \Phi \) we can dispense with the \( U \)'s by 3.1.

**Remark 3.4.** By our comments 3.2, if \( J_{1/2} \) is finitely spanned we need only take a finite number of powers \( U_{J_{1/2}}^{2m} \).

**Remark 3.5.** \( \text{Inv} (K_i) \) is Baer-radical modulo \( K_i \) since it is a union of \( H^*(K_i) \), where \( H^*(K_i) \) is Baer-radical modulo \( H^{*-1}(K_i) \) (being the sum over all \( x, y \in J_{1/2} \) of ideals \( K_i^{(x)} = K_i + V_x, K_i + U_x, K_i \) nilpotent modulo \( K_i \) by (2.13)). Once more this shows that if \( K_i \) is semiprime in \( J_i \) then \( \text{Inv} (K_i) = K_i \) and \( K_i \) is invariant.

We can, if compelled, write down explicitly the ideal generated by a diagonal Peirce ideal.

**Theorem 3.6.** If \( K_i \) is an ideal in a Peirce space \( J_i \) \((i = 1, 0)\) of a Jordan algebra \( J \), then the ideal it generates in \( J \) is

\[
I(K_i) = I_i \oplus I_{1/2} \oplus I_j
\]

\[
I_i = \text{Inv} (I_i) = (\mathcal{V} + \mathcal{U})(K_i) = \sum_{j,k=0}^{\infty} (V_{J_{1/2}J_{1/2}}^j + U_{J_{1/2}}^{2k}) K_i
\]

\[
I_{1/2} = V_{J_{1/2}} \text{Inv} (K_i) = V_{J_{1/2}} \mathcal{V} K_i = V_{J_{1/2}} \left( \sum_{j=0}^{\infty} V_{J_{1/2}J_{1/2}}^j \right) K_i
\]

\[
I_j = U_{J_{1/2}} \text{Inv} (K_i) = (\mathcal{V} + \mathcal{U}) U_{J_{1/2}} K_i = \text{Inv} (U_{J_{1/2}} K_i)
\]

\[
= \sum_{j,k=0}^{\infty} (V_{J_{1/2}J_{1/2}}^j + U_{J_{1/2}}^{2k}) U_{J_{1/2}} K_i
\]

**Proof.** The ideal generated by \( K_i \) coincides with the ideal generated by its invariant hull \( \text{Inv} (K_i) = (\mathcal{V} + \mathcal{U}) K_i \) by (3.3), so by
(1.11) \( I_i = \text{Inv}(K_i), I_{1/2} = V_{1/2} \text{Inv}(K_i), I_j = U_{1/2} \text{Inv}(K_i). \) Note that
\( U_{1/2} \text{Inv}(K_i) = U_{1/2}(\mathcal{Z} + \mathcal{Z})K_i = (\mathcal{Z} + \mathcal{Z})U_{1/2}K_i \) since \( U_{1/2}U_{1/2} = 
U_{1/2}U_{1/2} \) shows \( U_{1/2}\mathcal{Z} = \mathcal{Z}U_{1/2} \) and \( U_{1/2}V_{1/2}J_{1/2} + \frac{V_{1/2}J_{1/2}U_{1/2}}{U_{1/2}} \subset \)
\( U_{1/2}J_{1/2}J_{1/2}J_{1/2} \subset U_{1/2} \) by (0.3) shows \( U_{1/2}\mathcal{Z} = \mathcal{Z}U_{1/2}. \) Note further that
\( V_{1/2}\mathcal{Z} \subset V_{1/2} \mathcal{Z}, V_{1/2} \text{Inv}(K_i) = V_{1/2} \mathcal{Z}. \)

because
\[
V_{1/2}\mathcal{Z} \subset V_{1/2} \mathcal{Z}, V_{1/2} \text{Inv}(K_i) = V_{1/2} \mathcal{Z}.
\]

follows from the following obscure Jordan identity:

\[
V_zU_yU_z = V_{U(z)U(y)z} - V_zV_{U(y)z, z} + V_{U(z)U(y), y} - V_zV_y, V_y, z
- V_{U(z,y, z,y)} + V_zV_{y, U(y)z} + V_{(z,y), y} V_y, z
\]

(or else substitute 1 in (0.5), \( V_yV_y = V_zU_y + U_{U(y)} \) to see \( V_{1/2}U_{1/2} \subset 
V_{1/2}J_{1/2}J_{1/2} + \frac{V_{1/2}J_{1/2}J_{1/2}}{J_{1/2}} \subset V_{1/2} \mathcal{Z}, \) \( \mathcal{Z} \mathcal{Z} \subset \mathcal{Z} \mathcal{Z}. \))

**Example 3.7.** The largest invariant ideal contained in \( K_i \triangleleft J_i \)

is the invariant kernel

\[
\text{Inv ker}(K_i) = \{ z \in K_i | E(\mathcal{Z}, \mathcal{Z})z \subset K_i \}
= \{ z \in K_i | V_z^{n, m} z \in K_i \text{ for all } n, m \}.
\]

**Proof.** Certainly if \( z \) belongs to an invariant ideal \( I_i \triangleleft K_i \) so
do all \( V^nz \) and \( U_{m}z, \) so \( z \) belongs to \( \text{Inv ker}(K_i) \subset \mathcal{Z}. \) Conversely, \( \mathcal{Z} \)
is clearly a linear subspace which is invariant, \( E(\mathcal{Z}, \mathcal{Z}) \mathcal{Z} \subset \mathcal{Z}. \)
It remains to show \( \mathcal{Z} \) is an ideal.

\( \mathcal{Z} \) is outer: the identities (0.3), (0.2) show

\[
VU_{\mathcal{Z}} \subset U_{\mathcal{Z}} + U_{\mathcal{Z}}V \subset U_{\mathcal{Z}}E(\mathcal{Z}, \mathcal{Z}), U^{2}U_{\mathcal{Z}} \subset U_{\mathcal{Z}}X^{2} + U_{\mathcal{Z}}
+ VU_{\mathcal{Z}} \subset U_{\mathcal{Z}}X + U_{\mathcal{Z}} + U_{\mathcal{Z}}E(\mathcal{Z}, \mathcal{Z})V \subset U_{\mathcal{Z}}E(\mathcal{Z}, \mathcal{Z}),
\]

and hence by induction \( E(\mathcal{Z}, \mathcal{Z})(U_{\mathcal{Z}}Z) \subset U_{\mathcal{Z}}(E(\mathcal{Z}, \mathcal{Z})Z) \subset U_{\mathcal{Z}}K_i \subset K_i. \) Therefore
\( U_{\mathcal{Z}}Z \subset Z. \)

\( \mathcal{Z} \) is inner: the identities (0.3), (0.2) show \( VU_{\mathcal{Z}}\mathcal{Z} \subset U_{\mathcal{Z}}V \mathcal{Z} + 
U_{U(\mathcal{Z})} \mathcal{Z} \mathcal{Z} \subset U_{\mathcal{Z}} \mathcal{Z} \mathcal{Z} \) (since \( \mathcal{Z} \) is \( V \)-invariant), \( U^{2}(U_{\mathcal{Z}} \mathcal{Z}) \subset \{ U_{\mathcal{Z}}, X^{2} + U_{\mathcal{Z}} + 
VU_{\mathcal{Z}} \mathcal{Z} \mathcal{Z} \} \mathcal{Z} \) (since \( \mathcal{Z} \) is \( U \)-invariant) \( \subset U_{\mathcal{Z}} \mathcal{Z} \), hence by induction \( E(\mathcal{Z}, \mathcal{Z})(U_{\mathcal{Z}} \mathcal{Z}) \subset U_{\mathcal{Z}} \mathcal{Z} \subset U_{\mathcal{Z}} \mathcal{Z} \subset K_i \) and \( U_{\mathcal{Z}} \mathcal{Z} \subset Z_i. \)

**Example 3.8.** We give a straightforward example of Jordan
algebra having noninvariant Peirce ideals. Let \( D \) be an associative
algebra with involution *, and let $D'$ be an ample subspace ($D' \subset H(D, *)$) is symmetric, contains 1, and has $xD'x^* \subset D'$ for all $x \in D$: if $1/2 \in \Phi$ then $D' = H(D, *)$. Then the algebra $J = H(D_n, D')$ of hermitian $nxn$ matrices over $D$ with diagonal entries in $D'$ forms a Jordan algebra with idempotent $e = e_{11}$. Here a subspace $K_i = K'[11]$ of the Peirce space $J_i = D'[11]$ is an ideal iff $K'$ is a Jordan ideal in $D'$, 

(i) (outer ideal) $x'k'x' \in K'$ for all $x' \in D'$, $k' \in K'$

(ii) (inner ideal) $k'x^*k' \in K'$. 

On the other hand, such a $K_1$ is $V$-invariant iff $K$ is closed under traces,

(iii) (V-invariant) $t(DK') \subset K'$: $xk' + k'x^* \in K'$ for $x \in D$, $k' \in K'$

and $U$-invariant iff it is closed under norms,

(iv) (U-invariant) $xk'x^* \in K'$ for all $x \in D$, $k' \in K'$.

These follow from the general rules $V(a[1j], b^*[1j])e[11] = t(abc)[11]$ and

$$U(a[1j])U(b^*[1j])e[11] = abcb^*a^*[11]$$

and $U(1[a\ell], d[1k])U(b^*[1j], f^*[1k])e[11] = (abcf^*d^* + dfc^*a^*)[11]$. In this case $U$-invariance implies $V$-invariance (and conversely if $1/2 \in \Phi$), and the invariant hull of $K_1$ is

$$\text{Inv} (K_1) = K_1 + U_{1/2}U_{1/2}K_1 = \{ \sum xK'x^* \}[11].$$

For example, if we take $D = M_2(\Phi)$ a split quaternion algebra over a ring $\Phi$ and $D' = \Phi 1$, then $K'$ is an ideal of $D'$ iff (i) $\Phi^2K' \subset K'$, (ii) $\Phi K'^2 \subset K'$, and $K'$ is $V$-invariant iff (iii) $t(D)K' = \Phi K' \subset K'$, and $K'$ is $U$-invariant iff (iv) $n(D)K' = \Phi K' \subset K'$. If $1/2 \in \Phi$ or $\Phi$ is a field all ideals $K'$ of $D'$ are invariant, but if $\Phi = \mathbb{Z}[x]$, $K' = \mathbb{Z}x^2 + x^4\mathbb{Z}[x] + 2\mathbb{Z}[x]$ then one easily verifies that $K'$ is a Jordan ideal in $\mathbb{Z}[x]$ which is not an associative ideal (and hence not invariant).

In this example we obtained the invariant hull from a single application of $U_{1/2}U_{1/2}$ because the coordinates of $J_{1/2} = \sum D[1j]$ are closed under multiplication. To construct examples where the invariant hull requires all $V^a_{1/2}J_{1/2}$ and $U^m_{1/2}$ we take subalgebras where the coordinates of $J_{1/2}$ are not closed. From now on our examples will sit inside $H(D_2, D')$.

**Example 3.9.** (All $V$'s are necessary.) Let $D = \Lambda(V) \otimes \Phi[\epsilon]$ be the ring of dual numbers ($\epsilon^2 = 0$) over the exterior algebra $\Lambda(V)$ on an infinite-dimensional vector space $V$ over a field $\Phi$ of characteristic $\neq 2$, with canonical reversal involution fixing $V$. (Thus the symmetric elements are spanned by the elements of $\Lambda^n(V)$ for $n \equiv 0$ or $1 \mod 4$.)
Then the set $H(D_2)$ of all $2 \times 2$ matrices with entries in the associative coordinate ring $D$ forms a Jordan algebra. We take $ar{J}$ to be the subalgebra

$$
\bar{J} = \varepsilon H(D_2) + (V \wedge V)[12] = \varepsilon H(D)[11] + \{V \wedge V + \varepsilon D\}[12] + \varepsilon H(D)[22]
$$

and $J = \bar{J} + \Phi e[11]$ the subalgebra obtained by tacking on $e = 1[11]$. Thus $H(D_2) \supset J \supset \bar{J} \supset \varepsilon H(D_2)$.

Since $\bar{J}_1 = \varepsilon J_1$ is trivial $(U_{\bar{J}_1} \bar{J}_1 = \bar{J}_1^2 = 0$ since $\varepsilon^2 = 0$), any subspace $K_1 \subset \bar{J}_1$ is an ideal in $J_1$. However, only certain subspaces will be invariant:

$$
\begin{align*}
V_{u_1 \wedge u_2}[12, u_3 \wedge u_4]\varepsilon[11] &= 2u_1 \wedge u_2 \wedge u_3 \wedge u_4 \wedge k[11] \\
U_{u_1 \wedge u_2}[12, u_3 \wedge u_4]\varepsilon[11] &= -2u_1 \wedge u_2 \wedge u_3 \wedge u_4 \wedge k[11] \\
U_{u_1 \wedge v}[12]\varepsilon[11] &= 0 .
\end{align*}
$$

Thus a subspace $K_1 = \varepsilon K[11]$ will be invariant only if the subspace $K$ of $H(D)$ is closed under multiplication by the degree 4 part of the exterior algebra (generated by all $u_1 \wedge u_2 \wedge u_3 \wedge u_4$ for $u_i \in V$). If $K = \Phi v_0$ then $V_{u_1 \wedge v}[12, w_1 \wedge t_1][12] \cdots V_{v_n \wedge v_n}[12, w_n \wedge t_n][12]K_1 = \varepsilon \Phi u_1 \wedge v_1 \wedge w_1 \wedge t_1 \wedge \cdots \wedge u_n \wedge v_n \wedge w_n \wedge t_n \wedge v_0[11] \subset \varepsilon \Lambda^{n+1}(V)[11]$, from which it is clear that arbitrarily high powers of $V_{j/2, j/2}^1$ are needed to generate the arbitrarily long elements $\varepsilon u_1 \wedge u_2 \wedge \cdots \wedge u_4 \wedge v_0[11]$ in Inv $K_1$.

**Example 3.10.** (All $U$'s are necessary.) Again we take $H(D_2)$ for $D$ an associative algebra with involution, but this time $D$ is a “square root” of an exterior algebra $\Lambda(V)$ on an infinite-dimensional vector space $V$ over a field $\Lambda$ of characteristic 2. If $V$ has basis $\{v_1, v_2, \cdots\}$ we let $D = \Phi[x_1, x_2, \cdots]$ be a commutative polynomial ring (with identity involution) where $x_1^2 = v_1, v_2^2 = 0$. Note

$$
D^2 \subset \Phi[x_1^2, x_2^2, \cdots] = \Phi[v_1, v_2, \cdots] \cong \Lambda(V), (D^2)^2 = 0 .
$$

Let

$$
\bar{J} = H(D_2^2) + \{\sum \Phi x_i\}[12] = D^4[11] + \{\sum \Phi x_i + D^2 x_i\}[12] + D^4[22]
$$

and $J = \bar{J} + \Phi e[11]$. Again $\bar{J}_1 = D^2[11]$ is trivial since the characteristic is 2 and $(D^2)^2 = 0$, so any subspace $K_1 \subset \bar{J}_1$ is an ideal in $J_1$. Here $V$-invariance is automatic,

$$
$$

$U$-invariance of $K_1 = K[11]$ means closure of $K$ under even products of $v_i$'s, since

$$
$$
and if \( a = \sum \alpha_i x_i + \sum \beta_i x_i \) then \( \alpha^2 = \sum \alpha_i^2 v_i \). From this it is clear that arbitrarily large powers \( U_{x_1} U_{x_2} \cdots U_{x_n} v_0 \) are needed to obtain the invariant hull of \( K_t = \Phi v_0 \).

4. Simplicity of \( J_1 \) and \( J_o \). We use our constructions to show that Peirce subalgebras \( J_1 \) and \( J_o \) inherit simplicity from \( J \). The basic idea of the proof is easily stated. Since a simple algebra \( J \) contains no proper ideals \( K \), there are no proper projections in the Peirce subalgebras \( J_1 \) and \( J_o \), consequently by 1.11 there are no proper invariant ideals in \( J_1 \) or \( J_o \). Since \( J \) has unit element \( e \) there exist (by the usual Zornification) maximal ideals \( K \), necessarily strongly semiprime in \( J \) by (2.12), so any maximal \( K \) is invariant and therefore zero; but \( K = 0 \) maximal means \( J \) is simple.

For the nonunital algebra \( J_o \) we cannot use this argument, but we can make use of the simplicity of \( J_1 \): any ideal \( K \) in \( J_o \) is flipped into an ideal \( K_1 = U_{J_1} K_0 \) in \( J_1 \). If this image is zero the same holds for the invariant hull of \( K_o \), forcing this hull to be zero and \( K_0 = 0 \). If on the other hand the image is all of \( J_1 \) then the same holds for \( K_1 \); but the double flip of \( K_1 \) is contained in \( K_0 \), which forces \( K_0 = J_0 \). This means \( J_0 \) is simple.

Now to fill in the details.

Main Theorem 4.1. If \( e \) is an idempotent in a simple Jordan algebra \( J \) then the Peirce subalgebras \( J_t(e) \) and \( J_o(e) \) are also simple.

Proof. The result is vacuous if \( e = 0 \) (\( J_t = 0 \), \( J_o = J \)), so we may assume \( e \neq 0 \). Then \( J \) is not nil, \( \text{Nil}(J) \neq J \), so by simplicity \( \text{Nil}(J) = 0 \) and in particular \( J \) contains no trivial elements. Each \( J_t \) inherits this strong semiprimeness since an element trivial in \( J_t \) is trivial in all of \( J(U_{x_t} J = U_{x_j} J) \), therefore \( J_t \) is not trivial and will be simple if it has no proper ideals. We know \( J_t \) contains no proper invariant ideals, and we must deduce it has no proper ideals whatsoever.

We have already seen this is true for \( J_t \) thanks to its unit \( e \), so consider \( J_o \). Suppose we have an ideal \( K \) in \( J_o \). By the Flipping Lemma 1.10 the image \( K_1 = U_{J_1} K_0 \) is an ideal in \( J_1 \), so by what we have just shown it must either be \( J_1 \) or \( 0 \).

First consider the case \( K_1 = U_{J_1} K_0 = 0 \). Then \( K_0 \subset \text{Ker} U_{J_1} \), which by the Kernel Lemma 2.10 is an invariant ideal of \( J_o \). Such an invariant ideal can only be \( J_o \) or \( 0 \), and it is not all of \( J_o \) since \( U_{J_1} J_0 = 0 \) by (1.17), so \( U_{J_1} \) must be \( 0 \) and \( K_0 = 0 \) to begin with. So far we have shown that \( K_1 = 0 \) implies \( K_o = 0 \).

Now consider the case \( K_1 = U_{J_1} K_0 = J_1 \). Since \( J_0 \) is strongly semiprime it has no nilpotent ideals, so \( K_o = 0 \Rightarrow K_o' = U_k K_o = 0 \Rightarrow K_o'' = U_k K_o = 0 \). But by the previous case \( K_o'' = 0 \) implies \( K_o''' = \)
$U_{J^{1/2} K''}$ is nonzero and therefore all of $J_1$. Thus by (1.17) $J_0 = U_{J^{1/2} J_1} = U_{J^{1/2}(U_{J^{1/2} K''})}$. On the other hand, $U_{J^{1/2} U_{J^{1/2} K''}} = U_{J^{1/2} U_{J^{1/2}(U_{U_{(K_0)} J_0})} J_0} \subset K_0$ by (2.3), so we have $K_0 = J_0$. This shows $K_1 = J_1$ implies $K_0 = J_0$. Thus $K_0 \vartriangleleft J_0$ implies $K_0 = 0$ or $K_0 = J_0$, and $J_0$ too is simple.

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