REPRESENTATIONS OF WITT GROUPS

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This paper gives a tensor product theorem for the coordinate rings of the finite-dimensional Witt groups. This theorem leads to a demonstration of the equivalence of the representation theory of the Witt groups with that of certain truncated polynomial rings.

Introduction. The Steinberg tensor product theorem [1, Ch. A, §7] for a simply connected, semisimple algebraic group $G$ in characteristic $p$ displays irreducible $G$-modules as tensor products of Frobenius powers of infinitesimally irreducible $G$-modules (modules which are irreducible for the kernel $G^i$ of the Frobenius morphism of $G$).

A goal of modular representation theory is the expression of the coordinate ring of $G$ in terms of tensor products of Frobenius powers of $G$-modules which are suitably elementary for $G^i$. In this paper, we give a tensor product theorem for the finite-dimensional Witt groups. We produce a subcoalgebra $C$ of the coordinate ring $A$ of the $m$-dimensional Witt group $W_m$ which is isomorphic to the coordinate ring of the kernel $W_m^i$ of the Frobenius morphism. $A$ is the inductive limit of tensor products of Frobenius powers of $C$ [§3, Theorem].

One can see some things about the representations of $W_m$. First, every finite-dimensional representation of $W_m$ extends to a representation of $W_m$ on the same representation space [§5]. Second, a representation of $W_m$ on a finite-dimensional vector space $V$ is determined by a family $(f_1, \ldots, f_n)$ of commuting endomorphisms of $V$ such that $f_i^{p^m} = 0$. In other words, the representations of $W_m$ on $V$ may be studied via the representations of the algebras $k[x_1, \ldots, x_n]/(x_1^{p^m}, \ldots, x_n^{p^m})_n$ on $V$ [Theorem, §4]. In particular, the representations of $W_m$ which correspond to the representations of $k[x_1]/(x_1^{p^m})$ give canonical extensions for the representations of $W_m^i$.

This linear formulation of the representation theory of $W_m$ leaves one with the apparently difficult problem of determining the representation theory of $k[x_1, \ldots, x_n]/(x_1^{p^m}, \ldots, x_n^{p^m})$.

For the definition of the Witt groups, see [2, Ch. 5, §1].

NOTATION. Let $A$ denote the coordinate ring of the $m$-dimensional Witt group $W_m$, as a reduced, connected group scheme over the prime field $k = F_p$. For any subcoalgebra $C$ of $A$ which contains $k$, let $C^{(p^i)}$ be the image of $C$ under the $i$th-power of the Frobenius
morphism of $A$. We may form the inductive family of coalgebras 
$\{C \oplus C(p) \oplus \cdots \oplus C(p^n)\}_{n=0}^{\infty}$, where $C \oplus \cdots \oplus C(p^n) \hookrightarrow C \oplus \cdots \oplus C(p^n) \otimes C(p^{n+1})$ is the canonical morphism onto $C \oplus \cdots \oplus C(p^n) \otimes k$. Let 
$\lim_{n \to \infty} C \oplus C(p) \oplus \cdots \oplus C(p^n)$ be the coalgebra inductive limit of the family.

Let $\Pi: A \to A/M(p)A$ be the quotient morphism, where $M(p)$ is the image of the augmentation ideal $M$ under the Frobenius morphism. We show in §3 that there is a coalgebra splitting $s: A/M(p)A \to A$ of $\Pi$ such that $A$, as a coalgebra, is isomorphic to $\lim_{n \to \infty} C \oplus C(p) \oplus \cdots \oplus C(p^n)$ where $C = \text{image } s$.

0. We require some facts from [3, Def. 6] of K. Newman. Let $W_{m+1}$ be the $(m + 1)$-dimensional Witt group over $k = F_p$, with coordinate ring $A_{m+1}$. As an algebra, $A_{m+1}$ is the polynomial ring $k[X_1, X_p, X_p^2, \ldots, X_p^{m}]$ on $(m + 1)$-variables. Grade $A_{m+1}$ by letting $X_p^i$ have degree $p^i$. The coproduct $\Delta$ of $A_{m+1}$ is the following: $\Delta X_p^i = \sum_{o \leq j \leq i} Q_j \otimes Q_{p^i-j}$, where $Q_j$ is a homogeneous (relative to the grading) polynomial of degree $j$. In particular, $Q_0 = 1$, $Q_p^i = X_p^i$ and $\{Q_j\}_{j=0}^m$ is a sequence of divided powers.

Since degree $Q_j = j$, $Q_j$ lies in $k[X_1, X_p, \ldots, X_p^{m-1}]$ for $j < p^m$. The coordinate ring $A$ of $W_m$ may be identified with the sub-Hopf algebra $k[X_1, X_p, \ldots, X_p^{m-1}]$ of $A_{m+1}$.

1. The coalgebra splitting of $\Pi$. $M = (X_1, X_p, \ldots, X_p^{m-1})$ is the augmentation ideal of $A$. Let $C$ be the $k$-span of $\{Q_j\}_{j=0}^{m-1}$. $C$ is an irreducible coalgebra of dimension $p^m$, with $k \cdot X_1$ as its space of primitive elements. Since the coalgebra map $f: C \hookrightarrow A \xrightarrow{\Pi} A/M(p)A$ has an injective restriction to $k \cdot X_1$, $f$ is injective [5, Lemma 11.0.1]. Since $(A/M(p)A)^*$ is the restricted universal enveloping algebra of $(M/M^2)^*$ [3, 13.2.3], $\dim_k (A/M(p)A)^* = p^{\dim_k (M/M^2)^*} = p^m$. Therefore, $\dim_k (A/M(p)A) = p^m$ and $f$ is an isomorphism. $s = f^{-1}$ is the coalgebra splitting of $\Pi$ that we use.

2. The value of $\Pi$ at $Q_j$. Let $0 \leq j < p^m$. Write $j = \sum_{i=0}^{m-1} a_i p^i$ where $0 \leq a_i < p$.

**Lemma.** $\Pi(Q_j)$ is a nonzero scalar multiple of $\Pi(X_1^{a_0}X_p^{a_1} \cdots X_p^{a_{m-1}})$.

**Proof.** $Q_j$ is a linear combination of elements $X_1^{b_0}X_p^{b_1} \cdots X_p^{b_{m-1}}$ where $\sum b_i p^i = j$ by §0. If $\{b_i\} \neq \{a_i\}$, then $b_i \geq p$ for some $i$, and $\Pi(X_1^{b_0}X_p^{b_1} \cdots X_p^{b_{m-1}}) = 0$. Therefore, $\Pi(Q_j) \in k \cdot \Pi(X_1^{a_0}X_p^{a_1} \cdots X_p^{a_{m-1}})$, where the coefficient of $\Pi(X_1^{a_0}X_p^{a_1} \cdots X_p^{a_{m-1}})$ is nonzero since the map
3. The coalgebra structure of the coordinate ring. Give the set of monomials in $A$ the reverse lexicographic total order: $X_1^{a_0}X_2^{a_1} \cdots X_p^{a_{m-1}} > X_1^{b_0}X_2^{b_1} \cdots X_p^{b_{m-1}}$ if there is an index $k$ such that $a_k > b_k$ and $a_i = b_i$ for $i > k$.

Let $(a_i)_{i=0}^{m-1}$ be a sequence where $0 \leq a_i < p$, and let $(b_i)_{i=0}^{m-1}$ be a different sequence, where $0 \leq b_i$.

**Lemma.** If $\sum_{i=0}^{m-1} a_i p^i = \sum_{i=0}^{m-1} b_i p^i$, then $X_1^{a_0}X_2^{a_1} \cdots X_p^{a_{m-1}} > X_1^{b_0}X_2^{b_1} \cdots X_p^{b_{m-1}}$.

**Proof.** Let $k$ be the maximal index such that $a_k \neq b_k$. If $b_k > a_k$, then $\sum_{i=0}^{m-1} b_i p^i > \sum_{i=0}^{m-1} a_i p^i$ since $a_i < p$. Therefore, we must have $a_k > b_k$ and $X_1^{a_0} \cdots X_p^{a_{m-1}} > X_1^{b_0} \cdots X_p^{b_{m-1}}$.

Let $C$ be the coalgebra formed in §1.

**Theorem.** The map $\lim_{n} C \otimes C^{(p)} \otimes \cdots \otimes C^{(p^n)} \rightarrow A$, induced by multiplication; $C \otimes C^{(p)} \otimes \cdots \otimes C^{(p^n)} \rightarrow A$, is an isomorphism of coalgebras.

**Proof.** Denote the map by $g$.

Surjectivity of $g$. Suppose that monomials $X_1^{b_0}X_2^{b_1} \cdots X_p^{b_{m-1}}$ less than $X_1^{a_0}X_2^{a_1} \cdots X_p^{a_{m-1}}$ in the ordering lie in the image of $g$. We show that $X_1^{a_0}X_2^{a_1} \cdots X_p^{a_{m-1}}$ also lies there.

Write $a_i = \sum_j a_{ij} p^j$, where $0 \leq a_{ij} < p$. Let $t_k = \sum_{i=0}^{m-1} a_{ik} p^i$. By the lemmas of §2 and §3, 

$$Q_{t_k} = U_k \cdot X_1^{a_{0k}}X_2^{a_{1k}} \cdots X_p^{a_{m-1,k}} + Y_k,$$

where $Y_k$ is a linear combination of monomials of degree $t_k$ and less than $X_1^{a_{0k}} \cdots X_p^{a_{m-1,k}}$ in the ordering, and where $U_k$ is a nonzero scalar. Therefore,

$$\prod_{k=0}^{m-1} Q_{t_k} = \prod_{k=0}^{m-1} U_k \cdot X_1^{a_{0k}}X_2^{a_{1k}} \cdots X_p^{a_{m-1,k}} + Y,$$

where $Y$ is a linear combination of monomials which are less than $X_1^{a_0}X_2^{a_1} \cdots X_p^{a_{m-1}}$. Since $\prod_{k=0}^{m-1} Q_{t_k}$ and $Y$ lie in the image of $g$, so does $X_1^{a_0}X_2^{a_1} \cdots X_p^{a_{m-1}}$.

Injectivity of $g$. Since $g$ is surjective, so is $\Pi \circ g : \lim_{n} C \otimes C^{(p)} \otimes \cdots \otimes C^{(p^n)} \rightarrow A \otimes A/\{M^{(p^t)}A \}$ for any $t$; at the same time, $C^{(p^j)} \hookrightarrow A \rightarrow A/M^{(p^j)}A$ has image $= k$ if $j \geq t$. Therefore, $C \otimes C^{(p)} \otimes \cdots \otimes C^{(p^{t-1})} \rightarrow A \otimes A/M^{(p^t)}A$ is surjective. Since $\dim_k (A/M^{(p^t)}A) = p^{mt}$
by [4] or by inspection, and \( \dim_k (C \otimes C^{(p)} \otimes \cdots \otimes C^{(p_{t-1})}) = p^{m t} \),
\( \Pi \circ \text{mult.} \) is an isomorphism of coalgebras. In particular, \( C \otimes C^{(p)} \otimes \cdots \otimes C^{(p_{t-1})} \xrightarrow{\text{mult.}} A \) is injective. Hence, \( g \) is injective.

4. Representation theory of \( W_m \). The dual algebra \( U = (A/M^{(p)} A)^* \) is the restricted universal enveloping algebra of the abelian \( p \)-Lie algebra \( L = (M/M^p)^* \) [5, 13.2.3].

**Lemma.** There is a \( k \)-basis \( f_0, \ldots, f_{m-1} \) for \( L \), where \( f_i = f_{i+1} \) for \( i < m - 1 \) and \( f_{m-1}^p = 0 \).

**Proof.** Define \( f_j \) on the \( k \)-basis \( X_1, X_2, \ldots, X_p \) for \( M/M^2 \) by \( f_j(X_{p^i}) = \delta_{ij} \). We have the following to complete the proof.

1. If \( i \neq j + 1 \), then \( f_j(X_{p^i}) = (\text{mult} f_j)(d^{p-1}X_{p^i}) = 0 \), since \( d^{p-1}X_{p^i} \) is homogeneous of degree \( p^i \) under the grading of \( \text{mult} A \). \( A \) induced from the grading of \( A \), while \( \text{mult} f_j \) can be nonzero only at monomials in \( \text{mult} A \) of degree \( p^{j+1} \).

2. One may check that \( f_j(X_{p^{j+1}}) = 1 \).

To proof is complete.

By this lemma, the algebra map from the polynomial ring \( k[f] \) to \( U \) mapping \( f \) to \( f_i \) induces an isomorphism of \( k \)-algebras \( k[f]/(f^{p^m}) \cong U \).

Denote by \( R_n \) the set of isomorphism classes of finite-dimensional representations of \( W_m \) whose coefficients lie in \( C \otimes C^{(p)} \otimes \cdots \otimes C^{(p^{n+1})} \hookrightarrow A \). The canonical map \( C \otimes C^{(p)} \otimes \cdots \otimes C^{(p^n)} \hookrightarrow C \otimes C^{(p)} \otimes \cdots \otimes C^{(p^n)} \otimes C^{(p^{n+1})} \) induces \( R_n \hookrightarrow R_{n+1} \). Then \( R = \bigcup_n R_n \) is the set of isomorphism classes of finite-dimensional representations of \( W_m \).

Let \( B \) denote the quotient of the polynomial ring \( F_p[X_0, \ldots, X_n, \ldots] \) on generators \( \{X_i\}_{i=0}^n \) by the ideal \( (X_0^{p^n}, \ldots, X_n^{p^n}, \ldots) \). Denote by \( \hat{B} \) the set of isomorphism classes among those finite-dimensional representations of \( B \) in which all but a finite number of the \( X_i \) act as the zero endomorphism. Denote by \( \hat{B}_n \) the set of isomorphism classes of finite-dimensional representations of \( B \) in which all but a finite number of the \( X_i \) act as the zero endomorphism. Denote by \( \hat{B}_n \) the set of isomorphism classes of finite-dimensional representations of \( k[X_0, \ldots, X_n]/(X_0^{p^n}, \ldots, X_n^{p^n}) \). The map \( k[X_0, \ldots, X_n]/(X_0^{p^n}, \ldots, X_n^{p^n}) \to k[X_0, \ldots, X_n]/(X_0^{p^n}, \ldots, X_n^{p^n}) \), \( X_i \mapsto X_i \) for \( i \leq n \) and \( X_i \mapsto 0 \) for \( i > n \), induces \( \hat{B}_n \hookrightarrow \hat{B} \), and \( \hat{B} = \bigcup_n \hat{B}_n \).

**Theorem.** There is a canonical bijection \( R \to \hat{B} \), under which \( R_n \) and \( \hat{B}_n \) correspond.

**Proof.** Since \( C \cong A/M^{(p)} A \) as coalgebras, \( C^* \cong U \) as algebras. Since \( A \) is reduced, the Frobenius morphism on \( A \) is injective, and \( C \cong C^{(p)} \). Therefore,
The first isomorphism is induced by the maps \( I^7 \rightarrow (C(g, C(3, g) \cdot g^* C(p^n)) \) which are dual to the maps 
\[
\beta_j \text{ is the counit of } C^{(p)}; \text{ the second isomorphism is induced by } X_t \mapsto 1_0 \otimes \cdots \otimes 1_{t-1} \otimes f_1 \otimes 1_{t+1} \otimes \cdots \otimes 1_n, \text{ where } 1_j \text{ is the identity of } U_j. \text{ Here } u_j \text{ is the } j\text{th copy of } u \text{ in } \otimes^{n+1} u. \text{ Moreover,}
\]
\[
(2) \text{ under dualization, the canonical map } C \otimes C^{(p)} \otimes \cdots \otimes C^{(p^n)} \mapsto C \otimes C^{(p)} \otimes \cdots \otimes C^{(p^{n+1})} \text{ yields the map } k[f_0, \cdots, f_{n+1}]/(f_0^p, \cdots, f_{n+1}^p) \rightarrow k[X_0, \cdots, X_n]/(X_0^p, \cdots, X_n^p) \text{ where } X_i \mapsto X_i \text{ for } i \leq n \text{ and } X_{n+1} \mapsto 0.
\]
\[
The isomorphism \((C \otimes C^{(p)} \otimes \cdots \otimes C^{(p^n)})^* \cong k[X_0, \cdots, X_n]/(X_0^p, \cdots, X_n^p)\) of \((1)\) induces a bijection \(R_n \rightarrow \hat{B}_n\) such that \(R_n \rightarrow B_n\) commutes by \((2)\). Therefore, \(R \sim \hat{B}\).

5. Representations of \(W_m\). The coalgebra \(C\) constructed in \S 1 is isomorphic to the coordinate ring \(A/M^{(p)}A\) of \(W_m\) under the mapping \(\pi: A \rightarrow A/M^{(p)}A\) restricted to \(C\). Therefore, the representations of \(W_m\) with coefficients in \(C\) correspond to the representations of \(W_m^i\) via the isomorphism between the coefficient coalgebras \(C\) and \(A/M^{(p)}A\), and very finite-dimensional representation of \(W_m^i\) extends to a representation of \(W_m\) on the same representation space.

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