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REPRESENTATIONS OF WITT GROUPS

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This paper gives a tensor product theorem for the coordinate rings of the finite-dimensional Witt groups. This theorem leads to a demonstration of the equivalence of the representation theory of the Witt groups with that of certain truncated polynomial rings.

Introduction. The Steinberg tensor product theorem [1, Ch. A, §7] for a simply connected, semisimple algebraic group G in characteristic p displays irreducible G -modules as tensor products of Frobenius powers of infinitesimally irreducible G -modules (modules which are irreducible for the kernel G^1 of the Frobenius morphism of G).

A goal of modular representation theory is the expression of the coordinate ring of G in terms of tensor products of Frobenius powers of G -modules which are suitably elementary for G^1 . In this paper, we give a tensor product theorem for the finite-dimensional Witt groups. We produce a subcoalgebra C of the coordinate ring A of the m -dimensional Witt group W_m which is isomorphic to the coordinate ring of the kernel W_m^1 of the Frobenius morphism. A is the inductive limit of tensor products of Frobenius powers of C [§3, Theorem].

One can see some things about the representations of W_m . First, every finite-dimensional representation of W_m^1 extends to a representation of W_m on the same representation space [§5]. Second, a representation of W_m on a finite-dimensional vector space V is determined by a family $\{f_1, \dots, f_n\}$ of commuting endomorphisms of V such that $f_i^{p^m} = 0$. In other words, the representations of W_m on V may be studied via the representations of the algebras $\{k[x_1, \dots, x_n]/(x_1^{p^m}, \dots, x_n^{p^m})\}_n$ on V [Theorem, §4]. In particular, the representations of W_m which correspond to the representations of $k[x_1]/(x_1^{p^m})$ give canonical extensions for the representations of W_m^1 .

This linear formulation of the representation theory of W_m leaves one with the apparently difficult problem of determining the representation theory of $k[x_1, \dots, x_n]/(x_1^{p^m}, \dots, x_n^{p^m})$.

For the definition of the Witt groups, see [2, Ch. 5, §1].

NOTATION. Let A denote the coordinate ring of the m -dimensional Witt group W_m , as a reduced, connected group scheme over the prime field $k = F_p$. For any subcoalgebra C of A which contains k , let $C^{(p^i)}$ be the image of C under the i th-power of the Frobenius

morphism of A . We may form the inductive family of coalgebras $\{C \otimes C^{(p)} \otimes \dots \otimes C^{(p^n)}\}_{n=0}^\infty$, where $C \otimes \dots \otimes C^{(p^n)} \hookrightarrow C \otimes \dots \otimes C^{(p^n)} \otimes C^{(p^{n+1})}$ is the canonical morphism onto $C \otimes \dots \otimes C^{(p^n)} \otimes k$. Let $\varinjlim_n C \otimes C^{(p)} \otimes \dots \otimes C^{(p^n)}$ be the coalgebra inductive limit of the family.

Let $\Pi: A \rightarrow A/M^{(p)}A$ be the quotient morphism, where $M^{(p)}$ is the image of the augmentation ideal M under the Frobenius morphism. We show in §3 that there is a coalgebra splitting $s: A/M^{(p)}A \rightarrow A$ of Π such that A , as a coalgebra, is isomorphic to $\varinjlim_n C \otimes C^{(p)} \otimes \dots \otimes C^{(p^n)}$ where $C = \text{image } s$.

0. We require some facts from [3, Def. 6] of K. Newman. Let W_{m+1} be the $(m + 1)$ -dimensional Witt group over $k = F_p$, with coordinate ring A_{m+1} . As an algebra, A_{m+1} is the polynomial ring $k[X_1, X_p, X_{p^2}, \dots, X_{p^m}]$ on $(m + 1)$ -variables. Grade A_{m+1} by letting X_{p^i} have degree p^i . The coproduct Δ of A_{m+1} is the following: $\Delta X_{p^i} = \sum_{j=0}^{p^i} Q_j \otimes Q_{p^i-j}$, where Q_j is a homogeneous (relative to the grading) polynomial of degree j . In particular, $Q_0 = 1$, $Q_{p^i} = X_{p^i}$ and $\{Q_j\}_{j=0}^{p^m}$ is a sequence of divided powers.

Since degree $Q_j = j$, Q_j lies in $k[X_1, X_p, \dots, X_{p^{m-1}}]$ for $j < p^m$. The coordinate ring A of W_m may be identified with the sub-Hopf algebra $k[X_1, X_p, \dots, X_{p^{m-1}}]$ of A_{m+1} .

1. The coalgebra splitting of Π . $M = (X_1, X_p, \dots, X_{p^{m-1}})$ is the augmentation ideal of A . Let C be the k -span of $\{Q_j\}_{j=0}^{p^m-1}$. C is an irreducible coalgebra of dimension p^m , with $k \cdot X_1$ as its space of primitive elements. Since the coalgebra map $f: C \hookrightarrow A \xrightarrow{\Pi} A/M^{(p)}A$ has an injective restriction to $k \cdot X_1$, f is injective [5, Lemma 11.0.1]. Since $(A/M^{(p)}A)^*$ is the restricted universal enveloping algebra of $(M/M^{(p)})^*$ [3, 13.2.3], $\dim_k (A/M^{(p)}A)^* = p^{\dim_k (M/M^{(p)})^*} = p^m$. Therefore, $\dim_k (A/M^{(p)}A) = p^m$ and f is an isomorphism. $s = f^{-1}$ is the coalgebra splitting of Π that we use.

2. The value of Π at Q_j . Let $0 \leq j < p^m$. Write $j = \sum_{i=0}^{m-1} a_i p^i$ where $0 \leq a_i < p$.

LEMMA. $\Pi(Q_j)$ is a nonzero scalar multiple of $\Pi(X_1^{a_0} X_p^{a_1} \dots X_{p^{m-1}}^{a_{m-1}})$.

Proof. Q_j is a linear combination of elements $X_1^{b_0} X_p^{b_1} \dots X_{p^{m-1}}^{b_{m-1}}$ where $\sum b_i p^i = j$ by §0. If $\{b_i\}_i \neq \{a_i\}_i$, then $b_i \geq p$ for some i , and $\Pi(X_1^{b_0} X_p^{b_1} \dots X_{p^{m-1}}^{b_{m-1}}) = 0$. Therefore, $\Pi(Q_j) \in k \cdot \Pi(X_1^{a_0} X_p^{a_1} \dots X_{p^{m-1}}^{a_{m-1}})$, where the coefficient of $\Pi(X_1^{a_0} X_p^{a_1} \dots X_{p^{m-1}}^{a_{m-1}})$ is nonzero since the map

f of §1 is injective.

3. The coalgebra structure of the coordinate ring. Give the set of monomials in A the reverse lexicographic total order: $X_1^{a_0} X_p^{a_1} \dots X_p^{a_{m-1}} > X_1^{b_0} X_p^{b_1} \dots X_p^{b_{m-1}}$ if there is an index k such that $a_k > b_k$ and $a_i = b_i$ for $i > k$.

Let $\{a_i\}_0^{m-1}$ be a sequence where $0 \leq a_i < p$, and let $\{b_i\}_0^{m-1}$ be a different sequence, where $0 \leq b_i$.

LEMMA. If $\sum_{i=0}^{m-1} a_i p^i = \sum_{i=0}^{m-1} b_i p^i$, then $X_1^{a_0} X_p^{a_1} \dots X_p^{a_{m-1}} > X_1^{b_0} X_p^{b_1} \dots X_p^{b_{m-1}}$.

Proof. Let k be the maximal index such that $a_k \neq b_k$. If $b_k > a_k$, then $\sum_{i=0}^{m-1} b_i p^i > \sum_{i=0}^{m-1} a_i p^i$ since $a_i < p$. Therefore, we must have $a_k > b_k$ and $X_1^{a_0} \dots X_p^{a_{m-1}} > X_1^{b_0} \dots X_p^{b_{m-1}}$.

Let C be the coalgebra formed in §1.

THEOREM. The map $\lim_n C \otimes C^{(p)} \otimes \dots \otimes C^{(p^n)} \rightarrow A$, induced by multiplication; $C \otimes C^{(p)} \otimes \dots \otimes C^{(p^n)} \rightarrow A$, is an isomorphism of coalgebras.

Proof. Denote the map by g .

Surjectivity of g . Suppose that monomials $X_1^{b_0} X_p^{b_1} \dots X_p^{b_{m-1}}$ less than $X_1^{a_0} X_p^{a_1} \dots X_p^{a_{m-1}}$ in the ordering lie in the image of g . We show that $X_1^{a_0} X_p^{a_1} \dots X_p^{a_{m-1}}$ also lies there.

Write $a_i = \sum_j a_{ij} p^j$, where $0 \leq a_{ij} < p$. Let $t_k = \sum_{i=0}^{m-1} a_{ik} p^i$. By the lemmas of §2 and §3,

$$Q_{t_k} = U_k \cdot X_1^{a_{0k}} X_p^{a_{1k}} \dots X_p^{a_{m-1,k}} + Y_k,$$

where Y_k is a linear combination of monomials of degree t_k and less than $X_1^{a_{0k}} \dots X_p^{a_{m-1,k}}$ in the ordering, and where U_k is a nonzero scalar. Therefore,

$$\prod_{k=0}^{m-1} Q_{t_k}^{p^k} = \prod_{k=0}^{m-1} U_k^{p^k} \cdot X_1^{a_0} X_p^{a_1} \dots X_p^{a_{m-1}} + Y,$$

where Y is a linear combination of monomials which are less than $X_1^{a_0} X_p^{a_1} \dots X_p^{a_{m-1}}$. Since $\prod_{k=0}^{m-1} Q_{t_k}^{p^k}$ and Y lie in the image of g , so does $X_1^{a_0} X_p^{a_1} \dots X_p^{a_{m-1}}$.

Injectivity of g . Since g is surjective, so is $\Pi \circ g: \lim_n C \otimes C^{(p)} \otimes \dots \otimes C^{(p^n)} \xrightarrow{g} A \xrightarrow{[\Pi]} A/M^{(p^t)} A$ for any t ; at the same time, $C^{(p^j)} \hookrightarrow A \xrightarrow{\Pi} A/M^{(p^t)} A$ has image = k if $j \geq t$. Therefore, $C \otimes C^{(p)} \otimes \dots \otimes C^{(p^{t-1})} \xrightarrow{\text{mult.}} A \xrightarrow{\Pi} A/M^{(p^t)} A$ is surjective. Since $\dim_k (A/M^{(p^t)} A) = p^{mt}$

by [4] or by inspection, and $\dim_k(C \otimes C^{(p)} \otimes \dots \otimes C^{(p^{t-1})}) = p^{mt}$, $\Pi \circ \text{mult.}$ is an isomorphism of coalgebras. In particular, $C \otimes C^{(p)} \otimes \dots \otimes C^{(p^{t-1})} \xrightarrow{\text{mult.}} A$ is injective. Hence, g is injective.

4. Representation theory of W_m . The dual algebra $U = (A/M^{(p)}A)^*$ is the restricted universal enveloping algebra of the abelian p -Lie algebra $L = (M/M^2)^*$ [5, 13.2.3].

LEMMA. *There is a k -basis f_0, \dots, f_{m-1} for L , where $f_i^p = f_{i+1}$ for $i < m - 1$ and $f_{m-1}^p = 0$.*

Proof. Define f_j on the k -basis $X_1, X_p, \dots, X_{p^{m-1}}$ for M/M^2 by $f_j(X_{p^i}) = \delta_{ij}$. We have the following to complete the proof.

(1) If $i \neq j + 1$, then $f_j^p(X_{p^i}) = (\otimes^p f_j)(\Delta^{p-1} X_{p^i})$ is 0, since $\Delta^{p-1} X_{p^i}$ is homogeneous of degree p^i under the grading of $\otimes^p A$ induced from the grading of A , while $\otimes^p f_j$ can be nonzero only at monomials in $\otimes^p A$ of degree p^{j+1} .

(2) One may check that $f_j^p(X_{p^{j+1}}) = 1$.

To proof is complete.

By this lemma, the algebra map from the polynomial ring $k[f]$ to U mapping f to f_1 induces an isomorphism of k -algebras $k[f]/(f^{p^m}) \cong U$.

Denote by R_n the set of isomorphism classes of finite-dimensional representations of W_m whose coefficients lie in $C \otimes C^{(p)} \otimes \dots \otimes C^{(p^n)} \hookrightarrow A$. The canonical map $C \otimes C^{(p)} \otimes \dots \otimes C^{(p^n)} \hookrightarrow C \otimes C^{(p)} \otimes \dots \otimes C^{(p^{n+1})}$ induces $R_n \hookrightarrow R_{n+1}$. Then $R = \bigcup_n R_n$ is the set of isomorphism classes of finite-dimensional representations of W_m .

Let B denote the quotient of the polynomial ring $F_p[X_0, \dots, X_n, \dots]$ on generators $\{X_i\}_{i=0}^\infty$ by the ideal $(X_0^{p^m}, \dots, X_n^{p^m}, \dots)$. Denote by \hat{B} the set of isomorphism classes among those finite-dimensional representations of B in which all but a finite number of the X_i act as the zero endomorphism. Denote by \hat{B}_n the set of isomorphism classes of finite-dimensional representations of $k[X_0, \dots, X_n]/(X_0^{p^m}, \dots, X_n^{p^m})$. The map $k[X_0, \dots, X_n, \dots]/(X_0^{p^m}, \dots, X_n^{p^m}, \dots) \rightarrow k[X_0, \dots, X_n]/(X_0^{p^m}, \dots, X_n^{p^m})$, $X_i \mapsto X_i$ for $i \leq n$ and $X_i \mapsto 0$ for $i > n$, induces $\hat{B}_n \hookrightarrow \hat{B}$, and $\hat{B} = \bigcup_n \hat{B}_n$.

THEOREM. *There is a canonical bijection $R \rightarrow \hat{B}$, under which R_n and \hat{B}_n correspond.*

Proof. Since $C \cong A/M^{(p)}A$ as coalgebras, $C^* \cong U$ as algebras. Since A is reduced, the Frobenius morphism on A is injective, and $C \cong C^{(p^i)}$. Therefore,

(1) $(C \otimes C^{(p)} \otimes \dots \otimes C^{(p^n)})^* \cong \bigotimes^{n+1} U \cong k[X_0, \dots, X_n]/(X_0^{p^m}, \dots, X_n^{p^m})$.
 The first isomorphism is induced by the maps $U \rightarrow (C \otimes C^{(p)} \otimes \dots \otimes C^{(p^n)})^*$ which are dual to the maps $C \otimes C^{(p)} \otimes \dots \otimes C^{(p^n)} \xrightarrow{\varepsilon_0 \otimes \dots \otimes \varepsilon_{i-1} \otimes I \otimes \varepsilon_{i+1} \otimes \dots \otimes \varepsilon_n} C^{(p^i)}$, where ε_j is the counit of $C^{(p^j)}$; the second isomorphism is induced by $X_i \mapsto 1_0 \otimes \dots \otimes 1_{i-1} \otimes f_1 \otimes 1_{i+1} \otimes \dots \otimes 1_n$, where 1_j is the identity of U_j . Here u_j is the j th copy of u in $\bigotimes^{n+1} u$. Moreover,

(2) under dualization, the canonical map $C \otimes C^{(p)} \otimes \dots \otimes C^{(p^n)} \hookrightarrow C \otimes C^{(p)} \otimes \dots \otimes C^{(p^{n+1})}$ yields the map $k[f_0, \dots, f_{n+1}]/(f_0^{p^m}, \dots, f_{n+1}^{p^m}) \rightarrow k[X_0, \dots, X_n]/(X_0^{p^m}, \dots, X_n^{p^m})$ where $X_i \mapsto X_i$ for $i \leq n$ and $X_{n+1} \mapsto 0$.

The isomorphism $(C \otimes C^{(p)} \otimes \dots \otimes C^{(p^n)})^* \cong k[X_0, \dots, X_n]/(X_0^{p^m}, \dots, X_n^{p^m})$ of (1) induces a bijection $R_n \rightarrow \hat{B}_n$ such that
$$\begin{array}{ccc} R_n & \rightarrow & \hat{B}_n \\ \downarrow & & \downarrow \\ R_{n+1} & \rightarrow & \hat{B}_{n+1} \end{array}$$
 commutes

by (2). Therefore, $R \xrightarrow{\sim} \hat{B}$.

5. Representations of W_m^1 . The coalgebra C constructed in §1 is isomorphic to the coordinate ring $A/M^{(p)}A$ of W_m^1 under the mapping $\pi: A \rightarrow A/M^{(p)}A$ restricted to C . Therefore, the representations of W_m with coefficients in C correspond to the representations of W_m^1 via the isomorphism between the coefficient coalgebras C and $A/M^{(p)}A$, and every finite-dimensional representation of W_m^1 extends to a representation of W_m on the same representation space.

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