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Laurence Richard Boxer

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LAURENCE BOXER

We study the hyperspace (denoted 2_h^M) of ANR's of a (polyhedral) closed surface M. The topology of 2_h^M is induced by Borsuk's homotopy metric. We show the subpolyhedra of M are dense in 2_h^M. We obtain a necessary and sufficient condition for an arc in 2_h^M joining two points. We show that 2_h^M is an ANR ($\mathscr{M}$). We prove that the subspace of 2_h^M whose members are AR's has the homotopy type of M.

0. **Introduction.** For a finite-dimensional compactum X with metric ρ, let 2_h^X denote the space of nonempty compact ANR subsets of X. The topology of 2_h^X is induced by the metric ρ_h defined by Borsuk [3]. In [1] and [2], Ball and Ford studied several properties of 2_h^X, particularly for the case $X = S^2$. In this paper we generalize several of their results.

Throughout this paper, M will denote a (polyhedral) closed surface. We show the nonempty polyhedral subcompacta of M are dense in 2_h^M. We give a necessary and sufficient condition for the existence of an arc in 2_h^M joining two given members of 2_h^M. We show 2_h^M is an absolute neighborhood retract for metrizable spaces (ANR ($\mathscr{M}$)) and that the subspace of 2_h^M whose members are the compact AR subsets of M has the homotopy type of M.

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1. **Preliminaries.** Let ρ be a metric for M. We use the following notation: If $x \in M$ and $A \subset M$, then

$$B(x, r) = \{y \in M \mid \rho(x, y) < r\} ;$$

$\bar{A}$, $\operatorname{Int} A$, and $\operatorname{Bd} A$ are the closure, interior, and boundary of A (in M) respectively.

Euclidean n-space is denoted R^n. The interval $[0, 1]$ is denoted I. If $x, y \in R^n$ and $t \in R^1$, then $x + y$ will indicate the vector sum, and $t \cdot x$ will indicate scalar multiplication of x by t.

If A is a polyhedron, we will assume A is compact unless otherwise stated.

A *map* is a continuous function.

We use the following notation and terminology of [1] and [2]:

A δ-set or a δ-arc is a set or arc of diameter less than δ. A δ-map or a δ-embedding is a map or embedding that moves no point by as much as δ. The words "every δ-subset of A contracts to a point in an ε-subset of A" are denoted $s(A, \delta, \varepsilon)$.

Where more than one topology is considered on a set, the topology in which a sequence converges will be indicated by an obvious notation. For example, $a_n \underset{\rho}{\to} a_0$ indicates that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to a_0 in the topology of the metric ρ.

Let X be a finite-dimensional compactum. Let ρ be a metric for X. Let A and B be nonempty compact ANR subsets of X. The *Hausdorff metric* ρ_s is given by

$$\rho_s(A, B) = \max\{\sup\{\rho(a, B) \mid a \in A\}, \sup\{\rho(b, A) \mid b \in B\}\} .$$

The *homotopy metric* ρ_h is characterized in [3] by the following: Let A and $\{A_n\}_{n=1}^{\infty}$ be nonempty compact ANR subsets of a finite-dimensional compactum X. Then $A_n \underset{\rho_h}{\to} A$ if and only if

(a) $A_n \underset{\rho_s}{\to} A$, and

(b) given $\varepsilon > 0$, there is a $\delta > 0$ such that for all n, $s(A_n, \delta, \varepsilon)$.

We denote by 2_h^X the topological space whose members are the nonempty compact ANR subsets of X and whose topology is induced by the metric ρ_h. It is shown in [3] that 2_h^X is complete and separable, and that 2_h^X is a topological invariant of X. We mention here other useful results of Borsuk: If $\rho_h(A, B) < \varepsilon$, then there are ε-maps $f: A \to B$ and $g: B \to A$. For $C \in 2_h^X$, let $[C]_X$ denote the collection of all members of 2_h^X that have the same homotopy type as C. Then $[C]_X$ is open in 2_h^X. Since these sets partition 2_h^X, $[C]_X$ is also closed.

The terms *homotopy*, *deformation retraction*, *isotopy*, etc. will be used in standard fashion, except that it will be convenient not to insist that the interval be I. For example, if $c < d$, a deformation retraction of A onto B is a map $H: A \times [c, d] \to A$ such that $H_c = \mathrm{Id}_A$ and H_d is a retraction of A onto B. (We use the notation $H_t(a) = H(a, t)$ for all $(a, t) \in A \times [c, d]$.) It will occasionally be convenient to refer to the map H_d as a deformation retraction. A map $H: A \times [c, d] \to A$ is *strongly contracting* if $c \leqq u \leqq v \leqq d$ implies $H_u \circ H_v(A) \subset H_v(A) \subset H_u(A)$ ([1], p. 37).

The term *surface* will be used to refer to a (second countable) connected 2-manifold, with or without boundary. A *closed surface* is a compact surface without boundary. A *bounded surface* is a compact surface with boundary. We differ from [1] and [2] in that we will call an *annulus* any space homeomorphic to $\{(x, y) \in R^2 \mid 1 \leqq x^2 + y^2 \leqq 2\}$.

The following gives a useful criterion for convergence in 2_h^X:

LEMMA 1.1 ([1], 3.4, p. 38). *Let A and B be members of 2_h^X (X an arbitrary finite-dimensional compactum). Let $h: A \times I \to A$ be a strong deformation retraction of A onto B. Let $\{t_n\}_{n=1}^{\infty}$ be an increasing sequence in I converging to 1. Suppose that for each n, $A_n = h_{t_n}(A)$ is an* ANR. *If*

(a) *h is strongly contracting, or*

(b) *for all n, $h | A_n \times [t_n, t_{n+1}]$ is a strong deformation retraction of A_n onto A_{n+1}, then $A_n \underset{\rho_h}{\to} B$.*

REMARKS. Case (b) above is not proved in [1], but the proof is identical to that of (a). We will use both cases.

The next two lemmas will be used in questions of arcs.

LEMMA 1.2 ([1], 4.1, p. 43). *If $A_n \underset{\rho_h}{\to} A$ in 2_h^X and if for each n there is an ε_n-embedding $g_n: A_n \to X$ of A_n into X, where $\varepsilon_n \to 0$, then $g_n(A_n) \underset{\rho_h}{\to} A$.*

LEMMA 1.3 ([1], 4.2 *and* 4.3, p. 43). *If $A \in 2_h^X$ and $f: A \times I \to X$ is an isotopy, then $\{f_t(A) | t \in I\}$ contains an arc in 2_h^X from A to $f_1(A)$.*

The next two results will be used several times:

THEOREM 1.4 ([11], 3.4, pp. 382–383). *Let N be a compact surface with m boundary curves. Let L be a closed surface containing disjoint open disks $D_1, \cdots, D_m$ such that $N = L \backslash \bigcup_{j=1}^m D_j$. Let $r: N \to N$ be a deformation retraction of N, and let $R = r(N)$. Then $L \backslash R$ is a union of m simply-connected components $G_1, \cdots, G_m$, with $D_j \subset G_j$ for $j = 1, \cdots, m$.*

An immediate consequence of the above is:

COROLLARY 1.5. *Let N be a bounded surface. Let $R \subset \operatorname{Int} N$ be a bounded surface that is a deformation retract of N. Then each component of $\overline{N \backslash R}$ is an annulus.*

In the following theorems of Epstein, N will denote a surface, with or without boundary, compact or not.

THEOREM 1.6 ([8], 1.7, p. 85). *If a simple closed curve $S \subset N$ contracts to a point in N then S bounds a disk in N.*

THEOREM 1.7 ([8], A2, p. 106) (*stated in a different form*). *Sup-*

pose N is a polyhedral surface and $f: I \to N$ is an embedding with $f^{-1}(\mathrm{Bd}\, N) = \{0, 1\}$. Let U be a neighborhood of $f(I)$ in N. Then there is an ambient isotopy of N that is fixed on Bd N *and outside U and that changes f to a piecewise linear embedding.*

The following lemmas will be used in the next section.

LEMMA 1.8. *Let Y be a topological space, $L \subset Y$, and let β be an arc with endpoints u and v such that $\beta \subset L$. Suppose there is an open set D in $Y \backslash \{u, v\}$ and an arc $\bar{\gamma} \subset L$ with endpoints a and b such that $\{a, b\} \subset \mathrm{Bd}\, D$ and $\gamma = \bar{\gamma} \backslash \{a, b\}$ is a component of $L \cap D$. Then either $\gamma \cap \beta = \phi$ or $\bar{\gamma} \subset \beta$.*

Proof. Let $p: (I, 0, 1) \to (\beta, u, v)$ be a homeomorphism. (The notation means that p is a map from I to β such that $p(0) = u$ and $p(1) = v$.) Suppose $\gamma \cap \beta \neq \phi$. There is an $x \in \gamma$ and a $t_0 \in (0, 1)$ such that $p(t_0) = x$. Then $A = p^{-1}(\beta \cap D)$ is a nonempty open set in I contained in $(0, 1)$. Thus t_0 lies in a component (a_0, b_0) of A. We have $x \in p((a_0, b_0)) \subset \beta \cap D \subset L \cap D$, so $p((a_0, b_0))$ is a connected subset of $L \cap D$ containing x. Thus $p((a_0, b_0)) \subset \gamma$ and $\{p(a_0), p(b_0)\} \cap D = \phi$, so $\{p(a_0), p(b_0)\} \subset \mathrm{Bd}\, D$. The arc $B = p([a_0, b_0])$ has its interior in γ, but the endpoints of B are not in γ. Therefore $\bar{\gamma} = B \subset p(I) = \beta$.

The following is an immediate consequence of ([7], 4.2, p. 360):

LEMMA 1.9. *If A is an annulus with boundary curves T_1 and T_2, let $H: T_2 \times I \to A$ be a map such that $H_0 = \mathrm{Id}_{T_2}$ and $H_1(T_2) = T_1$. Then $H(T_2 \times I) = A$.*

We say Y *dominates* X if there are maps $f: X \to Y$ and $g: Y \to X$ such that $g \circ f$ is homotopic to Id_X. We write $\Delta X = \min\{\dim Y / Y$ is a finite simplicial complex that dominates $X\}$.

2. The role of the polyhedra. In [3], Borsuk asked the following questions: If X is a polyhedron, is the collection of all nonempty subpolyhedra of X dense in 2^X_h? What is the category (in the sense of Baire) of the collection of all nonempty subpolyhedra of X in 2^X_h? In [1], the first question was answered affirmatively for the case $X = S^2$, and the second question was given the following answer: If X is a connected polyhedron with no 1-dimensional open subset, the collection of all nonempty polyhedra properly contained in X is a first category subset of 2^X_h. It was also shown in [1] that the collection of nonempty topological polyhedra (i.e., homeomorphic images of polyhedra) properly contained in S^2 is a dense G_δ, hence second category, subset of $2_h^{S^2}$. We will extend the above to closed surfaces.

LEMMA 2.1. *If X is a finite-dimensional compactum and U is open in X, then $\mathscr{U} = \{C \in 2_h^X \mid C \subset U\}$ is open in 2_h^X.*

Proof. Let $\{A_n\}_{n=1}^\infty \subset 2_h^X \backslash \mathscr{U}$. Assume $A_n \underset{\rho_h}{\to} A_0$. For each n there exists $x_n \in A_n \backslash U$. Since X is compact we may assume (by taking a subsequence if necessary) that $x_n \to x_0 \in X \backslash U$. Since $A_n \underset{\rho_s}{\to} A_0$, we have $x_0 \in A_0$. Therefore $A_0 \notin \mathscr{U}$, so $\mathscr{U}$ is open.

We prove a theorem about the Baire category of the collection of topological polyhedra in M as a subset of 2_h^M. (Recall M is a (polyhedral) closed surface.)

THEOREM 2.2. *Let $\mathscr{T}$ be the collection of nonempty topological polyhedra properly contained in M. Then $\mathscr{T}$ is a second category subset of 2_h^M.*

Proof. Let D be a disk contained in M. By 2.1, $\mathscr{U} = \{Y \in 2_h^M \mid Y \subset \operatorname{Int} D\}$ is open in 2_h^M, and thus is topologically complete. Let $f\colon \operatorname{Int} D \to S^2$ be an embedding. Then the map $f_*\colon \mathscr{U} \to 2_h^{S^2}$ given by $f_*(Y) = f(Y)$ is an open embedding ([3], p. 198). Since the collection of nonempty topological polyhedra contained in S^2 is a dense G_δ subset of $2_h^{S^2}$ ([1], 3.12, p. 42), it follows that $\mathscr{U} \backslash \mathscr{T}$ is a first category subset of $\mathscr{U}$. The classical Baire category theorem implies $\mathscr{U} \cap \mathscr{T}$ is a second category subset of $\mathscr{U}$, and thus of 2_h^M. Hence $\mathscr{T}$ is a second category subset of 2_h^M.

The rest of this section is devoted to proving the following:

THEOREM 2.3. *The collection of nonempty subpolyhedra of M is dense in 2_h^M.*

To prove 2.3, we show in 2.4 that for a given $C \in 2_h^M$ we can split M into two pieces that join along simple closed curves such that the intersection of C with each piece is an ANR. Each of the pieces of M embeds in S^2. In 2.5, we use the fact that the result is known for S^2 to construct a sequence of polyhedra whose intersection is C satisfying the hypotheses of 1.1.

LEMMA 2.4. *Let q be a positive integer. Assume M is orientable with genus q or nonorientable with genus $2q$. Let $C \in 2_h^M$. Then there are compact subsurfaces X_1 and X_2 of M and simple closed curves $\alpha_1, \cdots, \alpha_{q+1}$ in M such that:*

(a) $M = X_1 \cup X_2$.

(b) *The* α_n *are pairwise disjoint.*

(c) $\mathrm{Bd}\, X_1 = \mathrm{Bd}\, X_2 = X_1 \cap X_2 = \bigcup_{n=1}^{q+1} \alpha_n$.

(d) X_1 *and* X_2 *both are homeomorphic to a sphere with* $q + 1$ *disjoint open disks removed.*

(e) $\bigcup_{n=1}^{q+1} \alpha_n \backslash C$ *has finitely many components.*

Proof. It is an easy consequence of the standard way to represent a surface that there are subsurfaces X_1' and X_2' of M and simple closed curves $\alpha_1', \cdots, \alpha_{q+1}'$ in M satisfying (a) through (d). It follows that for each n there is a two-sided collar N_n of α_n' in M such that the N_n are pairwise disjoint. For any n such that $\alpha_n' \backslash C$ has finitely many components, set $\alpha_n = \alpha_n'$. Thus we suppose α' is any of the α_n' such that $\alpha_n' \backslash C$ has infinitely many components. We write $N = N_n$. Clearly we may write $\alpha' \backslash C = \bigcup_{m=1}^{\infty} \gamma_m$, where the γ_m are distinct components of $\alpha' \backslash C$ and each $\bar{\gamma}_m$ is an arc whose endpoints a_m and b_m lie in C.

Let $Z = \limsup \{\bar{\gamma}_m\}_{m=1}^{\infty}$, i.e., Z is the set of all $x \in \alpha'$ such that every neighborhood of x meets infinitely many $\bar{\gamma}_m$. Then Z is closed (see [13], p. 10). Thus Z is a compact subset of α'. It is easily seen that $Z \subset C$.

Let w_0, w_1, and w_2 be distinct points of γ_1 such that w_0 lies in the arc $\overline{w_1 w_2}$ of γ_1 from w_1 to w_2. Let $f_0: (I, 0, 1) \to (\alpha' \backslash (\overline{w_1 w_2} \backslash \{w_1, w_2\}), w_1, w_2)$ be a homeomorphism. Since N is an annulus,

(1) there is a disk $B \subset N$ such that $N \backslash B$ is homeomorphic to $I \times (0, 1)$, $w_0 \in (N \backslash B) \cap \alpha' \subset \overline{N \backslash B} \cap \alpha' \subset \gamma_1$, and $Z \cup f_0(I) \subset \mathrm{Int}\, B$. Since ANR's are locally arcwise connected, (1) implies that for each $z \in Z$ there is a neighborhood U of z contained in $\mathrm{Int}\, B$ such that $U \cap C$ is arcwise connected. Since Z is compact,

(2) there are open sets $U_1, \cdots, U_p$ such that $Z \subset \bigcup_{k=1}^{p} U_k \subset \mathrm{Int}\, B$ and each $U_k \cap C$ is arcwise connected.

It is easily seen that for almost all m there is a k such that $\bar{\gamma}_m \subset U_k$. We assume $\bar{\gamma}_1, \cdots, \bar{\gamma}_{m_0}$ are those $\bar{\gamma}_m$ that fail to lie in any U_k. Define $\Gamma_0 = \phi$, and for $k \in \{0, 1, \cdots, p-1\}$ define

$$\Gamma_{k+1} = \left\{ \bar{\gamma}_m \subset U_{k+1} \mid \bar{\gamma}_m \notin \bigcup_{j=0}^{k} \Gamma_j \right\} .$$

Define $\Gamma_{p+1} = \{\bar{\gamma}_1, \cdots, \bar{\gamma}_{m_0}\}$. For each j let $\Gamma_j' = \{\gamma_m \mid \bar{\gamma}_m \in \Gamma_j\}$. Clearly $\Gamma_0, \Gamma_1, \cdots, \Gamma_{p+1}$ partition $\{\bar{\gamma}_m\}_{m=1}^{\infty}$. Let the endpoints a_m and b_m of $\bar{\gamma}_m$ satisfy $f_0^{-1}(a_m) < f_0^{-1}(b_m)$. For $m > 1$, $\bar{\gamma}_m = f_0([f_0^{-1}(a_m), f_0^{-1}(b_m)])$.

We begin an induction argument by observing that for $k = 0$ we have a map $f_k: (I, 0, 1) \to (\mathrm{Int}\, B, w_1, w_2)$ such that:

(3) If $t \in I$ and $f_k(t) \notin C$ then $f_k(t) = f_0(t)$.

(4) $f_k(I)\backslash C$ is a union of members of $\bigcup_{j=k+1}^{p+1} \Gamma'_j$.

Suppose for some $k < p$, $f_k\colon (I, 0, 1) \to (\operatorname{Int} B, w_1, w_2)$ is a map satisfying (3) and (4). If $f_k(I)\backslash C$ meets no member of Γ'_{k+1} we define $f_{k+1} = f_k$; then (3) and (4) are satisfied when k is replaced by $k+1$. Otherwise we define $c_k = \inf\{t \in I | f_k(t)$ belongs to a member of $\Gamma'_{k+1}\}$, and $d_k = \sup\{t \in I | f_k(t)$ belongs to a member of $\Gamma'_{k+1}\}$. By (4) and our choice of $\{w_1, w_2\}$, $0 < c_k < d_k < 1$. By (3) and (4), each of $f_k(c_k) = f_0(c_k)$ and $f_k(d_k) = f_0(d_k)$ must be an endpoint of some $\bar{\gamma}_m \in \Gamma_{k+1}$ or a member of Z. It follows that $\{f_k(c_k), f_k(d_k)\} \subset \overline{U_{k+1}} \cap C$.

If $\{f_k(c_k), f_k(d_k)\} \subset U_{k+1}$ then (2) implies there is an arc γ'_k in $U_{k+1} \cap C$ from $f_k(c_k)$ to $f_k(d_k)$.

If, say, $f_k(c_k) \notin U_{k+1}$ then there must be infinitely many members of Γ'_{k+1} that meet $f_k(I)$, for otherwise (4) implies $f_k(c_k)$ is an endpoint a_m of some $\bar{\gamma}_m \in \Gamma_{k+1}$ and thus $f_k(c_k) \in U_{k+1}$, contrary to assumption. Thus $f_k(c_k) \in Z \cap U_{k_1}$ for some k_1. There is a sequence $\{a_{m_r}\}$ of endpoints of members $\overline{\gamma_{m_r}}$ of Γ_{k+1} such that $f_k \circ f_0^{-1}(\overline{\gamma_{m_r}}) \not\subset C$ and $a_{m_r} \to f_k(c_k)$. Hence there is an r such that $a_{m_r} \in U_{k_1}$. By (2) there are arcs γ' in $U_{k_1} \cap C$ from $f_k(c_k)$ to a_{m_r} and γ'' in $U_{k+1} \cap C$ from a_{m_r} to $f_k(d_k)$. There is an arc $\gamma'_k \subset \gamma' \cup \gamma'' \subset C \cap \operatorname{Int} B$ from $f_k(c_k)$ to $f_k(d_k)$.

The other cases are treated as above. So in any case, $C \cap \operatorname{Int} B$ contains an arc γ'_k from $f_k(c_k)$ to $f_k(d_k)$. Let $f_{k+1}\colon (I, 0, 1) \to (\operatorname{Int} B, w_1, w_2)$ be determined by: $f_{k+1}|[c_k, d_k]$ is a homeomorphism of $([c_k, d_k], c_k, d_k)$ onto $(\gamma'_k, f_k(c_k), f_k(d_k))$; and $f_{k+1}(t) = f_k(t)$ for $t \in I\backslash[c_k, d_k]$. Clearly f_{k+1} is continuous. The construction shows (3) and (4) are satisfied when k is replaced by $k+1$.

With the induction completed, we have by (4) a map $f_p\colon (I, 0, 1) \to (\operatorname{Int} B, w_1, w_2)$ such that $f_p(I)\backslash C$ is a union of members of the finite set Γ'_{p+1}. Now $f_p(I)$ contains an arc β from w_1 to w_2. Let γ_m be a component of $f_p(I)\backslash C$. Apply 1.8, with $Y = M$, $L = f_p(I)$, $D = M\backslash(C \cup \{w_1, w_2\})$, $\bar{\gamma} = \bar{\gamma}_m$: We have $\bar{\gamma}_m \subset \beta$ or $\gamma_m \cap \beta = \phi$. Therefore $\beta\backslash C$ has finitely many components, and $\alpha = \beta \cup \overline{w_1 w_2}$ is a simple closed curve such that $\alpha\backslash C$ has finitely many components.

Let $h\colon \operatorname{Int} B \to R^2$ be a homeomorphism. Let $h'\colon (I, 0, 1) \to (\beta, w_1, w_2)$ be a homeomorphism. Let $g\colon ([-1, 1], 0, \{-1, 1\}) \to (\alpha', w_1, \{w_2\})$ be a relative homeomorphism such that $g(I) \subset \operatorname{Int} B$. Define $H\colon \alpha' \times I \to \operatorname{Int} N$ by

$$H(g(s), t) = \begin{cases} g(s) & \text{if } \quad -1 \leqq s \leqq 0; \\ h^{-1}[(1-t)\cdot h \circ g(s) + t \cdot h \circ h'(s)] & \text{if } \quad 0 \leqq s \leqq 1\,. \end{cases}$$

Clearly H is well-defined and continuous, $H_0 = \mathrm{Id}_{\alpha'}$, and H_1 is a homeomorphism of α' onto α. It follows from ([7], 2.1, p. 87) that there is a homeomorphism $T\colon N \to N$ such that $T(\alpha') = \alpha$ and $T(x) = x$ for all $x \in \operatorname{Bd} N$.

By applying this construction to each of the curves α'_n, we easily obtain a homeomorphism $P: M \to M$ taking $X'_1, X'_2, \alpha'_1, \cdots, \alpha'_{q+1}$ onto sets satisfying (a) through (e).

Theorem 2.3 follows from 1.1 and the following:

THEOREM 2.5. *Let $C \in 2^M_h$ be a proper subset of M. Then there is a sequence $\{A_n\}_{n=1}^\infty$ in 2^M_h such that for all n:*

(a) *Each component of A_n is a polyhedral bounded surface.*

(b) $C \subset A_{n+1} \subset \operatorname{Int} A_n$.

Also there is a sequence $0 = t_1 < t_2 < t_3 < \cdots$ with $\lim t_n = 1$ and a map $h: A_1 \times I \to A_1$ such that:

(c) *h is a strong deformation retraction of A_1 onto C.*

(d) *For each n, $h | A_n \times [t_n, t_{n+1}]$ is a strong deformation retraction of A_n onto A_{n+1}.*

Proof. We remark that the proof is long, so some of the technical details have been omitted. A more complete proof is in [5].

It is easy to see that there is no loss of generality in assuming C is connected. By sewing a Moebius band onto the boundary of a disk cut out of $M \backslash C$ if necessary, we can also assume that M is nonorientable of even genus, or orientable. In view of ([1], 3.2, 3.3, and 3.5, pp. 36–39) we assume $M \neq S^2$.

For a given connected $C \in 2^M_h$ with $C \neq M$, let $\alpha_1, \cdots, \alpha_{q+1}, N_1, \cdots, N_{q+1}, X_1, X_2$ be as in 2.4 and its proof. It follows from 2.4(e) and ([4], 2.12, p. 102) that $\hat{X}_1 = X_1 \cap C$ and $\hat{X}_2 = X_2 \cap C$ are ANR's. We may assume $\hat{X}_1 \neq \phi$. For $k = 1, 2$, $X_k \cup \bigcup_{j=1}^{q+1} N_j$ is homeomorphic to X_k, which is embeddable in S^2. If $\hat{X}_2 \subset \operatorname{Int}(\bigcup_{j=1}^{q+1} N_j)$ then $C \subset \operatorname{Int}(X_1 \cup \bigcup_{j=1}^{q+1} N_j)$, in which case we are done, by [1]. Thus we assume

(1) $\hat{X}_2 \not\subset \operatorname{Int}(\bigcup_{j=1}^{q+1} N_j)$.

Let Γ be the set of components γ of $\bigcup_{j=1}^{q+1} \alpha_j \backslash C$ such that $\gamma \subset \alpha_j$ implies $\gamma \neq \alpha_j$. From 2.4(e), Γ is a finite set. We argue by induction on the number of members of Γ.

If $\Gamma = \phi$ then for each $j \in \{1, 2, \cdots, q + 1\}$ either $\alpha_j \subset C$ or $\alpha_j \subset M \backslash C$. Since C is connected and $\hat{X}_1 \neq \phi$, if no α_j lies in C we have $C = \hat{X}_1$, contrary to (1). We assume

(2) $\bigcup_{j=1}^{p} \alpha_j \subset C$ for some p with $1 \leqq p \leqq q + 1$, and if $p < q + 1$ then $\bigcup_{j=p+1}^{q+1} \alpha_j \subset M \backslash C$.

Neither $\hat{X}_1$ nor $\hat{X}_2$ need be connected; nevertheless, the theorems of [1] cited above (and their proofs) imply there are sequences $\{B^k_n\}_{n=1}^\infty$ $(k = 1, 2)$ such that for all n:

(3) Each component of B^k_n is a polyhedral surface.

(4) $\hat{X}_k \subset B^k_{n+1} \subset \operatorname{Int} B^k_n \subset B^k_n \subset \operatorname{Int}(X_k \cup \bigcup_{j=1}^{q+1} N_j)$. Also there are

maps $h^k: B_1^k \times I \to B_1^k$ and a sequence $0 = t_1 < t_2 < t_3 < \cdots$ such that $\lim t_n = 1$,

(5) h^k is a strong deformation retraction of B_1^k onto $\hat{X}_k$, and for each n:

(6) $h^k | B_n^k \times [t_n, t_{n+1}]$ is a strong deformation retraction of B_n^k onto B_{n+1}^k.

(7) $h^k | (\text{Bd } B_n^k) \times [t_n, t_{n+1}]$ is an isotopy of $\text{Bd } B_n^k$ onto $\text{Bd } B_{n+1}^k$.

(8) If $y \in \text{Bd } B_n^k$ and $x \in h^k(\{y\} \times [t_n, t_{n+1}])$, then $h^k(\{x\} \times [t_n, t_{n+1}]) \subset h^k(\{y\} \times [t_n, t_{n+1}])$ and $h^k(x, t) = h^k(y, t)$ for $t \in [t_{n+1}, 1]$.

(9) For all $x \in \text{Bd } B_n^k$, $h^k(\{x\} \times I)$ is an arc and $h^k(\{x\} \times [0, 1))$ is a (noncompact) polyhedron.

(10) If D is a component of $B_n^k \backslash \hat{X}_k$ and E is a component of $\text{Bd } D$ such that $E \subset \hat{X}_k$, then there is a boundary curve β of B_n^k such that $\beta \subset D$ and $h_1^k(\beta) = E$.

From (2) and (4) we may assume for all n and for $k = 1, 2$,

(11) $\bigcup_{j=1}^p \alpha_j \subset \text{Int } B_n^k$ and $B_n^k \cap \bigcup_{j=p+1}^{q+1} \alpha_j = \phi$.

For all n, let $A_n = (B_n^1 \cap X_1) \cup (B_n^2 \cap X_2)$. We define a map h on $A_1 \times I$ by

$$h(x, t) = \begin{cases} h^1(x, t) & \text{if} \quad x \in B_1^1 \cap X_1 ; \\ h^2(x, t) & \text{if} \quad x \in B_1^2 \cap X_2 . \end{cases}$$

If $x \in (B_1^1 \cap X_1) \cap (B_1^2 \cap X_2) = \bigcup_{j=1}^p \alpha_j = \hat{X}_1 \cap \hat{X}_2$, then (5) implies $h^1(x, t) = x = h^2(x, t)$ for all $t \in I$. Therefore h is well-defined and continuous. It is easily seen that

(12) if $x \in B_1^k \cap X_k$ then $h(x, t) \in B_1^k \cap X_k$. It follows that $h(A_1 \times I) = A_1$.

By (11), if β is a boundary curve of B_n^k then $\beta \subset \text{Int } X_1$ or $\beta \subset \text{Int } X_2$. The union of those boundary curves of B_n^k that lie in $\text{Int } X_k$ is $(\text{Bd } A_n) \cap X_k$. It follows that A_n is a polyhedral bounded surface.

For all n, $C \subset A_{n+1} = (B_{n+1}^1 \cap X_1) \cup (B_{n+1}^2 \cap X_2) \subset [(\text{Int } B_n^1) \cap X_1] \cup [(\text{Int } B_n^2) \cap X_2] = \text{Int } (B_n^1 \cap X_1) \cup \bigcup_{j=1}^p \alpha_j \cup \text{Int } (B_n^2 \cap X_2) = \text{Int } A_n$.

It is clear that $h_0 = \text{Id}_{A_1}$ and $h_t | C = \text{Id}_C$ for all $t \in I$. Also $h_1(A_1) = h_1^1(B_1^1 \cap X_1) \cup h_1^2(B_1^2 \cap X_2) = (\text{by (5) and (12)}) \hat{X}_1 \cup \hat{X}_2 = C$. Thus h is a strong deformation retraction of A_1 onto C.

For all n, we see by (6) and (12) that $h | A_n \times [t_n, t_{n+1}]$ is a strong deformation retraction of A_n onto A_{n+1}.

By (12), analogues of (7) through (9) hold when we replace $(\hat{X}_k, \{B_n^k\}_{n=1}^\infty, h^k)$ with $(C, \{A_n\}_{n=1}^\infty, h)$.

If D is a component of $A_n \backslash C$ then by (11) D is a component of $B_n^k \backslash \hat{X}_k$ for some k. Then (10) and the construction imply $(C, \{A_n\}_{n=1}^\infty, h)$ satisfies the analogue of (10). This concludes our discussion of the case $\Gamma = \phi$.

Suppose the theorem is true whenever Γ has less than r members

$(r > 0)$. Now let Γ have r distinct members, $\gamma_1, \cdots, \gamma_r$. Topologically γ_r is an open interval in some α_j, say $\gamma_r \subset \alpha_1$. Let $\{z_1, z_2\}$ be the endpoints of $\gamma_r(z_1 = z_2$ if $\bar{\gamma}_r = \alpha_1)$. Let $C' = C \cup \bar{\gamma}_r$. Clearly C' is a connected ANR, and $\Gamma' = \{\gamma_1, \cdots, \gamma_{r-1}\}$ is the set of all components γ of $\bigcup_{j=1}^{q+1} \alpha_j \backslash C'$ such that $\gamma \subset \alpha_j$ implies $\gamma \neq \alpha_j$. The inductive hypothesis gives a sequence $\{B_n\}_{n=1}^{\infty} \subset 2_h^M$ such that for all n:

(13) B_n is a polyhedral bounded surface.

(14) $C' \subset B_{n+1} \subset \operatorname{Int} B_n$.

Also there is a map $\psi: B_1 \times I \rightarrow B_1$ and a sequence $0 = t_1 < t_2 < t_3 < \cdots$ such that $\lim t_n = 1$,

(15) ψ is a strong deformation retraction of B_1 onto C', and for all n:

(16) $\psi / B_n \times [t_n, t_{n+1}]$ is a strong deformation retraction of B_n onto B_{n+1}.

(17) $\psi / (\operatorname{Bd} B_n) \times [t_n, t_{n+1}]$ is an isotopy of $\operatorname{Bd} B_n$ onto $\operatorname{Bd} B_{n+1}$.

(18) If $y \in \operatorname{Bd} B_n$ and $x \in \psi(\{y\} \times [t_n, t_{n+1}])$ then $\psi(\{x\} \times [t_n, t_{n+1}]) \subset \psi(\{y\} \times [t_n, t_{n+1}])$ and $\psi(x, t) = \psi(y, t)$ for $t \in [t_{n+1}, 1]$.

(19) For all $x \in \operatorname{Bd} B_n$, $\psi(\{x\} \times I)$ is an arc and $\psi(\{x\} \times [0, 1))$ is a (noncompact) polyhedron.

(20) If D is a component of $B_n \backslash C'$ and E is a component of $\operatorname{Bd} D$ such that $E \subset C'$, then there is a boundary curve β of B_n such that $\beta \subset D$ and $\psi_1(\beta) = E$.

For all n we define $\varepsilon_n = \sup\{\operatorname{diam} \psi(\{x\} \times I) / x \in B_n\}$. By compactness, ε_n is finite, and we easily see

(21) $\lim \varepsilon_n = 0$.

Let D be a component of $B_1 \backslash C'$ such that $\bar{\gamma}_r$ lies in a boundary component E of D. From (20) there is a boundary curve β of B_1 such that $\beta \subset D$ and $\bar{\gamma}_r \subset \psi_1(\beta)$. It can be shown that:

(22) β contains a continuum β' such that $\psi_1(\beta') = \bar{\gamma}_r$. If β' is an arc whose endpoints are e_1 and e_2 then $\psi_1(\{e_1, e_2\}) = \{z_1, z_2\}$ and $\psi_1(\beta' \backslash \{e_1, e_2\}) = \gamma_r$.

Further, we show:

(23) If U is an open set contained in D such that $E \cap \operatorname{Bd} U \neq \phi$, then $U \cap \psi(\beta \times I) \neq \phi$.

For U meets a component U_n of $\overline{B_n \backslash B_{n+1}}$ for some n. By (14), (16), and 1.5, U_n is an annulus. From (16), (17), (18), and 1.9, $U_n = \psi(\beta \times [t_n, t_{n+1}])$, and (23) follows.

Let $y_0 \in \gamma_r$. By (23) there are continua $P_k (k = 1, 2)$ such that $\beta' = P_k$ satisfies (22) and $P_k \cap (\operatorname{Int} X_k) \cap B(y_0, \varepsilon_1) \neq \phi$. It can be shown that $P_1 \cap P_2 = \phi$. By (17), for all n,

(24) $\psi(P_1 \times \{t_n\}) \cap \psi(P_2 \times \{t_n\}) = \phi$.

It can be shown that not both of P_1 and P_2 are simple closed curves. Hence we assume P_1 is an arc. Then P_2 is an arc or a simple closed curve.

By (22) we may assume the endpoints a_1^1 and b_1^1 of P_1 satisfy $\psi_1(a_1^1) = z_1$, $\psi_1(b_1^1) = z_2$. If P_2 is an arc then we may assume its endpoints a_1^2 and b_1^2 satisfy $\psi_1(a_1^2) = z_1$, $\psi_1(b_1^2) = z_2$. If P_2 is a simple closed curve then $z_1 = z_2$, and by analogy with the above we choose $a_1^2 = b_1^2 \in P_2 \cap \psi_1^{-1}(z_1)$.

By (19), $\eta^k = \psi(\{a_1^k\} \times I)$ and $\xi^k = \psi(\{b_1^k\} \times I)$ are arcs. By (17) and (18) we have

(25) $\eta^1\backslash\{z_1\}$, $\eta^2\backslash\{z_1\}$, $\xi^1\backslash\{z_2\}$ (and $\xi^2\backslash\{z_2\}$ if $\xi^2 \neq \eta^2$) are pairwise disjoint.

Let $p_k \in P_k \cap \psi_1^{-1}(y_0)$, $k = 1, 2$. Let P_a^1 be the arc of P_1 from a_1^1 to p_1. Let P_b^1 be the arc of P_1 from p_1 to b_1^1. If $a_1^2 \neq b_1^2$, let P_a^2 and P_b^2 be the arcs of P_2 from a_1^2 to p_2 and from p_2 to b_1^2, respectively. If $a_1^2 = b_1^2$ then $z_1 = z_2$. Then let P_a^2 be the arc of p_2 from a_1^2 to p_2 contained in $P_2 \cap \psi_1^{-1}(\psi_1(P_a^1))$ and let P_b^2 be the other arc of P_2 from a_1^2 to p_2.

Clearly $T_1 = \bigcup_{k=1}^2 [\eta^k \cup P_a^k \cup \psi(\{p_k\} \times I)]$ and $T_2 = \bigcup_{k=1}^2 [\xi^k \cup P_b^k \cup \psi(\{p_k\} \times I)]$ are simple closed curves that are deformed by ψ into proper subsets of α_1. By 1.6, T_1 and T_2 bound disks M_1 and M_2 respectively in B_1. Clearly $M_k = \psi(T_k \times I)$.

There is an arc λ_1' in $M_1 \cap B(z_1, \varepsilon_1)$ from a_1^1 to a_1^2 such that $\{a_1^1, a_1^2\} = \lambda_1' \cap \operatorname{Bd} M_1$. Then $\lambda_1' \subset B_1 \cap B(z_1, \varepsilon_1)$ and $\lambda_1' \cap \operatorname{Bd} B_1 = \{a_1^1, a_1^2\}$. By (19), $M_1\backslash\{z_1, y_0\}$ is a (noncompact) polyhedron, so by 1.7 there is an ambient isotopy of M_1 that is fixed on $(M_1\backslash B(z_1, \varepsilon_1)) \cup \operatorname{Bd} M_1$ and that carries λ_1' onto a polyhedral arc λ_1. Similarly, there is a polyhedral arc μ_1 in $M_2 \cap B(z_2, \varepsilon_1)$ from b_1^1 to b_1^2 such that $\{b_1^1, b_1^2\} = \mu_1 \cap \operatorname{Bd} B_1$.

For all n, let $a_n^k = \psi(a_1^k, t_n) \in \operatorname{Bd} B_n$, and let $b_n^k = \psi(b_1^k, t_n) \in \operatorname{Bd} B_n$. Let $\eta_0^k = \eta^k$, $\xi_0^k = \xi^k$, $\eta_n^k = \psi(\{a_n^k\} \times [t_{n+1}, 1])$ (the arc of η^k from a_{n+1}^k to z_1), $\xi_n^k = \psi(\{b_n^k\} \times [t_{n+1}, 1])$ (the arc of ξ^k from b_{n+1}^k to z_2). Note that we have begun an induction argument by showing that for $n = 1$, the following statements (26) through (29) are valid:

(26) There are polyhedral arcs $\lambda_n \subset M_1 \cap B_n \cap B(z_1, \varepsilon_n)$ from a_n^1 to a_n^2, $\mu_n \subset M_2 \cap B_n \cap B(z_2, \varepsilon_n)$ from b_n^1 to b_n^2 such that:

(27) $\{a_n^1, a_n^2\} = \lambda_n \cap \operatorname{Bd} B_n = \lambda_n \cap \operatorname{Bd} M_1$.

$$\{b_n^1, b_n^2\} = \mu_n \cap \operatorname{Bd} B_n = \mu_n \cap \operatorname{Bd} M_2 .$$

(28) $\lambda_n \cap (\eta_n^1 \cup \eta_n^2) = \phi = \mu_n \cap (\xi_n^1 \cup \xi_n^2)$.

(For $n = 1$, (27) and (28) follow from observing which points are left fixed by the ambient isotopies.)

(29) $\lambda_n \cap \lambda_j = \phi = \mu_n \cap \mu_j$ for $j < n$.

Suppose $m > 0$ and (26) through (29) are valid for $n = 1, \cdots, m$. The inductive step is done as above, with obvious modifications. For example, to obtain λ_{m+1} satisfying (26) through (29), we work in the disk bounded not by T_1, but by the simple closed curve

$$\overline{u_m v_m} \cup \overline{u_m a_{m+1}^1} \cup \eta_m^1 \cup \eta_m^2 \cup \overline{v_m a_{m+1}^2} ,$$

where $\overline{u_m v_m}$ is the arc of λ_m whose endpoints u_m and v_m satisfy $u_m \in \psi(P_1 \times \{t_m\})$, $v_m \in \psi(P_2 \times \{t_m\})$, $\overline{u_m v_m} \setminus \{u_m, v_m\} \subset \operatorname{Int} B_{m+1}$; $\overline{u_m a^1_{m+1}}$ is the arc of $\psi(P_1 \times \{t_{m+1}\})$ from u_m to a^1_{m+1}; and $\overline{v_m a^2_{m+1}}$ is the arc of $M_1 \cap \psi(P_2 \times \{t_{m+1}\})$ from v_m to a^2_{m+1}. Thus (26) through (29) hold for all n.

Since $\lambda_n \subset M_1$, $\mu_n \subset M_2$, and $(\operatorname{Bd} M_1) \cap (\operatorname{Bd} M_2) \setminus \psi(\{p_1, p_2\} \times I) = \eta^2 \cap \xi^2$, (25) and (27) imply

$$(30) \qquad \lambda_n \cap \mu_j = \begin{cases} \phi \text{ if } n \neq j \text{ , or if } n = j \text{ and } \eta^2 \neq \xi^2 \text{ ;} \\ \{a^2_n = b^2_n\} \text{ if } n = j \text{ and } \eta^2 = \xi^2 \text{ .} \end{cases}$$

For $k = 1, 2$, let Q_k be the boundary curve of B_1 containing P_k. Let $Q^n_k = \psi(Q_k \times \{t_n\})$, $P^n_k = \psi(P_k \times \{t_n\})$. Let $E_n = [(Q^n_1 \cup Q^n_2) \setminus (P^n_1 \cup P^n_2)] \cup \lambda_n \cup \mu_n$. Clearly E_n is a polyhedron, and $E_n \cap E_j = \phi$ for $n \neq j$. If $Q_1 \neq Q_2$, then (17), (24), (27), and (30) imply E_n is a simple closed curve. (Note (30) implies if $\lambda_n \cap \mu_n = \{a^2_n\}$ then $Q^n_2 = P^n_2$, so $E_n = (Q^n_1 \setminus P^n_1) \cup \lambda_n \cup \mu_n$.) Similarly, if $Q_1 = Q_2$ then either E_n is a simple closed curve for all n or E_n is a disjoint union of two simple closed curves for all n.

For all n, let $J_n \subset M_1$ be the disk bounded by $\eta^1_{n-1} \cup \eta^2_{n-1} \cup \lambda_n$ and let $J'_n \subset M_2$ be the disk bounded by $\xi^1_{n-1} \cup \xi^2_{n-1} \cup \mu_n$. Define $A_n = [B_n \setminus (M_1 \cup M_2)] \cup J_n \cup J'_n$. To complete the proof, we must show (13) through (20) are satisfied when $(\{A_n\}^\infty_{n=1}, C)$ replaces $(\{B_n\}^\infty_{n=1}, C')$ and an appropriate map h replaces ψ.

We have

$$\operatorname{Bd} A_n = E_n \cup [(\operatorname{Bd} B_n) \setminus (Q^n_1 \cup Q^n_2)] \quad \text{and} \quad E_n \cap [(\operatorname{Bd} B_n) \setminus (Q^n_1 \cup Q^n_2)] = \phi \text{ .}$$

Therefore A_n is a polyhedral bounded surface. The analogue of (13) is satisfied.

Since $E_n \cap E_j = \phi$ for $n \neq j$, $(\operatorname{Bd} A_n) \cap (\operatorname{Bd} A_j) = \phi$. Clearly $z_1 \in J_{n+1} \subset J_n$ and $z_2 \in J'_{n+1} \subset J'_n$. It follows that $C \subset A_{n+1} \subset \operatorname{Int} A_n$. The analogue of (14) is satisfied.

It is easily seen that there are maps $h'\colon J_1 \times I \to J_1$ and $h''\colon J'_1 \times I \to J'_1$ such that for all $x \in \eta^1 \cup \eta^2$, $y \in \xi^1 \cup \xi^2$, $t \in I$,

(31) $h'(x, t) = \psi(x, t)$; $h''(y, t) = \psi(y, t)$; and such that h' and h'' satisfy analogues of (15) through (19):

(15′) h' is a strong deformation retraction of J_1 onto $\{z_1\}$, and for all n:

(16′) $h' | J_n \times [t_n, t_{n+1}]$ is a strong deformation retraction of J_n onto J_{n+1}.

(17′) $h' | \lambda_n \times [t_n, t_{n+1}]$ is an isotopy of λ_n onto λ_{n+1}.

(18′) If $x \in h'(\{y\} \times [t_n, t_{n+1}])$ for $y \in \lambda_n$, then $h'(\{x\} \times [t_n, t_{n+1}]) \subset h'(\{y\} \times [t_n, t_{n+1}])$ and $h'(x, t) = h'(y, t)$ for $t \in [t_{n+1}, 1]$.

(19′) For all $x \in \lambda_n$, $h'(\{x\} \times I)$ is an arc and $h'(\{x\} \times [0, 1))$ is a (noncompact) polyhedron.

Similar versions of (15′) through (19′) hold upon replacing $(h', \{J_n\}_{n=1}^{\infty}, z_1, \{\lambda_n\}_{n=1}^{\infty})$ by $(h'', \{J'_n\}_{n=1}^{\infty}, z_2, \{\mu_n\}_{n=1}^{\infty})$.

Define a map h on $A_1 \times I$ by

$$h(x, t) = \begin{cases} h'(x, t) & \text{if } \ x \in J_1 \ ; \\ h''(x, t) & \text{if } \ x \in J'_1 \ ; \\ \psi(x, t) & \text{otherwise} \ . \end{cases}$$

By (31), h is well-defined and continuous. From (17) and (18),

(32) if $x \in B_n \backslash (M_1 \cup M_2)$ then $\psi(\{x\} \times I) \subset B_n \backslash (M_1 \cup M_2 \backslash \{z_1, z_2\})$.

By (15), (15′), and (32), $h(A_1 \times I) = A_1$. Clearly $h(x, t) = x$ for all $(x, t) \in C \times I$, and $h_1(A_1) = C$. Thus h satisfies the analogue of (15).

For all n:

By (16), (16′), and (32), h satisfies the analogue of (16).

By (17), (17′), and (32), h satisfies the analogue of (17).

By (18) and (18′), h satisfies the analogue of (18).

By (19) and (19′), h satisfies the analogue of (19).

By (20) and our construction of E_n, h satisfies the analogue of (20). The proof of Theorem 2.5 is completed.

3. **Arcs.** Let X be a finite-dimensional compactum and let $\{C_0, C_1\} \subset 2_h^X$. Under what circumstances is there an arc in 2_h^X from C_0 to C_1? In [1], it was found that a necessary but insufficient condition is that C_0 and C_1 have the same homotopy type; and a sufficient but unnecessary condition is that C_0 and C_1 be isotopic in X. For $X = M$, we obtain a condition that is both necessary and sufficient:

THEOREM 3.1. *Let $\{C_0, C_1\} \subset 2_h^M \backslash \{M\}$. By 2.5, there exist $A_j \in 2_h^M (j = 0, 1)$ such that each component of A_j is a bounded surface, $C_j \subset \operatorname{Int} A_j$, and C_j is a strong deformation retract of A_j. Then there is an arc in 2_h^M from C_0 to C_1 if and only if there is an ambient isotopy of M taking A_0 onto A_1.*

First we prove:

LEMMA 3.2. *Suppose $C \in 2_h^M \backslash \{M\}$, and let $\{A_n\}_{n=1}^{\infty}$, $\{t_n\}_{n=1}^{\infty}$, and h be as in 2.5. Then there is an arc $\mathscr{A}$ in 2_h^M from A_1 to C containing* *each A_n such that if $A \in \mathscr{A}\backslash\{C\}$, each component of A is a bounded surface.*

Proof. Recall the notation in the statement of Theorem 2.5. In the proof of 2.5, we saw:

(1) $h|(\text{Bd } A_n) \times [t_n, t_{n+1}]$ is an isotopy of $\text{Bd } A_n$ onto $\text{Bd } A_{n+1}$.

It follows from (16) and (18) of the proof of 2.5 that

(2) if $x \in \text{Bd } A_n$ then $h(\{x\} \times [t_n, t_{n+1}]) = \gamma_x$ is an arc such that $\gamma_x\backslash\{x, h(x, t_{n+1})\} \subset (\text{Int } A_n)\backslash A_{n+1}$.

If $\varepsilon_n = \sup\{\text{diam } h(\{x\} \times I) | x \in A_n\}$, then $\lim \varepsilon_n = 0$, and by 1.1, $A_n \underset{\rho_h}{\to} C$, so it follows that there is a sequence of positive numbers δ_n such that

(3) $\lim \delta_n = 0$, and for all n, $s(A_n, 6\varepsilon_n, \delta_n)$.

Let P be a component of $\overline{A_n\backslash A_{n+1}}$. By 2.5(a), 2.5(b), 2.5(d), and 1.5, P is an annulus. Let the boundary curves of P be $\alpha_n \subset \text{Bd } A_n$ and $\alpha_{n+1} \subset \text{Bd } A_{n+1}$. There is a set $E = \{x_0, x_1, \cdots, x_{k-1}\} \subset \alpha_n$ of k distinct points numbered according to an orientation of α_n (let $x_k = x_0$) such that if β_j is the arc of α_n from x_{j-1} to x_j containing no other member of E, then $\text{diam } \beta_j < \varepsilon_n$. For each j, let $y_j = h(x_j, t_{n+1})$. By (2), $\gamma_j = h(\{x_j\} \times [t_n, t_{n+1}])$ is an arc from x_j to y_j such that $\gamma_j\backslash\{x_j, y_j\} \subset \text{Int } P$. By (1), the γ_j are pairwise disjoint for $j \in \{0, 1, \cdots, k-1\}(\gamma_k = \gamma_0)$ and (also by (1))$\zeta_j = h(\beta_j \times \{t_{n+1}\})$ is an arc of α_{n+1} from y_{j-1} to y_j not containing y_m if $y_m \notin \{y_{j-1}, y_j\}$. Clearly $\text{diam } \gamma_j \leqq \varepsilon_n$.

Let $\{y, y'\} \subset \zeta_j$. There exist $x, x' \in \beta_j$ such that $y = h(x, t_{n+1})$ and $y' = h(x', t_{n+1})$. Then $\rho(y, y') \leqq \rho(y, x) + \rho(x, x') + \rho(x', y') \leqq \varepsilon_n + \text{diam } \beta_j + \varepsilon_n < 3\varepsilon_n$. Therefore $\text{diam } \zeta_j < 3\varepsilon_n$.

Let S_j be the simple closed curve in P defined by $S_j = \gamma_{j-1} \cup \beta_j \cup \gamma_j \cup \zeta_j$. Then $\text{diam } S_j \leqq \text{diam } \gamma_{j-1} + \text{diam } \beta_j + \text{diam } \gamma_j + \text{diam } \zeta_j < \varepsilon_n + \varepsilon_n + \varepsilon_n + 3\varepsilon_n = 6\varepsilon_n$. By (3) and 1.6, S_j bounds a disk $K_j \subset A_n$ such that

(4) $\text{diam } K_j < \delta_n$.

Indeed $K_j \subset P$, for if K'_j is the disk in P bounded by S_j and $K'_j \neq K_j$, then $K_j \cap K'_j = S_j$ and $K_j \cup K'_j$ is a 2-sphere in A_n, which is impossible.

It is easily seen that there is a map $F: P \times I \to P$ that is a strongly contracting strong deformation retraction and a pseudoisotopy of P to α_{n+1} such that $F(K_j \times I) \subset K_j$ for all j. From (4) we have

(5) F_t is a δ_n-embedding for $0 \leqq t < 1$.

Apply the above construction to each component of $\overline{A_n\backslash A_{n+1}}$. In the above, $F_t | \alpha_{n+1} = \text{Id}_{\alpha_{n+1}}$ for all $t \in I$, so we may extend each F_t via the identity to obtain a map $F^n: A_n \times I \to A_n$ that is a strongly contracting strong deformation retraction and a pseudoisotopy of

A_n onto A_{n+1} moving no point by as much as δ_n. Let $a_n: I \to 2_h^M$ be defined by $a_n(t) = F^n(A_n \times \{t\})$. By 1.3, a_n is continuous for $0 \leqq t < 1$. By 1.1, a_n is continuous for $t = 1$.

Let $L: I \to 2_h^M$ be defined by

$$L(t) = \begin{cases} a_n\left[\dfrac{t - t_n}{t_{n+1} - t_n}\right] & \text{if} \quad t_n \leqq t \leqq t_{n+1} \,; \\ C & \text{if} \quad t = 1 \,. \end{cases}$$

Since $a_n(1) = A_{n+1} = a_{n+1}(0)$, L is well-defined; and L is continuous for $0 \leqq t < 1$. From (3), (5), and 1.2, L is continuous for $t = 1$. Since $L(0) = A_1$ and $L(1) = C$, $L(I)$ contains an arc in 2_h^M from A_1 to C. The second conclusion of the lemma follows from the fact that for all n, F^n is a pseudoisotopy of A_n onto A_{n+1}.

We show the existence of a basis with useful properties.

LEMMA 3.3. *Let $C \in 2_h^M \backslash \{M\}$ and let $\varepsilon > 0$. By* 1.1 *and* 2.5, *there exists A such that $\rho_h(A, C) < \varepsilon$, each component of A is a bounded surface, $C \subset \operatorname{Int} A$, and C is a strong deformation retract of A. There is a neighborhood $\mathscr{U}$ of C in 2_h^M such that $X \in \mathscr{U}$ implies $\rho_h(X, C) < \varepsilon$, $X \subset \operatorname{Int} A$, and X is a strong deformation retract of A. Further, if each component of $X \in \mathscr{U}$ is a bounded surface, then there is an ambient isotopy of M that carries A onto X.*

Proof. We may assume A is a polyhedron, and that ε is so small that two maps $f_0, f_1: C \to A$ such that $\rho(f_0, f_1) < \varepsilon$ are homotopic in A. Recall $[C]_M = \{X \in 2_h^M \mid X$ and C have the same homotopy type$\}$ is open. From 2.1 it follows that

$$\mathscr{U} = [C]_M \cap \{X \in 2_h^M \mid X \subset \operatorname{Int} A\} \cap \{X \in 2_h^M \mid \rho_h(X, C) < \varepsilon\}$$

is an open set in 2_h^M containing C.

We may assume C and A are connected (otherwise we apply the following by components). Let $X \in \mathscr{U}$. There is an ε-map $g: C \to X$. Let $i: C \to A$, $j: X \to A$ be inclusion maps. By choice of ε, $i_* = j_* \circ g_*$: $\Pi_1 C \to \Pi_1 A$. By choice of A, i_* is an isomorphism. Therefore $j_*: \Pi_1 X \to \Pi_1 A$ is a surjective homomorphism. But $\{X, A\} \subset [C]_M$, so $\Pi_1 X$ and $\Pi_1 A$ are isomorphic. Since A is a bounded surface, $\Pi_1 A$ is a finitely generated free group. Therefore j_* is an isomorphism (see [10], p. 59).

Recall the definition of ΔX given in §1. Since X and A have the same homotopy type, $\Delta X = \Delta A$. But $\Delta A \leqq 1$, since if A is a disk it has the homotopy of a point, while otherwise A has the homotopy type of a wedge of finitely many simple closed curves. With $N = \Delta A \leqq 1$, we apply Whitehead's theorem ([12], 1, p. 1133) and conclude $j: X \to A$ is a homotopy equivalence.

By 1.1 and 2.5 there is a polyhedral bounded surface $B \in \mathscr{U}$ such that $X \subset \operatorname{Int} B$ and X is a strong deformation retract of B. Applying the above to B, we conclude the inclusion of B into A is a homotopy equivalence. Hence B is a strong deformation retract of A (see [6], 3.2, p. 6). Thus X is a strong deformation retract of A.

If $X \in \mathscr{U}$ is a bounded surface, then by 1.5 each component of $\overline{A \backslash X}$ is an annulus. Let S be a component of Bd A. Let A' be the component of $\overline{A \backslash X}$ containing S. Let S' be the component of Bd A' that lies in X. There are annuli A_1 and A_2 that collar S in $\overline{M \backslash A}$ and S' in X respectively. Then $A'' = A_1 \cup A' \cup A_2$ is an annulus. There is an isotopy $h: A'' \times I \to A''$ of A'' onto itself such that $h_1(A' \cup A_2) = A_2$, $h_1(A_1) = A' \cup A_1$, and $h(z, t) = z$ for all $(z, t) \in (\operatorname{Bd} A'') \times I$. Apply this construction to each component of $\overline{A \backslash X}$ and extend via the identity on $M \backslash (\overline{A \backslash X})$ to get an ambient isotopy of M that carries A onto X.

Proof of Theorem 3.1. Suppose there is an ambient isotopy of M taking A_0 onto A_1. By 1.3, there is an arc in 2_h^M from A_0 to A_1. By 3.2, there are arcs in 2_h^M from A_0 to C_0 and from A_1 to C_1. Hence there is an arc in 2_h^M from C_0 to C_1.

Conversely, suppose there is an embedding $p: I \to 2_h^M$ such that $p(0) = C_0$ and $p(1) = C_1$. Since $p(I)$ is compact, 3.3 implies that there exist $0 \leqq t_0 < t_1 < \cdots < t_m \leqq 1$; $A_{t_n} \in 2_h^M$ such that each component of A_{t_n} is a bounded surface; and neighborhoods $\mathscr{U}_n$ of $p(t_n)$ in 2_h^M such that if $X \in \mathscr{U}_n$ and each component of X is a bounded surface then there is an ambient isotopy of M taking A_{t_n} onto X, and such that $\mathscr{U}_n \cap \mathscr{U}_{n+1} \neq \phi$ and $p(I) \subset \bigcup_{n=0}^m \mathscr{U}_n$. Further, 3.3 enables us to assume that $A_0 = A_{t_0}$ and $A_1 = A_{t_m}$.

By 1.1 and 2.5, for each $n < m$ there exists $B_n \in \mathscr{U}_n \cap \mathscr{U}_{n+1}$ such that each component of B_n is a bounded surface. There are ambient isotopies of M taking A_{t_n} and $A_{t_{n+1}}$ onto B_n. Therefore there is an ambient isotopy of M taking A_{t_n} onto $A_{t_{n+1}}$. Hence there is an ambient isotopy of M taking $A_0 = A_{t_0}$ onto $A_{t_m} = A_1$.

4. **Global properties.** The spaces $D(N)$ and $L(N)$ of deformation retracts (respectively, compact AR subsets) of a compact 2-manifold N were studied by Wagner in [11]. The topologies of these spaces may be described thus: $A_n \xrightarrow[D(N)]{} C$ ($A_n \xrightarrow[L(N)]{} C$) if and only if there are maps $r_0: N \to N$, $r_n: N \to N$ that are deformation retractions (that are retractions) of N onto C and A_n respectively such that $r_n \to r_0$ uniformly on N. We show these spaces are closely related to 2_h^M.

We will need the following lemma. In both its statement and its proof, it is similar to ([2], 3.1, pp. 212–213).

LEMMA 4.1. *If* $C \in 2_h^M \backslash \{M\}$, C *is connected, and* $\varepsilon > 0$, *there is a* $\delta > 0$ *and a neighborhood* $\mathscr{U}$ *of* C *in* 2_h^M *such that if* $\{A, B\} \subset \mathscr{U}$, $B \subset A$, *and* A *is a bounded surface, then every pair of points in* $\operatorname{Bd} A$ *that can be joined by a* δ*-arc in* $M\backslash B$ *can be joined by an* ε*-arc in* $\operatorname{Bd} A$.

Proof. By 3.3, there is a neighborhood $\mathscr{U}_1$ of C in 2_h^M and a bounded surface $N \subset M$ such that for all $X \in \mathscr{U}_1$ we have $X \subset \operatorname{Int} N$ and X is a strong deformation retract of N.

Since M is an ANR, there exists $\eta > 0$ such that $s(M, \eta, \varepsilon/4)$. Also there is a $\delta > 0$ such that:

(1) If N has more than one boundary curve then

$$\delta < \min \{\rho(S, T) \mid S \text{ and } T \text{ are distinct boundary curves of } N\} .$$

(2) $\delta < 1/2 \min \{\eta, \varepsilon\}$.

(3) There is a neighborhood $\mathscr{U}_2$ of C in 2_h^M such that if $X \in \mathscr{U}_2$ then $s(X, \delta, \eta/2)$.

Let $\mathscr{U}_3 = \{X \in 2_h^M \mid \rho_h(X, C) < \delta/2\}$. Let $\mathscr{U} = \mathscr{U}_1 \cap \mathscr{U}_2 \cap \mathscr{U}_3$. Clearly $\mathscr{U}$ is a neighborhood of C in 2_h^M.

Suppose $\{A, B\} \subset \mathscr{U}$ such that $B \subset A$ and A is a bounded surface. From 1.4 (with $R = B$) it follows that B separates each pair of boundary curves of N in N. Since each component of $\overline{N\backslash A}$ is an annulus, it follows that

(4) B separates each pair of distinct boundary curves of A in A.

Let p and q be distinct points of $\operatorname{Bd} A$ such that there is a δ-arc β from p to q in $M\backslash B$.

Suppose β meets distinct boundary curves T_1 and T_2 of A. It follows from (4) that β must contain a δ-arc β' from $p' \in T_1$ to $q' \in T_2$ such that $\beta' \cap A = \{p', q'\}$. For $n = 1, 2$, let B_n be the annular component of $\overline{N\backslash A}$ containing T_n and let T'_n be the component of $\operatorname{Bd} N$ that is contained in B_n. By 1.4, $T'_1 \neq T'_2$. By (4) and 1.4, there are distinct components B'_n of $N\backslash B$ such that $\operatorname{Int} B_n \subset B'_n$. Then $T_n \subset B_n \subset \overline{B'_n}$, so we must have $\beta' \cap \operatorname{Bd} B'_n \neq \phi$. Since $\operatorname{Bd} B'_n \subset T'_n \cup \operatorname{Bd} B$ and $\beta' \cap \operatorname{Bd} B \subset \beta' \cap B = \phi$, we have $\beta' \cap T'_n \neq \phi$ for $n = 1, 2$. The latter contradicts (1). We conclude that $\beta \cap \operatorname{Bd} A$ is contained in a single component J of $\operatorname{Bd} A$.

By $N_s(\beta)$ we will mean the set of all points in M whose distance from β is less than s. Since $\operatorname{diam} \beta < \delta$, there is an $s > 0$ such that $\operatorname{diam} N_s(\beta) < \delta$. By the proof of 2.4, we may assume $\beta \cap J$ has finitely many components. If γ is a component of $\beta \cap J$

that is not a single point, then γ is an arc with endpoints b, c. There is an arc $\gamma' \subset N_s(\beta)\backslash B$ from b to c such that $\gamma' \cap J = \{b, c\}$. If $\gamma_1, \cdots, \gamma_m$ are the components of $\beta \cap J$ that are arcs, then $\beta_1 = (\beta \backslash \bigcup_{n=1}^m \gamma_n) \cup \bigcup_{n=1}^m \gamma'_n$ meets J in but finitely many points and (by choice of s) contains a δ-arc β_2 from p to q. Thus (by replacing β by β_2 if necessary) we may assume $\beta \cap J$ is a finite set.

Suppose $\beta \cap J = \{p, q\}$. We consider two cases:

(I) Suppose $\beta\backslash\{p, q\} \subset M\backslash A$. Since diam $\beta < \delta$, (3) implies there is an $\eta/2$-arc ξ in A from p to q. We assume $\xi\backslash\{p, q\} \subset \operatorname{Int} A$. Then $K = \beta \cup \xi$ is a simple closed curve and diam $K < \delta + \eta/2 < \eta$ (by (2)). By 1.6 and our choice of η, K bounds a disk $L \subset M$ with diam $L < \varepsilon/4$.

Let $x \in \beta\backslash\{p, q\}$, $y \in \xi\backslash\{p, q\}$. For any fixed $r > 0$, $B(x, r) \cap (M\backslash A) \neq \phi \neq B(y, r) \cap \operatorname{Int} A$. Suppose L fails to contain an arc of J from p to q. Our choices of β and ξ imply $J \cap K = J \cap \operatorname{Bd} L = \{p, q\}$, so the assumption implies $J \cap L = \{p, q\}$. Thus $\phi = J \cap \operatorname{Int} L = (\operatorname{Bd} A) \cap \operatorname{Int} L$. Since $\phi \neq B(y, r) \cap \operatorname{Int} A$ meets $\operatorname{Int} L \cap \operatorname{Int} A$ and $\phi \neq B(x, r) \cap (M\backslash A)$ meets $\operatorname{Int} L \cap (M\backslash A)$, it follows that $\operatorname{Int} L = (\operatorname{Int} L \cap \operatorname{Int} A) \cup (\operatorname{Int} L \cap (M\backslash A))$ is disconnected. This is impossible, so L contains an arc of J from p to q that lies in $N_{\varepsilon/4}(\beta)$ (since $\beta \subset L$ and diam $L < \varepsilon/4$).

(II) Suppose $\beta\backslash\{p, q\} \subset \operatorname{Int} A$. Then $A = A_1 \cup A_2$, where A_1 is a bounded surface containing B, A_2 is (by (4) and the fact that $\beta \subset M\backslash B$) a bounded surface whose boundary is the union of β and an arc of J from p to q, and $A_1 \cap A_2 = \beta$. By choice of $\mathscr{U}_3$, there is a δ-map $f: A \to B$. If $z \in A_2$ then $f(z) \in B \subset A_1$, so by (3) there is an $\eta/2$-arc $\zeta \subset A$ from z to $f(z)$. Clearly ζ meets β. Hence $A_2 \subset N_{\eta/2}(\beta)$. In particular, the arc of J from p to q that lies in Bd A_2 must lie in $N_{\eta/2}(\beta)$.

Our choice of η implies $\eta/2 < \varepsilon/4$. In both (I) and (II), J contains an arc from p to q that lies in $N_{\varepsilon/4}(\beta)$.

More generally, if $\beta \cap J = \{p = p_1, \cdots, p_k = q\}$ where the p_n are numbered in order from p to q along β, then each subarc $\overline{p_n p_{n+1}}$ of β satisfies the condition of (I) or (II). For each $n < k$ there is an arc ζ_n of J from p_n to p_{n+1} in $N_{\varepsilon/4}(\beta)$. There is an arc $\zeta_0 \subset \bigcup_{n=1}^{k-1} \zeta_n \subset N_{\varepsilon/4}(\beta)$ of J from p to q. Observe diam $\zeta_0 \leqq$ diam $N_{\varepsilon/4}(\beta) \leqq \varepsilon/2 +$ diam $\beta < \varepsilon/2 + \delta < \varepsilon$ (by (2)).

We now strengthen 3.3.

LEMMA 4.2. *Let $C \in 2_h^M\backslash\{M\}$, $\varepsilon > 0$. Then there exist $N \in 2_h^M$ and a neighborhood $\mathscr{U}$ of C in 2_h^M such that each component of N is a bounded surface and such that for all $X \in \mathscr{U}$, $\rho_h(X, C) < \varepsilon$, $X \subset \operatorname{Int} N$, and there is a strong deformation retraction $h: N \times I \to N$ of N onto X such that for each $t \in I$, h_t is an ε-map.*

Proof. It follows from ([2], 2.1, p. 210) that there is no loss of generality in assuming C is connected.

There is a neighborhood $\mathscr{U}_1$ of C in 2^M_h and a $\delta > 0$ such that

(1) if $X \in \mathscr{U}_1$ then $s(X, \delta, \varepsilon/2)$.

There are positive numbers δ_1 and δ_2 such that

(2) $17\delta_1 + \delta_2 < \delta$

and (by 4.1) such that

(3) there is a neighborhood $\mathscr{U}_2$ of C in 2^M_h such that if $\{X, Y\} \subset \mathscr{U}_2$, $X \subset Y$, and Y is a bounded surface, then each pair of points in $\mathrm{Bd}\, Y$ joined by a $7\delta_1$-arc in $M \backslash X$ can be joined by a δ_2-arc in $\mathrm{Bd}\, Y$.

Clearly

(4) there is a neighborhood $\mathscr{U}_3$ of C in 2^M_h and a $\delta_3 > 0$ such that if $X \in \mathscr{U}_3$ then $s(X, \delta_3, \delta_1)$.

Let $\mathscr{U}_4 = \{X \in 2^M_h \mid \rho_h(X, C) < (1/2)\delta_3\}$. By 3.3 there exist a bounded surface $N \in \bigcap_{n=1}^4 \mathscr{U}_n$ and a neighborhood $\mathscr{U}_5$ of C in 2^M_h such that $X \in \mathscr{U}_5$ implies $X \subset \mathrm{Int}\, N$ and X is a strong deformation retract of N.

Let $\mathscr{U} = \bigcap_{n=1}^5 \mathscr{U}_n$. Clearly $\mathscr{U}$ is a neighborhood of C in 2^M_h. Fix $X \in \mathscr{U}$. By 1.1 and 2.5 there is a bounded surface $B \in \mathscr{U}$ such that $X \subset \mathrm{Int}\, B$ and there is a strong deformation retraction $g\colon B \times I \to B$ of B onto X such that g_t is an $\varepsilon/2$-map for all $t \in I$. Thus it suffices to show the existence of a strong deformation retraction $H\colon N \times I \to N$ of N onto B such that H_t is an $\varepsilon/2$-map for all $t \in I$.

By choice of $\mathscr{U}_4$ we have $\rho_h(N, B) < \delta_3$. It follows from (4) and our choice of $\mathscr{U}_5$ that for all $x \in \mathrm{Bd}\, N$ there is a δ_1-arc in N from x to some $y \in \mathrm{Bd}\, B$. By 1.5, each component P of $\overline{N \backslash B}$ is an annulus. Let $\mathrm{Bd}\, P = S \cup S'$, where S and S' are boundary curves of N and B respectively. It follows from 1.4 that B separates distinct boundary curves of N in N. Thus

(5) for all $x \in S$, there is a δ_1-arc β from x to some $y \in S'$, and we may assume $\beta \backslash \{x, y\} \subset \mathrm{Int}\, P$.

Suppose $\mathrm{diam}\, S < \delta$. By (1) and 1.6, S bounds a disk of diameter less that $\delta/2$ in N. Since N is connected, the disk must be N itself. In this case it is clear that we have a strong deformation $H\colon N \times I \to N$ of N onto B such that H_t is an $\varepsilon/2$-map for all $t \in I$. Thus we assume

(6) $\mathrm{diam}\, S \geqq \delta$.

There is a set $G = \{x_1, \cdots, x_k\} \subset S$ of k distinct points numbered according to an orientation of S (let $x_0 = x_k$) such that if α_p is the arc of S from x_{p-1} to x_p containing no other member of G, then

(7) $2\delta_1 < \rho(x_{p-1}, x_p)$ and $\mathrm{diam}\, \alpha_p < 5\delta_1$.

By (2) and (6), $k > 1$.

By (5), for each p there exists $y_p \in S'(y_0 = y_k)$ and a δ_1-arc $\beta_p (\beta_0 = \beta_k)$ in P from x_p to y_p such that $\beta_p \backslash \{x_p, y_p\} \subset \mathrm{Int}\, P$. By (7), $\beta_{p-1} \cap \beta_p = \phi$.

Since P is an annulus, it follows that the β_p are pairwise disjoint. By choice of B, $\beta_{p-1} \cup \alpha_p \cup \beta_p$ is an arc in $M\backslash X$ from $y_{p-1} \in S'$ to $y_p \in S'$, and (7) implies

(8) $\operatorname{diam}(\beta_{p-1} \cup \alpha_p \cup \beta_p) < \delta_1 + 5\delta_1 + \delta_1 = 7\delta_1$.

By (3), there is a δ_2-arc γ_p of S' from y_{p-1} to y_p.

We claim γ_p does not contain y_q if $y_q \notin \{y_{p-1}, y_p\}$. For it follows from the disjointness of the β_p that the points $y_1, \cdots, y_k$ are numbered according to an orientation of S'. If some γ_p contains y_q for $y_q \notin \{y_{p-1}, y_p\}$, then $\{y_1, \cdots, y_k\} \subset \gamma_p$. Let $x \in \alpha_n \neq \alpha_p$. Then $\rho(x, \gamma_p) \leqq \rho(x, y_n) \leqq \rho(x, x_n) + \rho(x_n, y_n) \leqq \operatorname{diam} \alpha_n + \operatorname{diam} \beta_n < 5\delta_1 + \delta_1 = 6\delta_1$. It follows that $\operatorname{diam} S \leqq \operatorname{diam} \alpha_p + \operatorname{diam}(S\backslash\alpha_p) < 5\delta_1 + \operatorname{diam} N_{6\delta_1}(\gamma_p) \leqq 5\delta_1 + 12\delta_1 + \operatorname{diam} \gamma_p < 17\delta_1 + \delta_2 < \delta$ (by (3)), contrary to (6). The claim is established.

Then $L_p = \beta_{p-1} \cup \alpha_p \cup \beta_p \cup \gamma_p (p = 1, \cdots, k)$ is a simple closed curve in N. By (8) and our choice of γ_p, $\operatorname{diam} L_p < 7\delta_1 + \delta_2$. By (1), (2), and 1.6, L_p bounds a disk D_p in N with $\operatorname{diam} D_p < \varepsilon/2$. As in the proof of 3.2, D_p is the disk of P bounded by L_p.

As in 3.2, there is a strong deformation retraction $K: P \times I \to P$ of P onto S' such that $K(D_p \times I) = D_p$ for all p. Thus K_t is an $\varepsilon/2$-map for all $t \in I$. As in 3.2, K can be extended to a strong deformation retraction $H: N \times I \to N$ of N onto B such that H_t is an $\varepsilon/2$-map for all $t \in I$.

THEOREM 4.3. *Let $\{A_n\}_{n=1}^{\infty}$ and C be points of $2_h^M \backslash \{M\}$. Then $A_n \underset{\rho_h}{\to} C$ if and only if there exists $N \in 2_h^M$ such that each component of N is a bounded surface and $A_n \xrightarrow[D(N)]{} C$.*

Proof. By 3.3, there is a compact 2-manifold with boundary $N \in 2_h^M$ and a neighborhood $\mathscr{U}$ of C in 2_h^M such that if $X \in \mathscr{U}$ then $X \subset \operatorname{Int} N$ and X is a strong deformation retract of N.

Suppose $A_n \underset{\rho_h}{\to} C$. Let $\varepsilon > 0$. By 4.2 there is a compact 2-manifold with boundary $B \in \mathscr{U}$ and a neighborhood $\mathscr{V}$ of C in 2_h^M with $\mathscr{V} \subset \mathscr{U}$ such that if $X \in \mathscr{V}$ then $X \subset \operatorname{Int} B$ and there is an $\varepsilon/2$-map $r: B \to B$ that is a strong deformation retraction of B onto X. Choose an m such that $n > m$ implies $A_n \in \mathscr{V}$.

Let $f: N \to N$ be a deformation retraction of N onto B. Let $f_n: B \to B$ be an $\varepsilon/2$-map that is a deformation retraction of B onto A_n for $n > m$. Let $f_0: B \to B$ be an $\varepsilon/2$-map that is a deformation retraction of B onto C. Define $r_n: N \to N$ for $n = 0, n > m$ by $r_n(x) = f_n(f(x))$. For all $x \in N$ and $n > m$, $\rho(r_n(x), r_0(x)) < \varepsilon$. Hence $A_n \xrightarrow[D(N)]{} C$.

Conversely, suppose $A_n \xrightarrow[D(N)]{} C$. There exist deformation retractions $r_n: N \to N$ of N onto $A_{n'}$, $r_0: N \to N$ of N onto C such that $r_n \to r_0$ uniformly on N.

If $x \in C$, $\rho(x, r_n(x)) \to \rho(x, r_0(x)) = 0$. Hence $\rho(x, A_n) \to 0$.

If $x_n \in A_n$, $\rho(x_n, r_0(x_n)) = \rho(r_n(x_n), r_0(x_n)) \to 0$. Hence $\rho(x_n, C) \to 0$. We conclude $A_n \underset{\rho_s}{\to} C$.

Let $\varepsilon > 0$. Let $\delta > 0$ be such that if $\{x, y\} \subset N$ and $\rho(x, y) < \delta$ then $\rho(r_0(x), r_0(y)) < \varepsilon/6$. Let $\delta' > 0$ be such that $s(N, \delta', \delta)$. Let $m > 0$ be such that $n > m$ implies that for all $x \in N$, $\rho(r_n(x), r_0(x)) < \varepsilon/6$.

If $\{x, y\} \subset N$, $\rho(x, y) < \delta$, and $n > m$, then $\rho(r_n(x), r_n(y)) \leqq \rho(r_n(x), r_0(x)) + \rho(r_0(x), r_0(y)) + \rho(r_0(y), r_n(y)) < \varepsilon/6 + \varepsilon/6 + \varepsilon/6 = \varepsilon/2$.

Let $K \subset A_n \subset N$, diam $K < \delta'$. There is a contraction $h\colon K \times I \to N$ of K to a point such that diam $h(K \times I) < \delta$. Therefore, for $n > m$, $r_n \circ h\colon K \times I \to N$ is a contraction of K to a point such that $r_n \circ h(K \times I) \subset A_n$ and diam $(r_n \circ h(K \times I)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence $s(A_n, \delta', \varepsilon)$ for $n > m$, so $A_n \underset{\rho_h}{\to} C$.

THEOREM 4.4. *2^M_h is an* ANR ($\mathscr{M}$).

Proof. If N and $\mathscr{U}$ are as above, the previous theorem implies the inclusion of the set $\mathscr{U}$ into $D(N)$ is an open embedding. Since $D(N)$ is an ANR ($\mathscr{M}$) ([11], 5.5, p. 389), it follows ([9], 3.1, p. 391) that $\mathscr{U}$ is an ANR($\mathscr{M}$). Since M is an isolated point of 2^M_h (because $[M]_M = \{M\}$) the assertion follows from the fact that a local ANR ($\mathscr{M}$) is an ANR($\mathscr{M}$) ([9], 3.3, p. 392).

THEOREM 4.5. *Let $AR^M_h = \{X \in 2^M_h \mid X$ is an* AR}. *Then AR^M_h is a component of 2^M_h.*

Proof. Since AR^M_h is the set of all members of 2^M_h with the homotopy type of a point, AR^M_h is open and closed in 2^M_h, and thus is a union of components of 2^M_h. We must show AR^M_h is connected.

Let $C_n \in AR^M_h (n = 0, 1)$. By 3.2 there is an arc in AR^M_h from C_n to N_n, where N_n is a disk. Let $p_n \in N$ and let $h^n\colon N_n \times I \to N_n$ be a pseudoisotopy of N_n onto p_n. Then (using 1.3) $\{h^n(N_n \times \{t\}) \mid t \in I\}$ contains an arc in AR^M_h from N_n to $\{p_n\}$. Let $h\colon I \to M$ be a map such that $h(0) = p_0$ and $h(1) = p_1$. By 1.3, $\{\{h(t)\} \mid t \in I\}$ contains an arc in AR^M_h from $\{p_0\}$ to $\{p_1\}$. Thus there is an arc in AR^M_h from C_0 to C_1.

THEOREM 4.6. *$AR^M_h = L(M)$ as topological spaces.*

Proof. Clearly they are equal as sets. Let $C \in AR^M_h$. As above, there is a disk $N \subset M$ such that $C \subset \operatorname{Int} N$ and C is a strong deformation retract of N. We know $A_n \underset{\rho_h}{\to} C$ if and only if $A_n \xrightarrow[D(N)]{} C$.

But $A_n \underset{D(N)}{\longrightarrow} C$ if and only if $A_n \underset{L(M)}{\longrightarrow} C$ ([11], 5.4, p. 388).

Clearly the map $j: M \to AR_h^M$ defined by $j(x) = \{x\}$ is an embedding. We have the following:

COROLLARY 4.7. *$j(M)$ is a deformation retract of AR_h^M. Thus AR_h^M has the same homotopy type as M.*

Proof. This follows from Theorem 4.6 and ([11], 5.5, p. 389).

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UNIVERSITY OF GEORGIA

Current address: Salem College
Winston-Salem, NC 27108

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