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**MULTIPLIERS FOR $|C, 1|$ SUMMABILITY OF FOURIER
SERIES**

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In the present paper we improve the conditions of all previously known theorems on the absolute $(C, 1)$ summability factors of Fourier series.

1. Let the formal expansion of a function $f(x)$, periodic with period 2π and integrable in the sense of Lebesgue over $[-\pi, \pi]$, in a Fourier-trigonometric series be given by

$$(1.1) \quad f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x).$$

We write

$$\phi(u) = f(x + u) + f(x - u) - 2f(x)$$

and throughout this paper A will denote a positive constant, not necessarily the same at each occurrence.

Whittaker [5], in 1930, proved that the series

$$\sum_{n=1}^{\infty} A_n(x)/n^\alpha, \quad \alpha > 0,$$

is summable $|A|$ almost everywhere.

Later, Prasad [4] demonstrated that the series

$$\sum_{n=n_0}^{\infty} A_n(x)/\mu_n,$$

where

$$\mu_n = \left(\prod_{\nu=1}^{k-1} \log^\nu n \right) (\log^k n)^{1+\varepsilon}, \quad \log^k n_0 > 0, \quad \varepsilon > 0,$$

and

$$\log^k n = \log(\log^{k-1} n), \dots, \log^2 n = \log \log n;$$

is summable $|A|$ almost everywhere.

Chow [2], on the other hand, has shown that the series $\sum \lambda_n A_n(x)$ is summable $|C, 1|$ almost everywhere, provided $\{\lambda_n\}$ is a convex sequence satisfying the condition $\sum n^{-1 \cdot \lambda_n} < \infty$.

Cheng [1], in 1948, established the following:

THEOREM A. *If*

$$\Phi(t) \equiv \int_0^t |\phi(u)| du = O(t)$$

as $t \rightarrow 0$, then the series

$$\sum_{n=2}^{\infty} A_n(x)/(\log n)^{1+\delta}, \quad \delta > 0,$$

is summable $|C, \alpha|$, $\alpha > 1$.

In a recent paper, Hsiang [3] has proved the following theorems:

THEOREM B. *If*

$$(1.2) \quad \Phi(t) = O(t) \quad (t \rightarrow +0),$$

then the series $\sum_{n=1}^{\infty} A_n(x)/n^\alpha$ is summable $|C, 1|$ for every $\alpha > 0$.

THEOREM C. *If*

$$(1.3) \quad \Phi(t) = O\left\{t/\prod_{\nu=1}^k \log^\nu(1/t)\right\}$$

as $t \rightarrow +0$, then the series

$$(1.4) \quad \sum_{n=0}^{\infty} A_n(x) / \left(\prod_{\nu=1}^{k-1} \log^\nu n\right) (\log^k n)^{1+\varepsilon}$$

is summable $|C, 1|$ for every $\varepsilon > 0$.

In the present paper we prove the following theorem, which includes the theorem of Cheng and both the theorems of Hsiang:

THEOREM. *If*

$$(1.5) \quad \varphi(t) \equiv \int_t^\delta \frac{|\phi(u)|}{u} du = O\{(\log^k(1/t))^\eta\} \quad \text{as } t \rightarrow +0,$$

$0 < \delta \leq \pi$, then the series (1.4) is summable $|C, 1|$ for $0 < \eta < \varepsilon$.

The conditions of our theorem are less stringent than those of Cheng and Hsiang.

2. The proof of the theorem is based on the following lemmas:

LEMMA 1. *Let $S_n(x)$ be the n th partial sum of the series (1.1), then under the condition (1.5), we have*

$$(2.1) \quad \sum_{\nu=0}^n |S_\nu(x) - f(x)| = O\{n(\log^k n)^\eta\}.$$

Proof. Let $\varepsilon_\nu = \text{sign}[S_\nu(x) - f(x)]$, so that $\varepsilon_\nu = \pm 1$ and it depends only upon x and ν , and is independent of t . Also, we write

$$K_n(t) = \sum_{\nu=0}^n \varepsilon_\nu \sin \nu t .$$

Thus, we have

$$\begin{aligned} \sum_{\nu=0}^n |S_\nu(x) - f(x)| &= \frac{2}{\pi} \int_0^\pi \frac{\phi(t)}{t} K_n(t) dt + o(n) \\ &= \frac{2}{\pi} \left[\int_0^{1/n} + \int_{1/n}^\delta + \int_\delta^\pi \right] + o(n) \\ &= I_1 + I_2 + I_3 + o(n) , \end{aligned}$$

say. Now,

$$\begin{aligned} (2.3) \quad I_1 &\leq \int_0^{1/n} |\phi(t)| \cdot O(n^2) dt \\ &= O(n^2) \int_0^{1/n} -t\varphi'(t) dt , \quad \varphi'(t) = \frac{d}{dt} \varphi(t) . \\ &= O(n^2) [-t\varphi(t)]_0^{1/n} + O(n^2) \int_0^{1/n} \varphi(t) dt \\ &= O\{n(\log^k n)^\gamma\} . \end{aligned}$$

Also, for $nt \geq 1$, we have

$$\begin{aligned} (2.4) \quad I_2 &\leq \int_{1/n}^\delta \frac{|\phi(t)|}{t} \cdot n dt \\ &= O\{n(\log^k n)^\gamma\} . \end{aligned}$$

Since, by Riemann-Lebesgue theorem,

$$\int_\delta^\pi \frac{\phi(t)}{t} \sin ntdt = o(1) ,$$

we have

$$(2.5) \quad I_3 = O(n) .$$

Combining (2.1), (2.2), \dots , (2.5), the lemma follows.

LEMMA 2. *Let*

$$t_n(x) = \frac{1}{(n+1)} \sum_{\nu=1}^n \nu A_\nu(x) .$$

Then

$$T_n(x) \equiv \sum_{\nu=1}^n |t_\nu(x)| = O\{n(\log^k n)^\gamma\}$$

and

$$\sum_{n=n_0}^{\infty} (\mu_n)^{-1} \cdot n^{-1} |t_n(x)| < \infty .$$

Proof. Let

$$\sigma_n(x) = \frac{1}{(n + 1)} \sum_{\nu=0}^n S_\nu(x) .$$

Thus, we have

$$\begin{aligned} \sigma_n(x) - f(x) &= \frac{1}{(n + 1)} \sum_{\nu=0}^n \{S_\nu(x) - f(x)\} \\ (2.6) \quad \implies |\sigma_n(x) - f(x)| &\leq \frac{1}{(n + 1)} \sum_{\nu=0}^n |S_\nu(x) - f(x)| \\ &= O\{(\log^k n)^\eta\} \end{aligned}$$

by Lemma 1.

Therefore, we find that

$$\begin{aligned} (2.7) \quad T_n(x) &= \sum_{\nu=1}^n |t_\nu(x)| \\ &= \sum_{\nu=1}^n |S_\nu(x) - \sigma_\nu(x)| \\ &\leq \sum_{\nu=1}^n |S_\nu(x) - f(x)| + \sum_{\nu=1}^n |\sigma_\nu(x) - f(x)| \\ &= O[n(\log^k n)^\eta] \end{aligned}$$

by (2.6) and Lemma 1.

Finally, by Abel's transformation, we have

$$\begin{aligned} (2.8) \quad \sum_{n=m}^M (\mu_n)^{-1} \cdot n^{-1} |t_n(x)| &= \sum_{n=m}^{M-1} T_n(x) \Delta\{(\mu_n)^{-1} \cdot n^{-1}\} \\ &\quad - (\mu_{m-1})^{-1} (m - 1)^{-1} T_{m-1}(n) + \mu_M^{-1} \cdot M^{-1} T_M(x) \\ &= \sum_{n=m}^{M-1} \Delta\{(\mu_n)^{-1}\} \cdot n^{-1} T_n(x) \\ &\quad + \sum_{m=m}^{M-1} (\mu_{n+1})^{-1} \cdot n^{-1} (n + 1)^{-1} T_n(x) + O(1) \\ &= \sum_{n=m}^{M-1} \Delta\{(\mu_n)^{-1}\} \cdot (\log^k n)^\eta \\ &\quad + \sum_{n=m}^{M-1} (\mu_{n+1})^{-1} (n + 1)^{-1} (\log^k n)^\eta + O(1) \\ &\leq \sum_{n=m}^{M-1} \frac{A \cdot (\log^k n)^\eta}{n \left(\prod_{\mu=1}^{k-1} \log^\mu n \right) (\log^k n)^{1+\varepsilon}} + O(1) \\ &= O(1) , \end{aligned}$$

for $m \rightarrow \infty$ and $M \rightarrow \infty$.

In view of (2.7) and (2.8) the lemma is proved.

3. Proof of the theorem. Let $\tau_n(x)$ denotes the n th Cesàro mean of the sequence $\{n(\mu_n^{-1}) \cdot A_n(x)\}$.

By Abel's transformation, we have

$$\begin{aligned}
 \tau_n(x) &= \frac{1}{(n+1)} \sum_{\nu=n_0}^n \nu(\mu_\nu)^{-1} \cdot A_\nu(x) \\
 (3.1) \quad &= \frac{1}{(n+1)} \sum_{\nu=n_0}^{n-1} \Delta(\mu_\nu)^{-1} \cdot (\nu+1)t_\nu(x) + (\mu_n)^{-1}t_n(x) \\
 &= J_1^{(n)}(x) + J_2^{(n)}(x),
 \end{aligned}$$

say. Now, by Lemma 2, we find that

$$\begin{aligned}
 \sum_{n=m_0}^m J_1^{(n)}(x)/n &\leq \sum_{n=m_0}^m n^{-1}(n+1)^{-1} \sum_{\nu=n_0}^{n-1} \Delta(\mu_\nu)^{-1}(\nu+1)|t_\nu(x)|, \quad \log^k m_0 > 0 \\
 (3.2) \quad &\leq A \sum_{\nu=m_0}^m \Delta(\mu_\nu)^{-1}(\nu+1)|t_\nu(x)| \sum_{n=\nu+1}^m n^{-1}(n+1)^{-1} \\
 &\leq A \sum_{\nu=m_0}^m \Delta(\mu_\nu)^{-1}|t_\nu(x)| \\
 &= A \sum_{\nu=m_0}^{m-1} \Delta^2[(\mu_\nu)^{-1}] \cdot T_\nu(x) + \Delta(\mu_m^{-1})T_m(x) + O(1) \\
 &= O(1).
 \end{aligned}$$

Also, we have

$$\begin{aligned}
 (3.3) \quad \sum_{n=m_0}^m J_2^{(n)}(x)/n &\leq \sum_{n=m_0}^m (\mu_n)^{-1} \cdot n^{-1}t_n(x)| \\
 &= O(1).
 \end{aligned}$$

From (3.1), (3.2), and (3.3), we have

$$\sum_{n=m_0}^m \frac{|\tau_n(x)|}{n} = O(1).$$

This completes the proof of the theorem.

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