

Pacific Journal of Mathematics

**BOUNDED ANALYTIC FUNCTIONS ON UNBOUNDED
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Let R be an unbounded finite sheeted covering surface over an open Riemann surface with an exhaustion condition. In this paper, the necessary and sufficient condition in order that $H^\infty(R)$ separates the points of R is given in term of branch points, where $H^\infty(R)$ is the algebra of bounded analytic functions on R .

A covering surface R over a Riemann surface G is said to be *unbounded* if for any continuous curve $\lambda; z = z(t)$ ($0 \leq t \leq 1$) in G and any point p_0 in R with $\pi(p_0) = z(0)$ there exists a continuous curve $\lambda; p = p(t)$ ($0 \leq t \leq 1$) in R such that $p(0) = p_0$ and $z(t) = \pi \circ p(t)$ ($0 \leq t \leq 1$), where π is the projection of R onto G . For an unbounded covering surface R over G , the number of points of $\pi^{-1}(z)$ is constant $\leq \infty$ for every $z \in G$ where branch points are counted repeatedly according to their orders. If such a number n is finite, R is said to be *n-sheeted*.

In [2], Selberg proved the following: Let R be an unbounded n -sheeted covering surface over the unit disk $|z| < 1$ and $\{\zeta_k\}$ the projections of branch points with the order of branching n_k over ζ_k . Let z_0 be a point in the unit disk over which there exist no branch points of R . Then there exists a single valued bounded analytic function f on R such that f takes distinct values at any two points over z_0 if and only if $\sum n_k g(\zeta_k, z_0) < \infty$, where $g(\cdot, z_0)$ is the Green's function on $|z| < 1$ with pole at z_0 . Yamamura [5] extended the above result to the case where base surfaces are finitely connected plane regions.

On the other hand, Stanton [3] gave another proof of the above Selberg theorem using the Widom results [4]. The purpose of this paper is, by using the Widom-Stanton approach, to establish a result generalizing the Yamamura, and hence the Selberg, theorem to the case where the base surface $|z| < 1$ is replaced by certain surfaces which may be of infinite connectivity and genus.

1. Let R be an open Riemann surface of hyperbolic type and $g_R(\cdot, p)$ the Green's function on R with pole at p . Denote by $H^\infty(R)$ the algebra of single valued bounded analytic functions on R . For any $\alpha > 0$, set $R_\alpha = R(\alpha, p) = \{q \in R; g_R(q, p) > \alpha\}$. It is easily seen that, for each α , R_α is connected and $R - R_\alpha$ has no compact components. Suppose that each R_α is relatively compact in R . The

surface R with this property is referred to as being *regular*. Let $\beta_R(\alpha) = \beta_R(\alpha, p)$ be the first Betti number of R_α . Consider the quantity

$$m(R) = m(R, p) = \exp \left\{ - \int_0^\infty \beta_R(\alpha) d\alpha \right\}.$$

Widom proved that if $m(R) > 0$, then $H^\infty(R)$ separates the points of R , i.e., for any two distinct points p and q in R , there exists an $f \in H^\infty(R)$ such that $f(p) \neq f(q)$; it is also shown that $m(R) > 0$ does not depend on the choice of points p in R . (See [3] and [4].)

2. Hereafter, we suppose that G is an open Riemann surface of hyperbolic type and R is an unbounded n -sheeted covering surface over G . Then R is also hyperbolic. Let $g_G(\cdot, z_0)$ be the Green's function on G with pole at $z_0 \in G$. Suppose that G is regular and satisfies the condition

$$\int_0^\infty \beta_G(\alpha) d\alpha < \infty$$

where $\beta_G(\alpha) = \beta_G(\alpha, z_0)$ is the first Betti number of $G_\alpha = G(\alpha, z_0) = \{z \in G; g_G(z, z_0) > \alpha\}$. Then, R is also regular.

THEOREM. *Under the assumption stated above, the following four conditions are equivalent by pairs:*

(i) $m(R) > 0$;

(ii) $H^\infty(R)$ separates the points of R ;

(iii) for any $z_0 \in G - \{\zeta_k\}$, where $\{\zeta_k\}$ is the set of projections of branch points of R , there exists an f in $H^\infty(R)$ such that f takes distinct values at any two points of R over z_0 ;

(iv) $\sum n_k g_G(\zeta_k, z_0) < \infty$

for $z_0 \in G - \{\zeta_k\}$, where n_k is the order of branching over ζ_k .

Since (i) \rightarrow (ii) has been proved, we only have to show (ii) \rightarrow (iii), (iii) \rightarrow (iv), and (iv) \rightarrow (i).

3. *Proof of (ii) \rightarrow (iii).* Let π be the projection of R onto G and set $\pi^{-1}(z_0) = \{p_1, \dots, p_n\}$ (distinct points) for $z_0 \in G - \{\zeta_k\}$. Since $H^\infty(R)$ separates the points of R , there exists an f_{ij} in $H^\infty(R)$ such that $f_{ij}(p_i) \neq f_{ij}(p_j)$, for any pair (i, j) with $i \neq j$ and $1 \leq i, j \leq n$. We set

$$F_i = \prod_{\substack{1 \leq j \leq n \\ j \neq i}} (f_{ij} - f_{ij}(p_j)) \quad (1 \leq i \leq n)$$

and

$$f = \sum_{i=1}^n c_i F_i$$

for suitable constants c_i specified below. Observe that $f \in H^\infty(R)$. We can choose constants c_i so as to satisfy $f(p_i) \neq f(p_j)$ for any $i \neq j$.

Proof of (iii) \rightarrow (iv). Let z_0 be an arbitrary point in $G - \{\zeta_k\}$ and f a function in $H^\infty(R)$ such that f takes distinct values at any two points over z_0 . Then, by the well known argument of algebraic functions, it is seen that f satisfies the irreducible equation

$$f^n + g_1(z)f^{n-1} + \cdots + g_n(z) = 0$$

where $g_1(z), \dots, g_n(z)$ are in $H^\infty(G)$. Let $D(z)$ be the discriminant of this equation. Observe that $D(z)$ is in $H^\infty(G)$, vanishes at every point in $\{\zeta_k\}$, and does not vanish at z_0 . Hence, by the Lindelöf principle (cf. [1]), we conclude

$$\sum n_k g_G(\zeta_k, z_0) < \infty.$$

Proof of (iv) \rightarrow (i). Let z_0 be a point in $G - \{\zeta_k\}$ and p_0 a point in R with $\pi(p_0) = z_0$. We set

$$R_\alpha = R(\alpha, p_0) = \{p \in R; g_R(p, p_0) > \alpha\}$$

and

$$V_\alpha = \{p \in R; h(p) > \alpha\}$$

where $h(p) = g_G(\pi(p), z_0)$. Denote by $\beta_R(\alpha)$ and $\gamma(\alpha)$ the first Betti numbers of R_α and V_α , respectively. We fix $\alpha_0 (> 0)$ such that V_{α_0} is connected. Then, V_α is also connected for every $\alpha \leq \alpha_0$. By the maximum principle, $h(p) \geq g_R(p, p_0)$, and therefore $V_\alpha \supset R_\alpha$. Also, by the maximum principle, $V_\alpha - R_\alpha$ has no relatively compact components in V_α . Therefore

$$(1) \quad \gamma(\alpha) \geq \beta_R(\alpha).$$

Consider each α with $0 < \alpha \leq \alpha_0$ such that there exist no branch points of R on ∂V_α and no critical points of $g_G(z, z_0)$ on ∂G_α , where ∂V_α and ∂G_α are the boundaries of V_α and G_α , respectively, and $G_\alpha = \{z \in G; g_G(z, z_0) > \alpha\}$. Let \hat{V}_α and \hat{G}_α be the doubles of V_α and G_α , respectively. Then, since \hat{V}_α can be considered as an unbounded n -sheeted covering surface over the compact surface \hat{G}_α , by the Riemann-Hurwitz and Euler-Poincaré formulas,

$$2(1 - \gamma(\alpha)) = 2(1 - \beta_\alpha(\alpha))n - 2b(\alpha)$$

where $\beta_G(\alpha)$ is the first Betti number of G_α and $b(\alpha)$ is the total sum of the branching order of branch points over G_α . Thus

$$(2) \quad \gamma(\alpha) = \beta_G(\alpha)n + b(\alpha) - n + 1.$$

Observe that the set of α such that there exist branch points of R on ∂V_α or critical points of $g_G(z, z_0)$ on ∂G_α is isolated. Hence, from (1) and (2), it follows that

$$(3) \quad \begin{aligned} \int_0^{\alpha_0} \beta_R(\alpha) d\alpha &\leq \int_0^{\alpha_0} \gamma(\alpha) d\alpha \\ &= n \int_0^{\alpha_0} \beta_G(\alpha) d\alpha + \int_0^{\alpha_0} b(\alpha) d\alpha + O(1). \end{aligned}$$

Observe that

$$\int_0^{\alpha_0} b(\alpha) d\alpha = \sum_{\zeta_k \in G - G_\alpha} n_k g_G(\zeta_k, z_0).$$

Therefore, by the assumption,

$$(4) \quad \int_0^{\alpha_0} b(\alpha) d\alpha < \infty$$

and also by the assumption

$$(5) \quad \int_0^{\alpha_0} \beta_G(\alpha) d\alpha < \infty.$$

From (3), (4), and (5), it follows that

$$\int_0^\infty \beta_R(\alpha) d\alpha = \int_0^{\alpha_0} \beta_R(\alpha) d\alpha + O(1) < \infty,$$

i.e.,

$$m(R) = \exp \left\{ - \int_0^\infty \beta_R(\alpha) d\alpha \right\} > 0.$$

ACKNOWLEDGMENT. The author wishes to express his thanks for helpful suggestions to Professor M. Nakai.

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Received November 4, 1977.

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