

# Pacific Journal of Mathematics

**A CHARACTERIZATION OF  $R^2$  BY THE CONCEPT OF MILD  
CONVEXITY**

SJUR FLAM

## A CHARACTERIZATION OF $\mathbf{R}^2$ BY THE CONCEPT OF MILD CONVEXITY

SJUR D. FLAM

**Let  $S$  be an open, connected set in a locally convex, Hausdorff topological vector space  $L$ . If the boundary of  $S$  contains exactly one point not a mild convexity point of  $S$  and this point is not isolated in  $\text{bd } S$ , then  $\dim L = 2$ .**

NOTATION.  $[S]$  denotes the convex hull of  $S$ .  $\langle S \rangle$  denotes the interior of  $[S]$  relative to the affine closure,  $\text{aff } S$ , of  $S$ .  $\text{int } S$ ,  $\text{cl } S$ , and  $\text{bd } S$  represent the interior, closure and boundary of  $S$ , respectively, while  $\text{ext } S$  and  $\text{exp } S$  denote the sets of extreme and exposed points of  $S$ .  $\text{codim } S$  denotes the codimension of  $\text{aff } S$ .

DEFINITION. Let  $S$  be a set in a topological vector space  $L$ . A point  $x$  is called a mild convexity point of  $S$  if there do not exist two points  $y$  and  $z$  such that  $x \in \langle y, z \rangle$  and  $[y, z] \sim \{x\} \subseteq \text{int } S$ . [1].

The proof of Theorem 2 proceeds through some lemmas. Easy proofs are omitted.

LEMMA 1. *A topological vector space over  $\mathbf{R}$  induces a locally convex, relative topology on every finite-dimensional linear subspace. Hence the relative topology on every finite-dimensional subspace is coarser than the standard Hausdorff topology on the subspace.*

*Proof.* Suppose the subspace  $M$  of  $L$  has finite dimension  $m$  and  $U$  is an arbitrary 0-neighborhood of  $L$ . Choose a balanced 0-neighborhood  $V$  such that

$$\sum_1^{m+1} V \subseteq U.$$

Then by Caratheodory's theorem [1]

$$V \cap M \subseteq [V \cap M] \subseteq U \cap M.$$

LEMMA 2. *Let  $S$  be an open set in a topological vector spaces. Suppose  $[x, y] \cup [y, z] \subseteq S$  and  $[x, y, z] \cap \text{bd } S$  contains mild convexity points of  $S$  only. Then  $\langle x, y, z \rangle \subseteq S$ .*

*Proof.* If  $x, y, z$  are collinear then there is nothing to prove; otherwise  $S$  intersects  $\text{aff } \{x, y, z\}$  in a set which is open relative to the standard Hausdorff topology by Lemma 1. Therefore

$[x, y, z] \sim S$  is compact relative to this topology and so is its convex hull  $C$ . It is known that  $C = [\text{ext } C]$ . If  $\text{ext } C \not\subseteq [x, z]$  then the inclusion  $\text{ext } C \subseteq \text{cl exp } C$  demonstrates the existence of a point  $e \in \text{exp } C \cap \langle x, y, z \rangle$ . Since  $\text{exp } C \subseteq \text{ext } C \subseteq [x, y, z] \sim S$  this point belongs to  $\text{bd } S$  and is not a mild convexity point of  $S$ . This contradiction implies  $\text{ext } C \subseteq [x, z]$  and the conclusion follows.

**LEMMA 3.** *If the nondegenerate interval  $[x, y]$  does not intersect an affine subspace  $M$  of a vectorspace, then there is a point  $x'$  such that  $x \in \langle x', y \rangle$  and  $[x', y] \cap M = \emptyset$ .*

**LEMMA 4.** *If  $[x, y] \cup [y, z] \cup [z, w]$  is contained in an open set belonging to a topological vector space over  $\mathbf{R}$  and  $u$  is an arbitrary vector, then  $y, z$  may be moved somewhat in the direction of  $u$  to the points  $y', z'$  so that  $[x, y'] \cup [y', z'] \cup [z', w]$  still belongs to the same open set.*

**LEMMA 5.** *If  $S$  is an open, connected set in a topological vector space over  $\mathbf{R}$  and  $T$  is a subset of the same space with  $\text{codim } T \geq 2$ , the  $S \sim T$  is polygonally connected.*

*Proof.*  $S$  is polygonally connected. If an interval  $[y, z]$  intersecting  $\text{aff } T$  belongs to a polygonal path, then by Lemma 4,  $y$  and  $z$  may be replaced by  $y'$  and  $z'$  so that the new path is in  $S \sim \text{aff } T$ .

**LEMMA 6.** *Let  $S$  be an open, connected set in a topological vector space over  $\mathbf{R}$ . Suppose that the set  $N$  of points in  $\text{bd } S$  which are not mild convexity points of  $S$  is empty or has codimension at least 3. Then if  $x, y \in S$  and  $[x, y] \cap \text{aff } N = \emptyset$  we have  $[x, y] \subseteq S$ .*

*Proof.* By Lemma 5 there is a polygonal path in  $S$  from  $x$  to  $y$  which does not intersect  $\text{aff } (N \cup x) \sim x$ . If  $[x, x_1], [x_1, x_2]$  are the first intervals in this path, then by application of the Lemmas 3 and 2 (in that order),  $[x, x_2]$  lies in  $S$  and clearly does not intersect  $\text{aff } (N \cup x) \sim x$ . Proceeding in this manner we eventually obtain  $[x, y] \subseteq S$ . A digression is given here.

**THEOREM 1.** *Suppose  $S$  is an open, connected set in a topological vector space over  $\mathbf{R}$ , and suppose  $\text{bd } S$  contains only mild convexity points. Then  $S$  is convex.*

**REMARK.** This theorem which follows immediately from Lemma 6 is established in [1] with the additional assumption that the space is Hausdorff.

LEMMA 7. *Let  $S$  be an open, connected set in a locally convex Hausdorff space over  $\mathbf{R}$ . Suppose the set  $N$  of points in  $\text{bd } S$  which are not mild convexity points has the property  $\text{codim cl aff } N \geq 3$ . Then for every  $x \in N$  there exists a closed hyperplane  $H$  and an  $x$ -neighborhood  $U$  such that  $U \sim H \subseteq S$ .*

*Proof.* Choose two points  $x_1, x_2$  both different from  $x$  such that  $x \in [x_1, x_2] \subseteq S \cup x$ . The set  $(\text{cl aff } N) \cup x_1$  is contained in a hyperplane  $H$ . Call the corresponding open halfspaces  $H^+$  and  $H^-$  respectively. Choose an  $x_i$ -neighborhood  $V_i \subseteq S$ . Then the union of  $U^+ = [(V_1 \cup V_2) \cap H^+]$ ,  $U^-$  (defined similarly) and  $H$  gives the required  $U$  by Lemma 6.

The announced result may be stated forthwith.

THEOREM 2. *Let  $S$  be an open connected set in a locally convex, Hausdorff space over  $\mathbf{R}$ . If  $\text{bd } S$  contains exactly one point which is not a mild convexity point of  $S$  and this point is not isolated in  $\text{bd } S$ , then the dimension of the space is 2.*

It is trivial to exhibit such a set in  $R^2$ , and it is easy to show that the set is starshaped.

#### REFERENCE

1. F. A. Valentine, *Convex Sets*, McGraw Hill (1964).

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