FINITE GROUPS WITH A STANDARD SUBGROUP ISOMORPHIC TO $PSU(4, 2)$

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The combined work of M. Aschbacher, G. Seitz, and I. Miyamoto classified finite groups $G$ with a standard subgroup $L$ isomorphic to $PSU(4, 2^n)$ such that either $n > 1$ or $C_0(L)$ has noncyclic Sylow 2-subgroups. In this paper, we study the case that $n = 1$ and $C_0(L)$ has cyclic Sylow 2-subgroups.

Introduction. A group $L$ is quasisimple if $L$ is its own commutator group and, modulo its center, $L$ is simple. A quasisimple subgroup $L$ of a finite group $G$ is standard if its centralizer in $G$ has even order, $L$ is normal in the centralizer of every involution centralizing $L$, and $L$ commutes with none of its conjugates. This definition of standard subgroups is equivalent to the original one given by M. Aschbacher in his fundamental paper [1].

I. Miyamoto has classified [23] finite groups $G$ containing a standard subgroup $L$ isomorphic to $PSU(4, 2^n)$ with $n > 1$ such that $C_0(L)$ has cyclic Sylow 2-subgroups. Part of his argument, however, failed to apply to $PSU(4, 2)$. This exceptional nature of $PSU(4, 2)$ may be explained by the isomorphism

$$PSU(4, 2) \cong PSp(4, 3) \cong PΩ(5, 3).$$

Because of this, certain groups of characteristic 3 have standard subgroups isomorphic to $PSU(4, 2)$.

In this paper, we prove the following theorem.

**Theorem.** Let $G$ be a finite group and suppose $L$ is a standard subgroup of $G$ with $L \cong PSU(4, 2)$. Furthermore, assume that $C_0(L)$ has cyclic Sylow 2-subgroups, and let $X$ denote the normal closure of $L$ in $G$. Then one of the following holds.

1. $X/O(X)$ is a simple group of sectional 2-rank 4.
2. $X \cong PSL(4, 4)$ or $PSU(4, 2) \times PSU(4, 2)$.
3. $N_0(L)/C_0(L) \cong Aut(L)$, and for each central involution $z$ of $L$, $C_0(z)$ has a quasisimple subgroup $K$ that satisfies the following conditions:
   1. $z \in K$ and $W = O_2(K)$ is cyclic of order 4.
   2. $K/\langle z \rangle$ is a standard subgroup of $C_0(z)/\langle z \rangle$ and $W$ is a Sylow 2-subgroup of $C_0(K)/\langle z \rangle$.
   3. Either $K/O(K) \cong SU(4, 3)$ or $K/Z(K)$ has a Sylow 2-subgroup isomorphic to a Sylow 2-subgroup of $PSL(6, q)$, $q \equiv 3 \pmod{4}$.  

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(3.4) $[K, O(C_0(z))] = 1.$

**Remark.** In Case (1), the structure of $X/O(X)$ can be determined by a theorem of D. Gorenstein and K. Harada [14]; we can show that $X/O(X)$ is isomorphic to $PSp(4, 3)$, $PSp(4, 9)$, $PSU(4, 3)$, $PSL(4, 3)$, or $PSL(5, 3)$. Case (3) occurs in the automorphism group of $PSU(5, 3)$ with $K \cong SU(4, 3)$.

The proof of the theorem begins with a study of fusion of an involution $t$ of $C_0(L)$. Let $A$ be the unique elementary abelian subgroup of order 16 of a Sylow 2-subgroup of $L$. We show that the conjugacy class of $t$ in $N_0(t)A$ contains 1, 6, or 16 elements. If it contains 1 or 6 elements, then after determining the possible structure of a Sylow 2-subgroup of $G$, we show $t \in G'$ by a transfer argument. It then follows that $N_x(A)/C_X(A) \cong A_5, \Sigma_5, A_6$ or $\Sigma_6$, and that $A \in Syl_2(C_X(A))$. If $N_x(A)/C_X(A) \cong A_5, \Sigma_5$ or $A_6$, a theorem of Harada [17] shows that $r(X) = 4$. When $N_x(A)/C_X(A) \cong \Sigma_6$, we appeal to a theorem of G. Stroth [26]. Using an additional information, we show that this case does not occur. The analysis of the case where there are 16 conjugates of $t$ follows the same line of arguments as in previous papers of Miyamoto and the author [11], [23] (we refer the reader to the introduction of [11]), although some additional argument is needed in the analysis of a subcase leading to Case (3) of the theorem.

Finally, we remark that the solvability of groups of odd order [6] is used implicitly throughout this paper.

**Notation and Terminology.** Our notation is standard and mainly taken from [12]. Possible exceptions are the use of the following:

- $m(X)$: the 2-rank of $X$.
- $r(X)$: the sectional 2-rank of $X$.
- $I(X)$: the set of involutions of $X$.
- $E^*(X)$: the set of maximal elementary abelian subgroups of $X$.
- $X^\omega$: the final term of the derived series of $X$.
- $J_2(X)$: the subgroup of $X$ generated by the abelian 2-subgroups of maximal rank.
- $X^2$: the subgroup of $X$ generated by the squares of elements of $X$.
- $E(X)$: the product of the quasisimple subnormal subgroups of $X$.
- $L(X)$: the 2-layer of $X$.
- $X$ wreath $Y$: the wreath product of $X$ by $Y$. 
**X* Y**  
**a central product of X and Y.**

**f(X mod Y)**  
**the preimage in X of f(X/Y), where f is a function from groups to groups.**

**Z_{2^n}**  
**the cyclic group of order 2^n.**

**E_{2^n}, n ≥ 2**  
**the elementary abelian group of order 2^n.**

**D_n, n ≥ 6**  
**the dihedral group of order n.**

**Q_8**  
**the quaternion group.**

**A_n, Σ_n, n ≥ 3**  
**the alternating and symmetric group of degree n.**

**F_q**  
**the field of q elements.**

**V(2, F)**  
**the vector space of 2-dimensional row vectors with coefficients in the field F.**

**M(4, F)**  
**the set of 4 × 4 matrices with entries in F.**

An **A_2^n**-subgroup is an abelian subgroup of order 2^n, while an **E_2^n**-subgroup is an elementary abelian subgroup of order 2^n. Suppose **G ≅ SL(2, 4) ≅ A_5.** Then **G** has two types of “natural” modules over **F_4.** The one is **V(2, F_4) viewed as an SL(2, 4)-module in an obvious way. We call this the natural module for G ≅ SL(2, 4).** The other is the unique nontrivial irreducible constituent of the permutation module for **A_5.** We call this the **natural module for G ≅ A_5.** We use the “bar” convention for homomorphic images. Thus if **G** is a group, **N** is a normal subgroup, and **G/N** denotes the factor group **G/N,** then for any subset **X** of **G,** **X** will denote the image of **X** under the natural projection **G → G/N.** A similar convention will be used when a group **G** has a permutation representation on a set **Ω,** where we write **X^Ω** instead of **X.**

1. In this section, we collect a number of preliminary lemmas to be used in later sections.

**LEMMA (1A).** Let **R** be a nonabelian 2-group with a cyclic maximal subgroup **Q,** and let **t ∈ I(Q) and u ∈ I(R − Q).** Then **u** is conjugate to **tu** in **R.**

**Proof.** This is a consequence of the classification of nonabelian 2-groups with a cyclic maximal subgroup. See Theorem 5.4.4. of [12].

**LEMMA (1B).** Let **G** be a group which contains a direct product **H × K** of subgroups **H** and **K.** Assume that **|G: HK| = 2** and that an element of **G − HK** interchanges **H** and **K.** Then **G − HK** contains involutions and they are all conjugate in **G.**

**Proof.** Let **g ∈ G − HK,** and let **g^2 = hk** with **h ∈ H** and **k ∈ K.**
Then \(hk = (hk)^g = k^g h^g\), so \(h^g = k\) and \(k^g = h\). Hence
\[
(gh^{-1})^2 = gh^{-1}gh^{-1} = g^g h^{-1}gh^{-1} = (hk)k^{-1}h^{-1} = 1.
\]
Thus \(G - HK\) contains an involution.

Now let \(g \in G - HK\) and \(g^2 = 1\). Let \(h \in H\) and \(k \in K\), and assume that \(ghk\) is an involution. Then \((hk)^g = (hk)^{-1}\), so \(h^g = k^{-1}\) and \(k^g = h^{-1}\). Hence \(h^{-1}gh = gg^{-1}h^{-1}gh = ghk\). That is, \(ghk\) is conjugate to \(g\). The proof is complete.

**Lemma (1C).** Let \(E\) be an elementary abelian 2-subgroup of a group \(G\), and let \(t\) be an involution of \(N_G(E)\). Then the following holds.

1. \(|E : C_E(t)| \leq |C_E(t)|\), and equality holds if and only if \(I(tE) = t^E\).
2. If \(|E : C_E(t)| \geq 4\), then
   \[N_G(\langle E, t \rangle) \leq N_G(\langle C_E(t), t \rangle) \cap N_G(E).\]

**Proof.** Commutation by \(t\) induces a homomorphism from \(E\) onto \([E, t]\), and so \(|[E, t]| = |E : C_E(t)|\). Also, \([E, t] \leq C_E(t)\). Hence \(|E : C_E(t)| \leq |C_E(t)|\). Since \(|I(tE)| = |C_E(t)|\) and \(|t^E| = |E : C_E(t)|\), equality holds if and only if \(I(tE) = t^E\).

Under the hypothesis of (2), \(E\) and \(\langle C_E(t), t \rangle\) are the only maximal elementary abelian subgroups of \(\langle E, t \rangle\), and they have different orders. Hence (2) follows.

**Lemma (1D).** Let \(G\) be a finite group and let \(g \in G\). Then \(|C_G(g)| \geq |G : G'|\).

**Proof.** For any \(x \in G\), \(g^{-1}g^x = [g, x] \in G'\). Hence \(|G : C_G(g)| = |g^G| \leq |G'|\).

**Lemma (1E).** Let \(R\) be an \(S_2\)-subgroup of a finite group \(G\) and \(S\) a normal subgroup of \(R\) with \(R/S\) abelian. Let \(x\) be an involution of \(R - S\) and suppose that each extremal conjugate of \(x\) in \(R\) is contained in \(xS\). Then \(x \in G'\).

**Proof.** Let \(T\) be a subgroup of \(R\) with \(S \leq T \leq R\) and \(x \in T\) subject to \(|T|\) maximal. Then since \(R/S\) is abelian, \(R/T\) is cyclic. Also, each extremal conjugate of \(x\) in \(R\) is contained in \(xT\). There-
fore, Lemma (1E) follows from [27], Corollary 5.3.2.

**Lemma (1F).** Let $T$ be an $S_2$-subgroup of a finite group $G$, and let $S$ be a normal subgroup of $T$ such that $T/S \cong D_8$ and $S \leq G^\circ$. Let $a \in I(T - S)$ and $b \in I(T - \langle a, S \rangle)$, and suppose $(ab)^2 = 1$, $a^g \cap \langle b, S \rangle = \varnothing$, $b^g \cap S = \varnothing$, and $(ab)^g \cap S = \varnothing$. Then $S \in \text{Syl}_2(G^\circ)$.

**Proof.** By Lemma (1E), $a \in G'$ and so $T \cap G' = S$, $\langle b, S \rangle$, or $\langle ab, S \rangle$. If $T \cap G' \neq S$, then $T \cap G'' = S$ again by Lemma (1E). Thus $S \in \text{Syl}_2(G^\circ)$.

**Lemma (1G).** Let $T$ be an $S_2$-subgroup of a finite group $G$, and let $S$ be a normal subgroup of $T$ such that $T/S \cong D_8$ and $S \leq G^\circ$. Let $Z/S = Z(T/S)$, and let $E/S$ and $F/S$ be the fours subgroups of $T/S$. Let $a \in I(Z - S)$ and $b \in I(E - Z)$, and suppose $a^g \cap F \leq aS$ and $b^g \cap F = \varnothing$. Then $S \in \text{Syl}_2(G^\circ)$.

**Proof.** By Lemma (1E), $b \in G'$ and so $E \cap G' = S$ or $Z$, since $bS \sim abS$ in $T$. If $E \cap G' = S$, then $T \cap G' = S$ as $S \leq T \cap G' \lhd T$. Suppose that $E \cap G' = Z$. Then either $T \cap G' = F$ or $T \cap G'/S$ is cyclic. Hence $a^g \cap T \cap G' \leq aS$ and so $a \in G''$ by Lemma (1E). Thus $T \cap G'' = S$. Therefore, $S \in \text{Syl}_2(G^\circ)$.

**Lemma (1H).** Let $A$ be a standard subgroup of a finite group $G$, and assume that $C_G(A)$ has a cyclic $S_2$-subgroup. Then the following holds.

1. $AO(G) \triangleleft G$ if and only if an involution $t$ of $C_G(A)$ is contained in $Z^*(G)$.
2. $AO(G)/O(G)$ is a standard subgroup of $G/O(G)$ and $C_G(AO(G)/O(G))$ has a cyclic $S_2$-subgroup.
3. If $AO(G) \triangleleft G$, then either $\langle A^o \rangle O(G)/O(G)$ is simple or $\langle A^o \rangle O(G)/O(G) \cong A/Z(A) \times A/Z(A)$. In each case, $C_G(\langle A^o \rangle O(G)/O(G)) = O(G)$.
4. If $AO(G) \triangleleft G$ and if there is a $t$-invariant 2-subgroup $P$ of $\langle A^o \rangle$ such that $1 \neq [P, t] \leq C_G(C_{o(t)}(t))$, then $[\langle A^o \rangle, O(G)] = 1$.

**Proof.** Let $t \in I(C(A))$ and let $\bar{G} = G/O(G)$. Then $\bar{t} \in I(\bar{G})$ and $\bar{A}$ is a quasisimple normal subgroup of $C(\bar{t})$. Let $x \in C(\bar{A}) \cap C(\bar{t})$. We may choose $x \in C(t)$. Then $[x, A] \leq A \cap O(G) \leq Z(A)$, so $[x, A] = 1$. Thus $C(\bar{A}) \cap C(\bar{t}) = C(\bar{A}) \cap C(\bar{t})$. Therefore, $C(\bar{A})$ has cyclic $S_2$-subgroups and (2) follows.

Assume that $\bar{A} \triangleleft \bar{G}$. Then $C(\bar{A}) \triangleleft \bar{G}$ and so $C(\bar{A})$ is a cyclic 2-group and $\bar{t} \in Z(\bar{G})$. Conversely, if $\bar{t} \in Z(\bar{G})$, then $\bar{A} \triangleleft C(\bar{t}) = \bar{G}$. This proves (1).
Assume that \( \bar{A} \trianglelefteq \bar{G} \). Then by a result of Aschbacher, \( F^*(\bar{G}) = \langle \bar{A}^o \rangle \) and either \( F^*(\bar{G}) \) is simple or \( \bar{A} \) is simple, \( F^*(\bar{G}) \cong \bar{A} \times \bar{A} \), and \( \bar{t} \) interchanges two components of \( F^*(\bar{G}) \). Let \( L = \langle A^o \rangle O(G) \) and assume that there is a \( t \)-invariant 2-subgroup \( P \) of \( L \) such that \( 1 \neq [P, t] \) and \( [[P, t], C_{O(G)}(t)] = 1 \). Then \( [[P, t], O(G)] = 1 \) by \( [11, (1J)] \). Hence \( C_L(O(L)) \trianglelefteq O(L) \). Since \( L = L/O(L) \) is simple or a direct product of simple groups interchanged by \( t \), it follows that \( L = C_L(O(L))O(L) \). Thus \( \langle A^o \rangle \cong C_L(O(L)) \) and (4) follows.

**Lemma (11).** Let \( K = PSL(n, q) \), \( n \geq 2 \), or \( PSU(n, q) \), \( n \geq 3 \), \( q \) odd, and let \( \alpha \) be an involutory automorphism of \( K \) that is not a product of an inner automorphism and a diagonal automorphism. Then \( C_K(\alpha) \) is solvable only if \( K = PSL(2, 9) \), \( PSL(3, 3) \), \( PSL(4, 3) \), \( PSU(3, 3) \), or \( PSU(4, 3) \). If \( C_K(\alpha) \) is not solvable, then the structure of \( C_K(\alpha)^o \) is given on the following table.

<table>
<thead>
<tr>
<th>( K )</th>
<th>( C_K(\alpha)^o )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( PSL(n, q) )</td>
<td>( PΩ^\pm(n, q) ), ( PSp(n, q) ), ( n ) even, ( PSL(n, p) ), ( q = p^2 ), ( PSU(n, p) ), ( q = p^2 )</td>
</tr>
<tr>
<td>( PSU(n, q) )</td>
<td>( PΩ^\pm(n, q) ), ( PSp(n, q) ), ( n ) even.</td>
</tr>
</tbody>
</table>

*Proof.* Consider the case \( K = PSL(n, q) \) first. Set \( G = GL(n, q) \) and \( H = SL(n, q) \). Let \( \tau \) be the transpose-inverse mapping of \( G \), and if \( q = p^2 \), let \( \sigma \) be the automorphism of \( G \) induced by that of \( F_2^q \) of order 2. Then \( \alpha \) is induced on \( K = H/Z(H) \) by an element \( x \) of \( \tau G, \sigma G \) or \( \tau \sigma G \) such that \( x^2 \in Z(G) \).

First, assume that \( x \in \tau G \). Then \( n \geq 3 \). Let \( x = \tau a \), \( a \in G \). Then as \( x^2 \in Z(G) \), it follows that \( ^t \alpha = a \) or \( -a \), where \( ^t \alpha \) is the transposed matrix of \( a \). We also have that

\[
C_\sigma(x) = \{ y \in G | ^t y a y = a \}.
\]

That is, \( C_\sigma(x) \) is the orthogonal or symplectic group defined by the symmetric or alternating matrix \( a \). Now \( \text{Aut} \langle \langle x, Z(G) \rangle \rangle \) is solvable, so \( N_\sigma(\langle x, Z(G) \rangle)^o \cong C_\sigma(x) \). Also, \( C_\sigma(x)^o \cong H \). Thus \( C_K(\alpha)^o \cong C_\sigma(x)^o Z(H)/Z(H) \), and so \( C_K(\alpha) \) is solvable only if \( (n, q) = (3, 3) \) or \( (4, 3) \), and if \( C_K(\alpha) \) is nonsolvable then \( C_K(\alpha)^o \cong PΩ^\pm(n, q) \) or \( PSp(n, q) \).

Next, consider the case \( x \in \sigma G \). Let \( x = \sigma a \), \( a \in G \). Then as \( x^2 \in Z(G) \), we see that \( e = a^* a \) is a scalar matrix such that \( a^{q-1} = 1 \).
Hence there is a scalar matrix $d \in G$ such that $d^{p+1}c = 1$, so that $(da)^\sigma da = 1$. Replacing $x$ by $xd$, we may assume that $a^*a = 1$. By [20, Proposition 3], there is an element $g \in G$ such that $a = g^*g^{-1}$. Thus $x^* = \sigma$ and we may assume from the outset that $x = \sigma$.

Therefore, $C_G(x) \cong GL(n, p)$, and so $C_K(\alpha)$ is solvable only if $(n, q) = (2, 9)$, and if $C_K(\alpha)$ is nonsolvable, then $C_K(\alpha)^\sigma \cong PSL(n, p)$.

Assume, therefore, $x \in \tau \sigma G$. Let $x = \tau \sigma a$, $a \in G$. As above, we may assume that $a^*a = 1$. That is, $a$ is a hermitian matrix. Thus $C_G(x)$ is the unitary group defined by $a$ over $F_q$, and so $C_K(\alpha)$ is solvable only if $(n, q) = (2, 9)$, and if $C_K(\alpha)$ is nonsolvable, then $C_K(\alpha)^\sigma \cong PSU(n, p)$.

Now consider the case $K = PSU(n, q)$. In this case, we set $G^* = GL(n, q^2)$, $G = U(n, q)$, and $H = SU(n, q)$. Let $\tau$ be the transpose-inverse mapping of $G^*$ and let $\sigma$ be the automorphism of $G^*$ induced by that of $F_q^2$ of order 2. Then we may regard $G = C_{G^*}(\sigma \tau)$, and assume that $\alpha$ is induced on $K = H/Z(H)$ by an element $x$ of $\sigma Z(G^*)G$ such that $x^2 \in Z(G^*)$. As before, we may assume that $x = \sigma a$, $a \in Z(G^*)G$, and $a^*a = 1$. Let $a = a_1a_2$ with $a_1 \in Z(G^*)$ and $a_2 \in G$. Then

\begin{equation}
(1) \quad a^* = a_1^{-q}a_2 = a_1^{-q-1}a = e^{-1}a,
\end{equation}

where $e = a_1^{q+1}$. Now there is an element $g \in G^*$ such that $a = g^*g^{-1}$ by [20, Proposition 3]. Hence by (1), $(g^*g^{-1})^* = e^{-1}(g^*g^{-1})$. That is,

\begin{equation}
(2) \quad eg^*g^{-*} = g^*g^{-1}.
\end{equation}

Now $(\sigma \tau)^* = \sigma \tau g^{-*}g$, so let $h = g^{-*}g$. Then $h^* = g^{-*}g^{-1}g^{-*} = e^{-1}h^{-1}$ by (2), so

\[t^*h = eh.\]

Hence

\[e = \pm 1.\]

Also,

\[h^* = g^{-*}g = e^{-1}g^{-*}g = eh\]

by (2). Choose an element $d \in Z(G^*)$ such that $d^{q-1} = e^{-1}$ and set $h_1 = dh$. Then $t^*h_1 = eh_1$ and $h_2^* = d^qeh = d^{q-1}eh_1 = h_1$. Thus $h_2$ is a symmetric or alternating matrix in $C_{G^*}(\sigma) = GL(n, q)$. Now $x^* = \sigma$ as $a^*a = 1$, so

\[C_G(x) = C_{G^*}(x) \cap C_{G^*}(\sigma \tau) \cong C_{G^*}(\sigma) \cap C_{G^*}(\sigma \tau^* h) = C_{G^*}(\sigma) \cap C_{G^*}(\tau h_1).\]
Thus $C_G(x) \cong O_x^\pm(n, q)$ or $Sp(n, q)$ by a previous discussion. Hence $C_K(\alpha)$ is solvable only if $(n, q) = (3, 3)$ or $(4, 3)$, and if $C_K(\alpha)$ is nonsolvable, then $C_K(\alpha)^\circ \cong P GL^\pm(n, q)$ or $PSp(n, q)$.

**Lemma (1J).** Let $E$ be an elementary abelian group of order 16 on which $M \cong SL(2, 4) \cong A_5$ acts. Let $R \in Syl_2(M)$.

1. If $|C_E(R)| = 4$, then $E$ is a natural module for $M \cong SL(2, 4)$.
2. If $|C_E(R)| = 2$, then $E$ is a natural module for $M = A_5$.

*Proof.* (1) follows from [11, (1K)]. Assume that $|C_E(R)| = 2$. Let $a_1, a_2, \ldots, a_5$ be the nontrivial fixed points on $E$ of $S_2$-subgroups of $M$ so that $\{a_1, a_2, \ldots, a_5\}$ is $M$-invariant. Since $M$ acts irreducibly on $E$, we have $a_1a_2\cdots a_5 = 1$ and $E = \langle a_1, a_2, \ldots, a_5 \rangle$. Now let $V$ be the direct product of $E$ and a group $\langle a \rangle$ of order 2, and let $M$ act on $V$ in an obvious fashion. Then, by the above remark, $\{aa_1, aa_2, \ldots, aa_5\}$ is an $M$-invariant set which generates $V$. Thus $V$ is a permutation module for $M \cong A_5$, and $E$ is a nontrivial irreducible constituent of $V$. This proves (2).

**Lemma (1K).** Let $E$ be an elementary abelian group of order $2^8$, and let $K$ and $L$ be subgroups of $\text{Aut}(E)$ such that $SL(2, 4) = K^L = SL(2, 16)$. Let $R \in Syl_2(K)$, and let $R \leq S \in Syl_2(L)$. Assume that $|C_K(S)| = 4$. Then there is no nontrivial $K$-invariant subgroup $A$ of $E$ such that $C_A(R) < C_K(S)$.

*Proof.* Let $W = C_K(S)$ and assume, by way of contradiction, that $A$ is a $K$-invariant subgroup of $E$ such that $1 \neq C_A(R) < W$. Clearly, $N_L(S)$ normalizes $W$. As $N_K(R) \leq N_L(S)$ and $N_K(R)$ centralizes $C_A(R)$ which is a subgroup of $W$ of order 2, we have that $[N_K(R), W] = 1$. As $|N_L(S)/S| = 15$ and $N_K(R)S/S$ is an $S_2$-subgroup of $N_L(S)/S$, it follows that $[N_L(S), W] = 1$.

Let $s \in I(L - N_L(S))$ and set $H = N_L(S) \cap N_L(S^*)$. Notice that $H$ is a complement for $S$ in $N_L(S)$. Furthermore, $W \cap W^* = C_L(L) = 1$, as $L = \langle S, S^* \rangle$ and $L$ acts irreducibly on $E$ by [8, (4B)].

Now $[H, WW^*] = 1$, as $[H, W] = 1$ by the first paragraph and $H^* = H$. For any $w \in W^*$, let $\hat{w} = wW^*$. Then as $\langle H, s \rangle \leq C_L(\hat{w})$ and $\langle H, s \rangle$ is a maximal subgroup of $L$, we have that $C_L(\hat{w}) = \langle H, s \rangle$. Consequently, $|\hat{w}| = |L:\langle H, s \rangle| = 136$. As $136 \times 2 = 272 > 255 = |E^4|$, it follows that $\hat{w}_1 \sim \hat{w}_2$ for any $w_1, w_2 \in W^*$. Choose $x \in L$ such that $\hat{w}_1 = \hat{w}_2$. Then $\langle H, s \rangle^x = C_L(\hat{w}_1)^x = C_L(\hat{w}_2) = \langle H, s \rangle$, and so $x \in N_L(\langle H, s \rangle) = \langle H, s \rangle$. This is a contradiction as we may choose $\hat{w}_1 \neq \hat{w}_2$. 


Now we define some subgroups of $SL(4, 4)$. Let $M^*, R^*, D^*$, and $E^*$ be the groups consisting of the following matrices, respectively.

$\begin{pmatrix} A & \frac{1}{a} \\ 1 & -1 \end{pmatrix}$, $A \in SL(2, 4)$, and $I$ is the $2 \times 2$ unit matrix,

$\begin{pmatrix} 1 & a & 1 \\ a & 1 & 1 \end{pmatrix}$, $a \in F_4$,

$\begin{pmatrix} a^{-1} & a^{-1} & a \\ a & a & 1 \end{pmatrix}$, $a \in F_4 - \{0\}$,

$\begin{pmatrix} 1 & 1 \\ a & b & c \\ d & 1 \end{pmatrix}$, $a, b, c, d \in F_4$.

Thus $R^* \in \text{Syl}_2(M^*)$, and $M^*$ and $D^*$ normalize $E^*$. Let $f^*$ be the field automorphism of $SL(4, 4)$ and let $t^*$ be the graph-field automorphism of $SL(4, 4)$. That is, $f^*$ is induced by the involution of Aut$(F_4)$ and $t^*$ is the transpose-inverse mapping followed by $f^*$ and conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let $L^* = M^*M^{t*}$.

We shall consider the following situation.

**Hypothesis (1.1).** $E$ is an elementary abelian group of order $2^8$, and $N$ is a subgroup of Aut$(E)$ which has a normal subgroup $L$ satisfying the following conditions.

1. $L = M \times M^t$, $t \in I(N)$, $M \cong SL(2, 4)$.
2. $C_N(L) = O(N)$.
3. For $R \in \text{Syl}_2(M)$, $W = C_E(RR^t)$ is a fours group.

**Lemma (1L).** Assume Hypothesis (1.1). Furthermore, assume the following.

4. $C_E(M) = 1$.
5. For a complement $H$ for $R$ in $N_M(R)$, $[W, hht] = 1$ for all $h \in H$.

Then there is a monomorphism $\sigma$ from the semidirect product
of $N$ and $E$ into $\langle M^*, E^*, D^*, t^*, f^* \rangle$ such that $M^* = M^*, R^* = R^*$, $O(N)^* \leq D^*$, $E^* = E^*$, $t^* = t^*$, and $f^* = f^*$ if $f$ is an element of $C(t) \cap N_M(M)$ acting as a field automorphism on $C_L(t) \cong SL(2, 4)$.

Proof. Let $r \in I(N_M(H))$ and set $s = rtrt$. We use the additive notation for $E$. As $M = \langle R, R^* \rangle$, the condition (4) implies that $C_E(R) \cap C_E(R^*) = \{0\}$. In particular, $W \cap W^* = \{0\}$, and as $W \cap W^* \leq C_E(R^*)$, $|C_E(R^*)| = |C_E(R^t)| \geq 2^4$. As $C_E(R^*) \cap C_E(R^t) = \{0\}$, we conclude that

$$E = C_E(R^*) \bigoplus C_E(R^t).$$

Also,

$$C_E(R^t) = W \bigoplus W^t$$

and $C_E(R^t) = C_E(R^t)^{trt} = W^{trt} \bigoplus W^s$.

Furthermore, as $C_E(R^*) \cap C_E(R^t) = W$ has order 4, Lemma (1J) shows that $C_E(R^*)$ and $C_E(R^t)$ are natural modules for $M \cong SL(2, 4)$. This proves that we can identify $M$ with $M^*$ so that $E \cong E^*$ as modules for $M$. More precisely, if $w \in C_E(t)^*$, $H = \langle h \rangle$, and $F = \{0, 1, x, x^2\}$, then $E$ and $E^*$ can be identified by the mapping which associates with $w^h + w^{hr} + w^{hr^t} + w^{hs}$, where $a, b, c, d \in \{0, 1, 2\}$, the matrix

$$\begin{pmatrix}
1 & 1 \\
x^c & x^a & 1 \\
x^d & x^b & 1
\end{pmatrix}$$

and the action of an element of $M$ on $E$ identified with $E^*$ is the conjugation by the corresponding element of $M^*$. In this identification, $R^*$ corresponds to $R$.

Using the condition (5), we have that for each $i \in \{0, 1, 2\}$,

$$(w^{h+i})^t = w^{h-i},$$

$$(w^{h+i})^t = w^{h-i}r^t,$$

$$(w^{h+i})^t = w^{h-i}_r,$$

$$(w^{h+i})^t = w^{h-i}_s.$$  

This shows that we can identify $t$ with $t^*$. Thus $\langle M, t \rangle E \cong \langle M^*, t^* \rangle E^*$.

Suppose $O(N) \neq 1$. The $A \times B$-lemma [12, Theorem 5.3.4] shows that $O(N)$ acts regularly on $W^*$. Hence $|O(N)| = 3$ and there is an element $z \in O(N)$ such that $w^z = w^h$. Then a computation similar to the above shows that $O(N)$ can be identified with $D^*$.

If $LO(N) = N_M$, then $N = \langle M, O(N), t \rangle$ so the above paragraphs prove the lemma. Suppose, therefore, that $LO(N) < N_M$.  


Now let \( f \) be an element of \( N(M) \) satisfying the following conditions:

\[ (*) \quad f \text{ inverts } H, f \in C(s), \text{ and } f \in C(w). \]

The second condition implies that \( f \) centralizes \( r \) and \( trt \). Therefore, by a computation similar to that in previous paragraphs, we can show that \( f \) can be identified with \( f^* \).

Suppose that \( C(M') \neq MO(N) \). Then there is an involution \( f \in C(M') \cap N(R) \) that satisfies the first two conditions in \((*)\). The \( f \) normalizes \( RR' \) and so acts on \( W \). Hence if \( O(N) \neq 1 \), there is an element \( z \in O(N) \) such that \( w^f = w^z \), and so \( fz^{-1} \) satisfies \((*)\). Assume that \( O(N) = 1 \). Then \([f, tft] \in C(M) \cap C(M') \leq C(L) = O(N) = 1 \), so that \((ft)^t = 1 \). Thus \( \langle f, t \rangle \) is a 2-group acting on \( W \), and so it centralizes some nontrivial element of \( W \). As \( C_w(t) = \langle w \rangle \), it follows that \( w^f = w \). Therefore, we can always choose an element \( f \in C(M') \) that satisfies \((*)\). By the above paragraph, \( f \) acts as the field automorphism on \( E \). It follows that \([f, t] \) centralizes \( E \), and therefore \([f, t] = 1 \). But then \( f = tft \in C(M') \cap C(M) = O(N) \), which is a contradiction. Therefore, \( C(M') = MO(N) \). This implies that \( LO(N) \) has index 2 in \( N(M) \).

Let \( K/L \) be an \( S_2 \)-subgroup of \( N/L \) with \( t \in K \). Notice that \( K/L \cong E_4 \). As \( I(Lt) = t^2 \) by Lemma (1B), \( K = LC_t(t) \) and so \( |C_k(t) \cap N(M) : C_L(t) \cap N(M)| = 2 \). As \( C_L(t) = \{xtxt \mid x \in M \} \cong M \cong SL(2, 4) \), \( N(M) \cap C(C_L(t)) = C(L) = O(N) \) and it follows that \( C_k(t) \cap N(M) \cap C(C_L(t)) = 1 \). Thus we may choose an involution \( f \in C(t) \cap N(M) \) which acts on \( C_L(t) \) as the field automorphism. Then \( f \) acts as the field automorphism both on \( M \) and on \( M' \). In particular, \( f \) inverts \( H \) and centralizes \( s \). Moreover, \( f \in N(RR') \), so \( f \) centralizes \( \langle w \rangle = C_w(t) \). Thus \( f \) satisfies \((*)\) and therefore \( f \) can be identified with \( f^* \). As \( N = \langle M, O(N), t, f \rangle \), we have proved the lemma.

**Lemma (1M). Assume Hypothesis (1.1). Furthermore, assume the following conditions.

(4) \( C_E(M) \neq 1 \).
(5) \( W \cap W^{rtrt} = 1 \) for \( r \in I(M - R) \).

Then \( E = C_E(M) \times C_E(M') \), and \( C_E(M') \) is a natural module for \( M \cong A_5 \).

**Proof.** Set \( s = rtrt \). Then

\[
W^s = (C_E(R) \cap C_E(R'))^{rtrt} = C_E(R)^s \cap C_E(R')^{rtrt}
= C_E(R)^r \cap C_E(R)^{rt}.
\]
As $M = \langle R, R' \rangle$, we may deduce as follows:

\[
C_E(M) \cap C_E(M') = C_E(R) \cap C_E(R') \cap C_E(R'^t) \\
= \left( C_E(R) \cap C_E(R') \right) \cap \left( C_E(R'^t) \right) \\
= W \cap W' \\
= 1.
\]

In particular, $M$ acts on $C_E(M')$ nontrivially, and so $|C_E(M')| \geq 2^4$. As $|E| = 2^8$, we must have that $E = C_E(M) \times C_E(M')$. Moreover, as $R$ normalizes $C_E(M)$ and $C_E(M')$, it follows that

\[
C_E(R) = \left( C_E(M) \cap C_E(R) \right) \times \left( C_E(M') \cap C_E(R) \right) \\
= C_E(M) \times \left( C_E(M') \cap C_E(R) \right).
\]

Therefore,

\[
W = C_E(R) \cap C_E(R')
= \left( C_E(M) \cap C_E(R') \right) \times \left( C_E(M') \cap C_E(R) \right).
\]

Since $|W| = 4$, we conclude that $|C_E(M') \cap C_E(R)| = 2$. Thus, $C_E(M')$ is a natural module for $M \cong A_5$ by Lemma (1J).

**Lemma (1N).** Let $t$ be an involution of a finite group $G$, and assume that $C(t)$ has a normal subgroup $L$ isomorphic to $SL(2, 4)$ such that $\langle t \rangle \in \text{Syl}_2(C(L) \cap C(t))$. Furthermore, assume that an $S_3$-subgroup $R$ of $L$ is contained in an $N(R) \cap C(t)$-invariant $E_{16}$-subgroup $S$ of $G$. Then $X = \langle L^o \rangle$ is isomorphic to $SL(2, 16)$ or $SL(2, 4) \times SL(2, 4)$, $C(X) = O(G)$, and $S \in \text{Syl}_2(X)$.

**Proof.** Let bars denote images in $G/O(G)$. Then by Lemma (1H), $\bar{L}$ is a standard subgroup of $\bar{G}$ and $C(\bar{L})$ has a cyclic $S_3$-subgroup. Let $H$ be an $S_3$-subgroup of $N_\ell(R)$. Then commutation by $t$ induces an $H$-isomorphism $S/R \to R$, and since $R = [R, H]$, it follows that $S = [S, H]$. Thus $S \leq X$, and in particular, $m(X) \geq 4$. Appealing to [16], we now get that $\bar{X} \cong SL(2, 16)$, $SL(2, 4) \times SL(2, 4)$ or $PSL(3, 4)$. If $\bar{X} \cong PSL(3, 4)$, then we must have that $\bar{t}$ acts on $\bar{X}$ as a graph automorphism. But then $\bar{t}$ does not normalize any $E_{16}$-subgroup of $\bar{X}$. Therefore, $\bar{X} \cong SL(2, 16)$ or $SL(2, 4) \times SL(2, 4)$ and so $S \in \text{Syl}_2(X)$. Since $R = [S, t] \leq L$, (3) and (4) of Lemma (1H) show that $C(X) = O(G)$ and $X \cong SL(2, 16)$ or $SL(2, 4) \times SL(2, 4)$.

**Lemma (1P).** Let $G$ be a finite group and $t$ an involution of $G$. Assume that $C(t) = K \times \langle t \rangle \times O(C(t))$ with $K \cong Sp(4, 2)$. Assume
furthermore that $G$ has a $t$-invariant subgroup $M$ isomorphic to the commutator subgroup of a maximal parabolic subgroup of $\text{Sp}(4, 4)$ and that conjugation by $t$ induces the same automorphism of $M$ as the involutory field automorphism of $\text{Sp}(4, 4)$. Then $E(G) \cong \text{Sp}(4, 4)$ and $C(E(G)) = O(G)$.

Proof. Let $S$ be a $t$-invariant $S_2$-subgroup of $M$ and let $T = \langle S, t \rangle$. We show that $I(St) = t^r = t^o \cap T$. Our assumption on the action of $t$ on $M$ in particular implies that $I(St) = t^r$, so $I(St) \subseteq t^o \cap T$. By assumption, $m(\mathcal{C}_s(x)) = 6$ for any $x \in I(S)$ and so, as $m(\mathcal{C}(t)) = 4$, $t^o \cap S = \emptyset$. Thus $t^o \cap T = I(St)$.

Let $T \leq U \in \text{Syl}_2(N(T))$. Then as $t^o \cap T = t^r$, $U = T\mathcal{C}_r(t)$. By hypothesis, $\mathcal{C}_s(t)$ is isomorphic to an $S_2$-subgroup of $\text{Sp}(4, 2)$, so $\mathcal{C}_r(t) \in \text{Syl}_2(C(t))$. Therefore, $\mathcal{C}_r(t) = C_r(t)$ and $U = T$. This shows that $T \in \text{Syl}_2(G)$.

Since $t^o \cap S = \emptyset$, Lemma (1E) shows that $t \in G'$, and since $M = M' \leq G'$, it follows that $S \in \text{Syl}_2(G')$. Thus, $X = \langle K'^o \rangle$ has $S_2$-subgroups of class at most 2. Now, $K' \cong A_6$ is standard in $G$ and $C(K')$ has cyclic $S_2$-subgroups. Moreover, $K'\text{O}(G) \not\subseteq G$ by Lemma (1H) as $t \in Z^*(G)$. Hence if bars denote images in $G/\text{O}(G)$, the same lemma shows that $C(\bar{X}) = 1$ and either $\bar{X}$ is simple or $\bar{X} \cong A_6 \times A_6$. In the first case, $\bar{X}$ is of known type by [9], and in either case $\bar{G}^\infty = \bar{X}$. Thus $\bar{M} = \bar{M}^\infty \leq \bar{X}$ and $\bar{S} \in \text{Syl}_2(\bar{X})$. Therefore, $\bar{X} \cong \text{Sp}(4, 4)$. Let $E$ be an $E_6$-subgroup of $S$. By hypothesis, $[E, t] = C_E(t) \cong E_6$, and hence $[[E, t], O(C(t))] = 1$ by the structure of $C(t)$. Therefore, $E(G) \cong \text{Sp}(4, 4)$ and $C(E(G)) = O(G)$ by (3) and (4) of Lemma (1H).

**Lemma (1Q).** Let $G$ be a finite simple group containing an $E_6$-subgroup $A$ such that $N(A)/C(A) \cong A_6$ and $A \in \text{Syl}_2(C(A))$. Then $G \cong M_{22}$, $\text{PSL}(4, q) \ (q \equiv 5 \text{ mod } 8)$, or $\text{PSU}(4, q) \ (q \equiv 3 \text{ mod } 8)$.

**Proof.** The proof of Lemma 12 of [17] shows that $G$ has $S_2$-subgroups of type $\hat{A}_8$ or $\hat{A}_{10}$. Then by [13] and [21], $G$ is isomorphic to one of the following groups: $\text{Mc}$, $M_{22}$, $M_{23}$, $\text{PSL}(4, q) \ (q \equiv 5 \text{ mod } 8)$, $\text{PSU}(4, q) \ (q \equiv 3 \text{ mod } 8)$, and $\text{Ly}$. The groups $\text{Mc}$, $M_{23}$, and $\text{Ly}$ have no $E_6$-subgroup whose automizer is isomorphic to $A_6$ (see a table on p. 543 of [7] and Proposition 9.1 of [13]). Thus we have the result.

**Lemma (1R).** Let $\hat{G}$ be a finite group and $\hat{Z}$ a subgroup of $Z(\hat{G})$ isomorphic to $Z_4$. Set $G = \hat{G}/\hat{Z}$ and let $A$ be an $E_8$-subgroup of $G$ satisfying the following conditions.

1. $N_0(A)/C_0(A) \cong \Sigma_4$. 

(2) \( A \in \text{Syl}_2(C_0(A)) \).
(3) \( |G : N_0(A)| \) is even.
(4) The preimage of \( A \) in \( \hat{G} \) is not abelian.

Furthermore, let \( t \) be an involution acting on \( \hat{G} \) and \( G \) in the following fashion.

(5) \( A \leq C_0(t) \leq N_0(A) \).
(6) \( C_0(t)C_0(A)/C_0(A) \cong \Sigma_3 \) wreath \( Z_2 \).
(7) \( N_0(A)/A = C_{N_0(A)/A}(t) \cdot C_0(A)/A \).
(8) \([\bar{Z}, t] \neq 1\).

Then there is a quasisimple characteristic subgroup \( \hat{H} \) of \( \hat{G} \) containing \( \bar{Z} \) such that \( C_{\hat{G}}(\hat{H}) = \bar{Z}O(\hat{G}) \). Either \( H/O(H) \cong SU(4, 3) \) or \( \hat{H}/Z(\hat{H}) \) has \( S_2 \)-subgroups isomorphic to those of \( PSL(6, q) \), \( q \equiv 3 \mod 4 \).

**Proof.** Let bars denote images in \( G/O(G) \). Assume that \( \bar{Q} = O_2(\bar{G}) \neq 1 \). Then \( \bar{Q} \cap C(\bar{A}) \neq 1 \) and so, as \( C(\bar{A}) = \bar{A}O(C(\bar{A})) \) by (2), it follows that \( 1 \neq \bar{Q} \cap \bar{A} \triangleleft N(\bar{A}) \). The condition (1) implies that \( N(\bar{A}) \) acts irreducibly on \( \bar{A} \). Therefore, \( \bar{A} \leq \bar{Q} \), but \( \bar{A} \neq \bar{Q} \) as \( |G : N(\bar{A})| \) is even. But now \( \bar{A} < N_0(\bar{A}) \triangleleft N(\bar{A}) \), which is a contradiction because \( O_2(N(\bar{A})) = \bar{A} \) by (1). Thus, \( O_2(\bar{G}) = 1 \).

By the above, \( F^*(\bar{G}) \) is a product of nonabelian simple groups. Let \( \bar{K} = F^*(\bar{G}) \), \( \bar{A} \leq \bar{T} \in \text{Syl}_2(\bar{G}) \), and \( \bar{U} = \bar{T} \cap \bar{K} \). Then \( 1 \neq \bar{U} \triangleleft \bar{T} \) by (6).

Hence we have that \( \bar{U} \cap \bar{A} = 1 \) and then, as \( \bar{U} \cap \bar{A} = \bar{K} \cap \bar{A} \triangleleft N(\bar{A}) \), we have that \( \bar{A} \leq \bar{U} \triangleleft \bar{K} \) just as above. However, \( \bar{A} \neq \bar{U} \) by (3), so \( \bar{A} \triangleleft N_0(\bar{A}) \leq N_\bar{K}(\bar{A}) \triangleleft N(\bar{A}) \). It now follows from (1) that \( N_{\bar{K}}(\bar{A})/C_{\bar{K}}(\bar{A}) \cong A_6 \) or \( \Sigma_6 \). Let \( \bar{L} \) be a component of \( \bar{K} \) and let \( \bar{V} = \bar{U} \cup \bar{L} \). Then \( 1 \neq \bar{V} \triangleleft \bar{A} = \bar{L} \cap \bar{A} \triangleleft N_{\bar{K}}(\bar{A}) \) and then \( \bar{A} \leq \bar{V} \triangleleft \bar{L} \) as before. As \( C(\bar{A}) \) is solvable, we conclude that \( \bar{K} \) is simple.

Now the conditions (5), (6), and (7) imply that there is an \( S_2 \)-subgroup \( S \) of \( N(\bar{A}) \) such that \( 1 \neq [S, t] \leq A \). Also, \( [C_{O_2(t)}(S), A] \leq [O(C_0(t)), O_2(C_0(t))] = 1 \). Therefore, \( [O(G), [S, t]] = 1 \) by [11, (1J)]. Thus, \( C_4(O(G)) \neq 1 \) and, since \( N(\bar{A}) \) is irreducible on \( \bar{A} \), we have \( [O(G), A] = 1 \).

Let \( K \) be the full inverse image of \( F^*(\bar{G}) \) in \( G \). Then \( A \leq C_K(O(K)) \). In particular, \( C_K(O(K)) \nleq O(K) \) and so, since \( K/O(K) \) is simple, we have that \( K = C_K(O(K))O(K) \). Thus \( K \) is a central product of \( K^\circ \) and \( O(K) \). Now we set \( H = K^\circ \). Then \( H \) is quasi-simple and \( Z(H) = O(H) \). Furthermore, \( A \leq O^r(K) = H \) and consequently, \( N_H(A)/C_H(A) \cong A_6 \) or \( \Sigma_6 \).

Now define \( \hat{H} \) and \( \hat{A} \) to be the subgroups of \( \hat{G} \) such that \( \hat{H}/\hat{Z} = H \) and \( \hat{A}/\hat{Z} = A \), respectively. Then, clearly \( \hat{H} \triangleleft \hat{G} \). We show that \( \hat{H} \) is perfect. Suppose false. Then there is a subgroup \( \hat{J} \) of \( \hat{H} \) of index 2 such that \( \hat{H} = \hat{J}\hat{Z} \). Let \( \hat{B} = \hat{A} \cap \hat{J} \). Then \( |\hat{B}| = 32 \), \( \hat{B}/\hat{Z} \cap \hat{B} \cong E_6 \), and \( A_6 \) acts on \( \hat{B}/\hat{Z} \cap \hat{B} \) nontrivially. This forces \( \hat{B} \) to be
We check that $\hat{H}$ is the desired subgroup of $\hat{G}$. By definition, 
$\hat{Z} \leq \hat{H}$ and $C_{\hat{G}}(\hat{H}) = \hat{Z}_0(\hat{G})$ since $\hat{H} = F^*(\hat{G})$ is simple. To prove the second assertion, assume first that $N_{\hat{H}}(A)/C_{\hat{H}}(A) \cong A_e$. Then $\hat{H}/Z(H) \cong M_{22}$, $PSL(4, q)$ ($q \equiv 5 \mod 8$) or $PSU(4, q)$ ($q \equiv 3 \mod 8$) by Lemma (1Q). The Schur multipliers of these simple groups are known [5], and so we can determine the structure of $\hat{H}$. We see that $\hat{H}/O(\hat{H}) \cong SL(4, q)$ or $SU(4, q)$. As (5) and (6) imply that $C_{\hat{o}}(t)$ is solvable, Lemma (1I) and (8) show that $\hat{H}/O(\hat{H}) \cong SU(4, 3)$. Therefore, assume that $N_{\hat{H}}(A)/C_{\hat{H}}(A) \cong \Sigma_6$. In this case, a similar argument and the theorem of [26] yield that $\hat{H}/Z(\hat{H})$ has $S_2$-subgroups of type $PSL(6, q)$, $q \equiv 3 \mod 4$. The proof is complete.

2. In this section, we fix notation for $L = PSU(4, 2) \cong SU(4, 2)$ and set down some facts about $L$ and its automorphisms.

By choosing a suitable basis of the underlying hermitian space, we identify the elements of $L$ with the $4 \times 4$ matrices $x$ with entries in $F_4$ satisfying

$$(2.1) \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ \bar{x} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$
and $\det x = 1$,

where $'x$ denotes the transposed matrix of $x$ and $\bar{x}$ is the matrix obtained by squaring each entries of $x$.

Denote by $P$ the group of matrices

$$(2.2) \quad \begin{pmatrix} 1 & & & \\ a & 1 & & \\ c & b & 1 & \\ d & a^2b + c^2 & a^2 & 1 \end{pmatrix}$$

where $b^2 = b$ and $d^2 = ac^2 + a^2c + d$. Define $A_1$ to be the group of matrices (2.2) with $b = 0$, and define $A_2$ to be the group of matrices (2.2) with $a = 0$. Let $Z$ be the group of matrices (2.2) with $a = b = c = 0$.

Let $e$ be a primitive cube root of unity in $F_4$ and set

$$a_1 = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & & \\ & e^2 & 1 \\ & & e \end{pmatrix}.$$
Denote by $H$ the group generated by the matrix
\[
\begin{pmatrix}
e \\
e^2 \\
e^2 \\
e
\end{pmatrix}
\]
Denote by $K_1$ the group of matrices
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}, \quad \begin{pmatrix}a & b \\
c & d\end{pmatrix} \in \text{SL}(2, 2)
\]
and denote by $K_2$ the group of matrices
\[
\begin{pmatrix}a & b \\
c & d
\end{pmatrix}, \quad \begin{pmatrix}a & b \\
c & d\end{pmatrix} \in \text{SL}(2, 4)
\]
Now we list some facts about $L$ and its automorphisms. Proofs will be mostly omitted because the assertions are consequences of straightforward calculations involving matrices.

**Lemma (2A).**
(1) \(|P| = 64\) and \(P \in \text{Syl}_4(L)\).

(2) \(P\) is generated by the involutions \(a_1, a_2, b_0, b_1, b_2, b_3\), and the following commutator relations hold:

\[
\begin{align*}
[a_1, b_2] &= b_0, \\
[a_1, b_3] &= b_0b_1, \\
[a_2, b_1] &= b_0, \\
[a_2, b_3] &= b_0b_2.
\end{align*}
\]

All other commutators are trivial.

(3) \(A_1\) is generated by \(a_1, a_2, b_1, b_2\).

(4) \(A_2\) is generated by \(b_0, b_1, b_2, b_3\).

(5) \(Z(P) = Z = \langle b_0 \rangle, \ Z_5(P) = \langle b_0, b_1, b_2 \rangle\).

(6) \(\mathcal{S}_0(P) = \{A_2\}\).

(7) \(\mathcal{S}^*(P/Z) = \{A_1/Z, A_2/Z\}\).

(8) \(P = A_1A_2\).

In the above lemma, (1) follows from the fact that \(|L| = 2^8 \cdot 3^4 \cdot 5\).

**Lemma (2B).**

(1) \(N_L(P) = HP\).

(2) The following relations hold:

\[
\begin{align*}
a_i^j &= a_2, \\
a_2^j &= a_1 a_2, \\
b_i^j &= b_2, \\
b_2^j &= b_0 b_2.
\end{align*}
\]

\(j\) centralizes other generators of \(P\) listed in Lemma (2A)(2).

(3) \(H\) acts regularly on \((P/A_2)^*, (A_1/A_1 \cap A_2)^*,\) and \((A_1 \cap A_2/Z)^*\).

**Lemma (2C).**

(1) \(N_L(A_1) = (K_1 \times H)A_1\).

(2) \(A_1 \cong D_8 \times D_8 \cong Q_8 \times Q_8\) and \(Z(A_1) = Z = \langle b_0 \rangle\).

(3) Under the action of \(N_L(A_1), (A_1/Z)^*\) decomposes into two orbits of lengths 9 and 6, the former corresponding to involutions of \(A_1 - Z\) and the latter corresponding to elements of order 4 of \(A_1\). \(O_3(K) \times H = \langle s_1b_0 \rangle \times \langle j \rangle\) acts regularly on the orbit of length 9.

(4) \(C_L(A_1/Z) = A_1\).

(5) \(O^{3,2'}(K, A_1) = A_1\).

(4) and (5) above are consequences of (1), (2), and (3).

**Lemma (2D).**

(1) \(N_L(A_2) = K_2A_2\).

(2) \(A_2\) is a natural module for \(K_2 \cong A_5\).

(3) \(C_L(A_2) = A_2\).

(4) Under the action of \(K_2, A_2^*\) decomposes into two orbits of lengths 5 and 10, the former consisting of \(c_1, c_2, c_3, c_4, \) and \(c_5\).

**Lemma (2E).**
(1) L has two conjugacy classes of involutions, and we may choose \( b_0 \) and \( b_1 \) as the representatives of these classes.

(2) \( C_P(b_0) = P \) and \( C_L(b_0) = N_L(A_4) \).

(3) \( C_P(b_1) = \langle a_1, A_2 \rangle \) and \( C_L(b_1) = \langle a_1, s_e \rangle A_2 \).

(4) Involutions of \( A_1 - Z \) are conjugate to \( b_i \) in \( N_L(A_4) \).

(5) Central involutions of \( L \) contained in \( A_2 \) are \( c_1, c_2, c_3, c_4, c_5 \), and so they are all conjugate in \( N_L(A_2) \).

Let \( A = \text{Aut}(L) \) and identify \( L \) with \( \text{Inn}(L) \). Then \( A = \langle f \rangle L \), where \( f \) is the automorphism of \( L \) induced by the automorphism of \( F_4 \) of order 2. Let \( R = \langle f \rangle P \).

**Lemma (2F).**

(1) \( R \in \text{Syl}_4(A) \).

(2) The following relations hold:

\[
a_1^f = a_1, \quad a_2^f = a_1 a_2,
\]

\[
b_0^f = b_0, \quad b_1^f = b_1, \quad b_2^f = b_1 b_2, \quad b_3^f = b_3.
\]

(3) \( r(R) = 4 \).

(4) \( Z(R) = Z(P) = Z, \quad R' = \langle a_1, b_0, b_1, b_2 \rangle \).

(5) \( R \) has exactly four \( E_{16} \)-subgroups: \( A_2, \quad \langle C_{A_1}(f), f \rangle = \langle a_1, b_0, b_1, f \rangle, \quad \langle C_{A_2}(f), f \rangle = \langle b_1 b_2, b_0, f \rangle, \) and \( \langle C_{A_2}(f), f^a \rangle = \langle b_0, b_1, b_2, a_1 f \rangle \).

All these are self-centralizing in \( R \).

(6) \( J_r(R) = \langle C_{A_1}(f), A_2, f \rangle = \langle a_1, b_0, b_1, b_2, b_3, f \rangle, \quad ZJ_r(R) = \langle b_0, b_1 \rangle \).

For the proof of (3) above, see [17, Lemma 2]. (6) is a direct consequence of (5).

**Lemma (2G).**

(1) \( N_A(A_4) = \langle f \rangle N_Z(A_4) \).

(2) \( N_A(A_4)/A_4 \cong K_4 \times \langle f \rangle H \cong \Sigma_3 \times \Sigma_3 \).

(3) \( C_A(A_4)/Z) = A_1 \).

(4) \( O_2(N_A(A_4)) = A_1 \).

(2), (3), and (4) above are consequences of (1) and Lemma (2C). See Lemma (2D) for the proof of the next lemma.

**Lemma (2H).**

(1) \( N_A(A_2) = \langle f \rangle N_Z(A_2) \).

(2) \( N_A(A_4)/A_4 \cong \langle f \rangle K_2 \cong \Sigma_5 \).

(3) \( C_A(A_3) = A_2 \).

(4) \( O_2(N_A(A_2)) = A_2 \).

**Lemma (2I).** \( N_A(\langle C_{A_4}(f), f \rangle) = \langle f \rangle K_i A_i \).
Proof. Observe that $b_0$ is the only central involution of $L$ contained in $A_i$. By Lemma (2E)(2), we have

$$N_A(\langle C_{A_i}(f), f \rangle) \leq N_A(C_{A_i}(f)) \leq C_A(b_0) = \langle f \rangle N_L(A_i).$$

Thus, using Lemma (2C)(1), we obtain the result.

**Lemma (2J).**

1. $C_A(C_{A_i}(f)) = \langle A_i, f \rangle$.
2. $N_A(\langle C_{A_i}(f), f \rangle) = \langle f, a_i, s_i, A_i \rangle$.

**Proof.** Use Lemma (2E)(3) to prove (1). Once (1) is proved, then $N_A(\langle C_{A_i}(f), f \rangle) \leq N_A(C_{A_i}(f)) \leq N_A(\langle A_i, f \rangle) \leq N_A(A_i)$, hence (2) follows easily.

**Lemma (2K).**

1. $C_L(f) \cong Sp(4, 2) \cong \Delta_6$.
2. $C_L(fb_0) = C_L(f) \cap C_L(b_0) = \langle a_i, b_0, b_1, b_3, s_i \rangle$.
3. If $x \in \Gamma(A - L)$, then $x \sim f$ or $fb_0$ in $A$ and $x^4 \cap C_L(x) \neq \{x\}$.
4. If $x \in \Gamma(N_A(P) - L)$ and $C(x) \cap N_L(A_i)$ is an extension of $E_s$ by $SL(2, 2)$, then $x \in f^4$.

**Proof.** For the proof of (1), (2), and (3), see [3, § 19]. For (4), suppose $(fb_0)^g = x, g \in L$. Since $C_L(fb_0)$ is also an extension of $E_s$ by $SL(2, 2)$ by (2), we have $C_L(fb_0)^g = C(x) \cap N_L(A_i)$, hence $\langle a_i, b_0, b_1 \rangle^g = O_4(C_L(fb_0)^g) = O_4(C(x) \cap N_L(A_i)) = C(x) \cap A_3$. Since $b_0 \in C(x) \cap A_3$ and since $b_0$ is strongly closed in $A_i$ with respect to $L$ by Lemma (2E), we have $b_0^g = b_0$, hence $g \in C_L(b_0) = N_L(A_i)$. But $C_L(fb_0)^g \leq N_L(A_i) \cap N_L(A_i) = N_L(P)$, a contradiction. Therefore, $x \in f^4$.

3. In this section, we begin the proof of the theorem stated in the introduction.

Let $G$ be a finite group which contains a standard subgroup $L$ isomorphic to $PSU(4, 2)$, and assume that $C(L)$ has a cyclic $S_2$-subgroup.

We identify $L$ with the group of $4 \times 4$ matrices $x$ satisfying (2.1). The symbols used in § 2 for various objects defined for $PSU(4, 2)$ will retain their meaning for the balance of the paper. Thus $P$ is an $S_2$-subgroup of $L$ consisting of matrices (2.2).

Let $t$ be an involution of $C(L)$ and set $C = C(t)$. We first prove the following.

**Lemma (3A).** If $t^g \cap LC_0(L) = \{t\}$, then $r(\langle L^g \rangle) = 4$. 
Proof. Assume that $t^g \cap L C_c(L) = \{t\}$. Let $T \in \text{Syl}_2(C_c(L))$, $Q = PT$, and $Q \leq R \in \text{Syl}_2(C)$. Then $t \in Z(R)$ and $Z(R) \leq Q$ by Lemma (2F). Therefore, $t^g \cap Z(R) = \{t\}$ by our assumption, and hence $N(R) \leq C$. This implies that $R \in \text{Syl}_2(G)$.

Now if $t \in Z^*(G)$, then $LO(G) \triangleleft G$ by Lemma (1H). Therefore, we may assume that $t^g \cap R \neq \{t\}$ by [10].

Let $t \neq u \in t^g \cap R$. Then $u \notin Q$ by our assumption, and so $|R: Q| = 2$. Notice that $Q/P \cong T$ is cyclic by our hypothesis. Hence if $R/P$ is nonabelian, then $uP \sim tuP$ in $R$ by Lemma (1A), and so $t^g \cap tuP \neq \emptyset$. If $R/P$ is abelian, then by Lemma (1E), either $t^g \cap \langle tu \rangle P \neq \emptyset$ or $t \in G'$. In the latter case, $R \cap G' = P$ or $P \langle tu \rangle$ as $P \leq L \leq G'$. Hence $r(\langle L^g \rangle) = 4$ by Lemma (2F). Therefore, we may assume that $t^g \cap tuL \neq \emptyset$ for all $u \in t^g \cap C$, $u \neq t$.

Suppose $tu \in t^g$ for all $u \in t^g \cap C$ with $u \neq t$. Let $t^g \in C - \{t\}$. If $t \in L^g C(L^g)$, then there exists an element $x \in C_{L^g}(t^g)$ with $tx \in t^g$ by Lemma (2K). Then $x = t(tx) \in t^g$, so $x^{2^{-1}} \in t^g \cap L$, contrary to our assumption. If $t \in L^g C(L^g)$, then $t \neq t^{2^{-1}} \in t^g \cap L C_c(L)$, contrary to our assumption. Thus there is a conjugate $t^g \in C - \{t\}$ such that $tt^g \neq t$.

Choose $t^g \in C - \{t\}$ so that $tt^g \neq t$, and let $t^h \in tt^g L$. If $C_L(t^h) \cong C_L(tt^g) = C_L(t^g)$, then $t \sim t^h \sim tt^g$ by Lemma (2K), a contradiction. Hence $C_L(t^h) \neq C_L(t^g)$. If $R/P$ is nonabelian, we may choose $h \in gR$ by Lemma (1A). But then $C_L(t^h) \cong C_L(t^g)$, a contradiction. Therefore, $R/P$ is abelian.

Now $Z(R) \leq Q$ by Lemma (2F), so $P \langle tt^g \rangle$ contains no extremal conjugates of $t$ in $R$. Thus $t \notin G'$ by Lemma (1E), and $r(\langle L^g \rangle) = 4$ as before. The proof is complete.

In view of Lemma (3A), we shall make the following hypothesis.

**Hypothesis (3.1).** $t^g \cap L C_c(L) \neq \{t\}$.

We next prove

**Lemma (3B).** Under Hypothesis (3.1), $\langle t \rangle \in \text{Syl}_2(C_c(L))$.

**Proof.** Let $T \in \text{Syl}_2(C_c(L))$ and let $t \neq t^g \in L C_c(L)$. We may assume $t^g \in PT$ so $T \leq C(t^g) = C^g$. Lemma (2E) shows that $C_L(t^g) = L \cap C^g$ contains an $E_{16}$-subgroup $A$. The image of $A \times T$ in $C^g/C_c(L)^g$ has rank at least 4 and its exponent is equal to that of $T$ as $T \cap C_c(L)^g = 1$. Thus Lemma (2F)(5) forces $|T| = 2$.

**Definition (3.1).** Let $Q = P \langle t \rangle$, and $B_i = A_i \langle t \rangle$ for $i \in \{1, 2\}$.

**Lemma (3C).** We have $t^g \cap L = \emptyset$.
Proof. This is obvious if \( t^g \cap LC_0(L) = \{ t \} \). Therefore, we may assume Hypothesis (3.1). Suppose \( t^g \in L \) for some \( g \in G \). By Lemma (2E), we may assume \( t^g = b_0 \) or \( b_1 \), so that \( C_p(t^g) \subseteq \text{Syl}_2(C_L(t^g)) \) and \( t^g \) has a square root in \( P \). In particular, \( t \) has a square root in \( C \). Hence, if \( Q \leq R \subseteq \text{Syl}_2(C) \), then \( R/P \cong \mathbb{Z} \) by Lemma (3B). Thus \( I(C) \leq L \langle t \rangle \). But then \( C_p(t^g) = \Omega_1(C_p(t^g)) \leq L^g \langle t^g \rangle \), and therefore, \( t^g \in C_p(t^g) \leq L^g \). This is a contradiction proving the lemma.

**Lemma (3D).** If \( C \) contains an \( S_2 \)-subgroup of \( G \), then \( r(\langle L^g \rangle) = 4 \).

**Proof.** We may assume Hypothesis (3.1) by Lemma (3A). Let \( Q \leq R \subseteq \text{Syl}_2(C) \), so that \( R \subseteq \text{Syl}_2(G) \). Suppose that \( t \in G' \). As \( |R/P| \) is at most 4 by Lemma (3B), Lemmas (1E) and (3C) show that there is an element \( u \in t^g \cap (R - Q) \) and, moreover, \( \langle u \rangle P \) contains an extremal conjugate \( v \) of \( t \) in \( R \). However, since \( Z(R) \leq Q \), we have \( v \in P \), which is impossible by Lemma (3C). Therefore, \( t \not\in G' \) and so \( r(\langle L^g \rangle) = 4 \) as in the third paragraph of the proof of Lemma (3A).

**Lemma (3E).** \( N(B_2) \leq N(A_2) \).

**Proof.** If \( N(B_2) \leq C \), then \( N(B_2) \) normalizes \( B_2 \cap L = A_2 \). If \( N(B_2) \not\leq C \), then \( \Omega = t^{N(B_2)} \neq \{ t \} \). By Lemma (3C), \( \Omega \leq A_t \), so \( A = \langle ab | a, b \in \Omega \rangle \) is a nonidentity \( N(B_2) \)-invariant subgroup of \( A_2 \). As \( K_2(\leq N(B_2)) \) acts irreducibly on \( A_3, A_2 = A \). Thus \( N(B_2) \leq N(A_2) \).

**Lemma (3F).** \( |C(A_2) \cap N(B_2): C(B_2)| \) is a power of 2.

**Proof.** As \( C(A_2) \cap N(B_2) \) stabilizes the series \( 1 < A_2 < B_2 \), the assertion follows from [12, Corollary 5.3.3].

**Lemma (3G).** Let \( \Omega = t^{N(B_2)} \). Then \( \Omega = \{ t \}, \{ t, c_1t, c_2t, c_3t, c_4t, c_5t \} \) or \( A_t \). If \( \Omega \neq \{ t \} \), \( N(B_2)^g \) is a primitive permutation group on \( \Omega \), and \( C(\Omega) = C(B_2) \).

**Proof.** By Lemma (3C), \( \Omega \leq A_t \). Under the action of \( K_2 \), which is contained in \( N_0(B_2) \), \( A_4 \) decomposes into two orbits of lengths 5 and 10, the former consisting of \( c_1, c_2, c_3, c_4, \) and \( c_5 \). Hence it is enough to show that \( |\Omega| \neq 11 \). Suppose \( |\Omega| = 11 \). Then by Lemmas (3E) and (3F), \( C(A_2) \cap N(B_2) = C(B_2) \) and then \( N(B_2)/C(B_2) \) is isomorphic to a subgroup of \( \text{Aut}(A_2) \cong GL(4, 2) \). This is a contradiction because \( |GL(4, 2)| \) is not divisible by 11.

**Lemma (3H).** Let \( f \in I(C - LC_0(L)) \) and suppose that the action of \( f \) on \( L \) is induced by the involution of \( \text{Aut}(F_4) \). If
If an element \( g \) of \( G \) interchanges \( B_2 \) and \( \langle C_{A_2}(f), f, t \rangle \), then \( g \) normalizes their intersection \( \langle b_0, b_1, b_2, t \rangle \) and so \( t^g = t \) for some \( h \in N(B_2) \) by hypothesis. However, \( gh \in C \) and \( \langle C_{A_2}(f), f, t \rangle^{gh} = B_2 \) which is a contradiction as \( \langle C_{A_2}(f), f, t \rangle \not\subseteq \langle L, t \rangle \) while \( B_2 \subseteq \langle L, t \rangle \).

**Lemma (31).** Let \( f \) be as in (3H) and suppose that \( \langle C_{A_2}(f), f, t \rangle^g = B_2 \) for some \( g \in G \). Then \( A_i \leq O^{2,v}(N(B_2)) \).

**Proof.** As \( \langle L, f, t \rangle = L \langle f \rangle \times \langle t \rangle \) and as \( K_A = N_L(\langle C_{A_2}(f), f \rangle) \) by Lemma (2I), we have that \( X = N_{L \langle f \rangle \langle t \rangle}(\langle C_{A_2}(f), f, t \rangle) \) is equal to \( \langle K_A, f, t \rangle \). Thus \( O^{2,v}(X) = O^{2,v}(K_A) = A_i \) by Lemma (2C), and hence \( A_i \leq O^{2,v}(N(\langle C_{A_2}(f), f, t \rangle)) \). Therefore, \( A_i \leq O^{2,v}(N(B_2)) \).

**Lemma (3J).** Under Hypothesis (3.1), the following conditions hold.

1. \( N(Q) \leq N(B_1) \cap N(B_2) \).
2. \( m(C) = 5 \).
3. \( C \) does not have an \( E_{1_{2 \leq 2}} \)-subgroup \( X \) such that \( SL(2, 2) \times SL(2, 2) \subseteq N_C(X)/C_C(X) \).

**Proof.** By Lemma (2A), \( \mathcal{S}(Q/Z(Q)) = \{B_1/Z(Q), B_2/Z(Q)\} \), hence (1) follows. (2) is a direct consequence of Lemma (2F)(5). By the same lemma, if \( X \) is an \( E_{1_{2 \leq 2}} \)-subgroup of \( C \), then \( X \sim B_2 \), \( \langle C_{A_2}(f), f, t \rangle \), or \( \langle C_{A_2}(f), f, t \rangle \) in \( C \), where \( f \) is an involution acting on \( L \) as a field automorphism. Hence \( N_C(X)/C_C(X) \sim \Sigma_5 \) or \( Z_2 \times SL(2, 2) \) by Lemmas (2H)—(2J). Thus (3) holds.

4. In this section, we shall work under the following hypothesis.

**Hypothesis (4.1).** \( t^{N(B_2)} = \{t\} \).

We prove the following theorem.

**Theorem (4A).** Under Hypothesis (4.1), \( r(\langle L \rangle) = 4 \).

The proof involves a series of reductions. First, if \( t^g \cap LC_c(L) = \{t\} \), then Theorem (4A) holds by Lemma (3A). Therefore, we assume...
that $G$ satisfies Hypothesis (3.1). Then $\langle t \rangle \in \text{Syl}_2(C_c(L))$ by Lemma (3B).

**Lemma (4B).** If $t \in G'$, then Theorem (4A) holds.

**Proof.** By Hypothesis (4.1), $N(B_4) \leq C$ so that $N(B_4) \cap C(A_4) = C(B_4)$. This implies that $B_4 \in \text{Syl}_2(C(A_4))$ as $C(B_4) = B_4 \triangleleft C$ by Lemmas (2H) and (3B). Hence $N(A_4) = N(B_4)C(A_4) = N(B_4)C(A_4)$ by a Frattini argument, and so $N(A_4)/C(A_4) \cong A_5$ or $\Sigma_5$ by Lemmas (2D) and (2H). We also have that $N_L(A_5) \leq N(G)(A_4)$ since $L \leq G'$. Therefore, $N_{\sigma}(A_4)/C_{\sigma}(A_4) \cong A_5$ or $\Sigma_5$. Also, $A_5 \leq C_{\sigma}(A_4) \triangleleft C(A_5)$. Since $B_4 \in \text{Syl}_2(C(A_5))$ and $t \in G'$, it follows that $A_5 \in \text{Syl}_2(C_{\sigma}(A_5))$. Thus, $r(G') = 4$ by [17, Theorem 3] and hence $r(\langle L^2 \rangle) = 4$. The proof is complete.

Let $Q \leq R \in \text{Syl}_2(C)$. The following lemma follows from Lemma (3D).

**Lemma (4C).** If $R \in \text{Syl}_2(G)$, then Theorem (4A) holds.

In view of Lemmas (4B) and (4C), we shall form now on assume that

$t \in G'$ and $R \in \text{Syl}_2(G)$.

We shall eventually derive a contradiction from this hypothesis.

**Lemma (4D).** There is an involution $f \in C$ whose action on $L = \text{PSU}(4, 2)$ is induced by the automorphism of $F_4$ of order 2.

**Proof.** It is enough to show that $I(R - Q) \neq \emptyset$. Since $R \in \text{Syl}_2(G)$, $N(R) \not\leq C$ so that $N(R) \not\leq N(B_4)$ as $N(B_4) \not\leq C$ by Hypothesis (4.1). If $I(R) \leq I(Q)$, then $B_4$ would be the only $E_{27}$-subgroup of $R$ by Lemma (2A), and so $N(R) \leq N(B_4)$. Therefore, $I(R - Q) \neq \emptyset$, as required.

We assume without loss of generality that $f \in R$. Notice that $R = Q\langle f \rangle$. Let $S \in \text{Syl}_2(N(R))$. Then $R < S$, so we may choose $g \in S - R$.

**Lemma (4E).** The following conditions hold.

1. $S = R\langle g \rangle$ and $g^2 \in R$.
2. $t^a = b_0t$ and $b_0^2 = b_0$.
3. $g$ interchanges $B_2$ and $\langle C_{A_4}(f), f, t \rangle$ by conjugation.
(4) \( g \in N(A_1) \cap N(B_1). \)

Proof. As \( C_8(t) = R < S, \{t\} < t^g. \) Also, \( t^g \leq Z(R). \) As \( Z(R) = \langle b_0, t \rangle \) by Lemma (2F) and as \( t \not\sim b_0 \) by Lemma (3C), it follows that \( t^g = \{t, b_0t\}. \) Therefore, \( |S: R| = 2 \) and \( S \leq C(b_0). \) Hence (1) and (2) follow.

By Lemma (2F), \( B_2, \langle C_{A_1}(f), f, t \rangle, \langle C_{A_2}(f), f, t \rangle, \) and \( \langle C_{A_3}(f), f, t \rangle \rangle, \) where \( x \in P - C_{A_1}(f)A_2, \) are the only \( E_{32}\)-subgroups of \( R. \) Since \( N(B_2) \leq C \) by Hypothesis (4.1), \( B_2 \neq B_2' \triangleleft R. \) Thus (3) holds. Then Lemma (31) shows that \( A_i' \leq O_{3, 2}^2(N(B_2)). \) Since \( N(N(B_2)) = N_L(A_3) \) by Lemma (2D). Hence \( A_i' \leq R \cap N_L(A_3) = P. \) Also, \( b_0 = b_0' \in A_i'. \) Since \( A_i/B_0 \) is the only \( E_{16}\)-subgroup of \( P/\langle b_0 \rangle \) by Lemma (2A), we have that \( A_i' = A_i. \) Since \( b_1 = \langle A_i, t \rangle \) and \( t^g = b_0t \in A_i t, g \in N(B_1). \) The proof is complete.

**Lemma (4F).** We may choose \( f \) so that the following conditions hold.

1. \( g \) interchanges \( A_1 \cap A_2 \) and \( C_{A_1}(f) \) by conjugation.
2. \( g \) interchanges \( P \) and \( \langle A_1, f \rangle \) by conjugation.
3. \( g \in N(\langle P, f \rangle). \)
4. \( t^g \cap \langle P, f \rangle = \emptyset. \)

Proof. Using Lemma (4E), we may deduce as follows:

\[
(A_1 \cap A_2)^g = (A_1 \cap B_0)^g = A_1 \cap \langle C_{A_1}(f), f, t \rangle = C_{A_1}(f).
\]

Since \( g^2 \in R \leq N(A_1 \cap A_2), C_{A_1}(f)^g = A_1 \cap A_2. \) Now \( A_i^g \) is a maximal subgroup of \( \langle C_{A_1}(f), f, t \rangle \) containing \( C_{A_1}(f). \) Since \( t^g \cap L = \emptyset \) by Lemma (3C), \( A_i^g \neq \langle C_{A_1}(f), t \rangle. \) Therefore, \( A_i^g = \langle C_{A_1}(f), f \rangle \) or \( \langle C_{A_1}(f), ft \rangle. \) Replacing \( f \) by \( ft \) in the latter case, we may choose \( f \) so that \( A_i^g = \langle C_{A_1}(f), f \rangle. \) Then

\[
P^g = (A_1A_2)^g = A_1\langle C_{A_1}(f), f \rangle = \langle A_1, f \rangle,
\]

and \( \langle A_1, f \rangle^g = P \) as \( g^2 \in R \leq N(P). \) Hence \( g \) normalizes \( \langle P, A_1, f \rangle = \langle P, f \rangle. \) Since \( A_i^g = \langle C_{A_1}(f), f \rangle \) and \( t^g \cap A_2 = \emptyset, t^g \cap \langle C_{A_1}(f), f \rangle = \emptyset. \) By Lemma (2K), every involution of \( Pf \) is conjugate to an element of \( C_{A_1}(f)f. \) Therefore, \( t^g \cap \langle P, f \rangle = \emptyset. \) The proof is complete.

**Lemma (4G).** The following conditions hold.

1. \( N(R) \leq N(B). \)
Proof. Since $Z(B_i) = \langle b_0, t \rangle$ by Lemma (2C), $t^{N(B_i)} \leq \{t, b_0 t\}$. By Lemma (4E), $g \in N(B_i) - C$. Hence $|N(B_i) : N_C(B_i)| = 2$ and $N(B_i) = N_C(B_i) \langle g \rangle$. Similarly, $N(R) = N_C(R) \langle g \rangle$. Since $N_C(R) \leq N_C(B_i)$ by Lemma (3J), (1) follows. Now $R \in \text{Syl}_2(N_C(B_i))$, so $S = R \langle g \rangle \in \text{Syl}_2(N(B_i))$. The proof is complete.

Lemma (4H). $I(S) \not\subseteq I(R)$.

Proof. Suppose this is false. Then $\Omega_1(S) = R$, so $N(S) \leq N(R)$, and Lemma (4G) yields that $S \in \text{Syl}_2(G)$. Also, $t^a \cap S = t^a \cap R \leq \langle P, f \rangle t$ by Lemma (4F)(4). As $\langle P, f \rangle \triangleleft S$ and $|S/\langle P, f \rangle| = 4$ by Lemma (4F), Lemma (1E) forces $t \not\in G'$ against our hypothesis. Therefore, $I(S) \not\subseteq I(R)$.

Now let bars denote images in $C(b_0)/\langle b_0 \rangle$. Then $S$ acts on $\tilde{A}_i$ by Lemma (4E). In the following two lemmas, we collect necessary information on this action. Notice that we may choose $\tilde{a}_1, \tilde{b}_2, \tilde{a}_2, \tilde{b}_1$ as a basis of $\tilde{A}_i$.

Lemma (4I). The following conditions hold.

(1) $\tilde{a}_i^{b_2} = \tilde{a}_1 \tilde{b}_2$, $\tilde{b}_2^{b_2} = \tilde{b}_2$, $\tilde{a}_2^{b_2} = \tilde{b}_2 \tilde{a}_2$, $\tilde{b}_1^{b_2} = \tilde{b}_1$.

(2) $\tilde{a}_i^{b} = \tilde{a}_i$, $\tilde{b}_2^{b} = \tilde{b}_2 \tilde{b}_1$, $\tilde{a}_2^{b} = \tilde{a}_2 \tilde{b}_2$, $\tilde{b}_1^{b} = \tilde{b}_1$.

(3) $\tilde{a}_i^{b_2f} = \tilde{a}_i \tilde{b}_2$, $\tilde{b}_2^{b_2f} = \tilde{b}_2 \tilde{b}_1$, $\tilde{a}_2^{b_2f} = \tilde{a}_2 \tilde{b}_2 \tilde{b}_1$, $\tilde{b}_1^{b_2f} = \tilde{b}_1$.

(4) $C_{\tilde{A}_i}(b_2) = \langle \tilde{b}_2, \tilde{b}_1 \rangle$.

(5) $C_{\tilde{A}_i}(f) = \langle \tilde{a}_i, \tilde{b}_2 \rangle$.

(6) $C_{\tilde{A}_i}(b_2f) = \langle \tilde{a}_1 \tilde{b}_2, \tilde{b}_1 \rangle$.

Proof. (1), (2), and (3) follow from relations listed in Lemmas (2A) and (2F). (4), (5), and (6) are consequences of (1), (2), and (3), respectively.

Now choose $f$ as in Lemma (4F). So far $g$ was an arbitrary element of $S - R$. We now prove

Lemma (4J). We may choose $g$ so that $g^s \in A_i$ and the following relations hold:

$$\tilde{a}_i = \tilde{b}_2, \tilde{b}_2 = \tilde{b}_1, \tilde{a}_2 = \tilde{a}_2, \tilde{b}_1 = \tilde{b}_1.$$

For $g$ satisfying these relations, we have that

$$C_{\tilde{A}_i}(g) = \langle \tilde{a}_1 \tilde{b}_2, \tilde{a}_2, \tilde{b}_1 \rangle.$$
Proof. Lemma (41) shows that $b_3, f$, and $b_2f$ have the following matrix forms with respect to the basis $\bar{a}_1, \bar{b}_2, \bar{a}_2, \bar{b}_1$ of $\bar{A}_1$, respectively.

\[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix}
\]

Choosing a suitable element $g \in S - R$, we determine the matrix form of $g$. By Lemma (4F), $g$ interchanges $\bar{A}_1 \cap \bar{A}_2 = \langle \bar{b}_1, \bar{b}_2 \rangle$ and $C_{\bar{A}_1}(f) = \langle \bar{a}_1, \bar{b}_1 \rangle$, and so $g$ normalizes $\langle \bar{b}_1 \rangle$. Therefore, $g$ has the following matrix form.

\[
\begin{pmatrix}
1 & a \\
1 & b \\
c & d & 1 & e \\
1 & 1
\end{pmatrix}
\]

By Lemma (4H), we may assume from the outset that $g^2 \in A_1$. Then $g$ induces an involutory automorphism on $\bar{A}_1$, and so the square of the matrix of $g$ is equal to the unit matrix. Hence we have that $a = b$ and $c = d$. Thus $g$ has the following matrix form.

\[
\begin{pmatrix}
1 & a \\
1 & a \\
c & c & 1 & e \\
1 & 1
\end{pmatrix}
\]

Now $P^g = \langle A_1, f \rangle$ by Lemma (4F), so $gbfg \equiv f \mod A_1$. This implies that

\[
\begin{pmatrix}
1 & a \\
1 & a \\
c & c & 1 & e \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & a \\
1 & a \\
c & c & 1 & e \\
1 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix}.
\]

Hence we have that $a = c$, and so $g$ has the following matrix form.

\[
\begin{pmatrix}
1 & a \\
1 & a \\
a & a & 1 & e \\
1 & 1
\end{pmatrix}
\]

We compute that $b_2fg$ has the following matrix form.
Hence replacing \( g \) by \( b_sfg \) if \( a = 1 \), we may assume that \( g \) has the following matrix form.

\[
\begin{pmatrix}
1 & a + 1 \\
1 & a + 1 \\
a + 1 & a + 1 & 1 & e + 1 \\
1 & 1 & e & 1
\end{pmatrix}
\]

This implies that \( a_2^e = a_2b_i^e \) or \( a_2b_i^e \). Since \( a_2^e \) is an involution, it follows that \( e = 0 \). This implies that the relations listed in Lemma (4J) hold. The latter half of the lemma follows from this easily.

Now choose \( g \) as in Lemma (4J). We next prove

**Lemma (4K).** The following conditions hold.

1. \( \langle P, f, g \rangle | A_1 \cong D_b \) and \( Z(\langle P, f, g \rangle | A_1) = \langle A_1, b_nf \rangle | A_1 \).
2. \( \tilde{S} = \langle \tilde{P}, \tilde{f}, \tilde{g} \rangle \times \langle \tilde{t} \rangle \).
3. \( Z(S) = \langle b_0 \rangle \).
4. \( Z(\tilde{S}) = \langle \tilde{b}_0, \tilde{b}_1, \tilde{t} \rangle \).

**Proof.** By the choice of \( g, g^t \in A_1 \) and \( g \) interchanges \( P = \langle A_1, b_1 \rangle \) and \( \langle A_1, f \rangle \). Hence (1) follows. By Lemma (4E)(2), \( \tilde{t} \in Z(\tilde{S}) \). Since \( \langle P, f, g \rangle \cap R = \langle P, f \rangle, \ t \in \langle P, f, g \rangle \). Thus (2) holds. Now \( Z(S) \leq C_8(t) = R \), so \( Z(S) \leq Z(R) = \langle b_0, \tilde{t} \rangle \). Since \( \tilde{t} = b_0 \tilde{t} \) by Lemma (4E), (3) follows. By (2), \( Z(\tilde{S}) = Z(\tilde{P}, \tilde{f}, \tilde{g}) \times \langle \tilde{t} \rangle \). Since \([b_0 \tilde{f}, \tilde{A}_1] \neq 1 \) and \( \langle A_1, b_0f \rangle | A_1 \subseteq Z(\langle \tilde{P}, \tilde{f}, \tilde{g} \rangle | A_1) \), we have that \( C_{(\tilde{P}, \tilde{f}, \tilde{g})(\tilde{A}_1)} = \tilde{A}_1 \). Hence \( Z(\tilde{P}, \tilde{f}, \tilde{g}) = C_{\tilde{A}_1}(\tilde{b}_0, \tilde{f}, \tilde{g}) = \langle \tilde{b}_1 \rangle \) by Lemmas (4I) and (4J). Thus \( Z(\tilde{S}) = \langle \tilde{b}_1, \tilde{t} \rangle \). This proves (4).

**Lemma (4L).** \( S \in Syl_2(G) \).

**Proof.** Assume that \( S \in Syl_2(G) \). Then \( \langle P, f, g \rangle \) contains an extremal conjugate \( u \) of \( t \) in \( S \) by Lemma (1E), since \( t \in G' \). Since \( t^g \cap \langle P, f \rangle = \varnothing \) by Lemma (4F), \( u = g \) or \( b_sfg \mod A_1 \), and we may assume that \( u = g \mod A_1 \). Then \( C_{\tilde{A}_1}(u) = \langle \tilde{a}_1, \tilde{b}_3, \tilde{a}_2, \tilde{b}_1 \rangle \) by Lemma (4J) and \( C_{(P, f, g)/(A_1)}(u) = \langle A_1, g, b_0fg \rangle | A_1 \), so \( |C_{(P, f, g)}(u)| \leq 2^e \) and \( |C_S(u)| \leq 2^e \). However, \( |C_S| = |R| = 2^e \). This is a contradiction. Therefore, \( S \notin Syl_2(G) \).

Now let \( T \in Syl_2(N(S)) \).
Lemma (4M). The following conditions hold.

1. $|T: S| = 2$.
2. $t^r = \langle b_0, b_1 \rangle t$.
3. $T \in \text{Syl}_2(G)$.

Proof. By Lemma (4L), $S < T$ and so $t^r = |T: C_T(t)| = |T: R| \geq 4$. On the other hand, $t^r \leq Z_3(S) = \langle b_0, b_1, t \rangle$ by Lemma (4K), so $t^r \leq \langle b_0, b_1 \rangle t$ since $t^a \cap L = \emptyset$. Hence (1) and (2) follow.

Now $Z(T) = \langle b_0 \rangle$ since $Z(T) \leq C_T(t) \leq S$ and $Z(S) = \langle b_0 \rangle$. Hence $Z_3(T) \leq N_T(B_1) = S$ by Lemma (4G)(2), and so $Z_3(T) \leq Z_3(S) = \langle b_0, b_1, t \rangle$. Now (2) shows that $\langle b_0, b_1 \rangle \not< T$, so $\langle b_0, b_1 \rangle \leq Z_3(T)$. It also follows from (2) and Lemma (4E)(2) that $t^h = b_1t$ or $b_0b_1t$ for $h \in T - S$. This implies that $t \not\in Z_3(T)$. Therefore, $Z_3(T) = \langle b_0, b_1 \rangle$.

Let $X = Z_3(T)$. Then $X \leq N_T(B_1) = S$, and $[X, S] \leq \langle b_0, b_1 \rangle$. Hence $[\bar{X}, \bar{S}] \leq \langle \bar{b}_1 \rangle = Z(\bar{T})$. Now $\langle \bar{b}_1, \bar{t} \rangle = Z(\bar{S}) \not< \bar{T}$, so $\langle \bar{b}_1, \bar{t} \rangle \leq \bar{X}$. In particular, $\bar{t} \in \bar{X}$ and so, if $\bar{Y} = \bar{X} \cap \langle \bar{P}, \bar{f}, \bar{g} \rangle$, then $\bar{X} = \bar{Y} \langle \bar{t} \rangle$ by Lemma (4K)(2). We have that

$$[\bar{Y}, \langle \bar{P}, \bar{f}, \bar{g} \rangle] \leq \langle \bar{b}_1 \rangle = \bar{A}_1.$$ 

Hence $\bar{Y} \leq \bar{Z}(\langle \bar{P}, \bar{f}, \bar{g} \rangle) \mod \bar{A}_1 \leq \langle \bar{A}_1, \bar{b}_3, \bar{f} \rangle$ by Lemma (4K)(1). From Lemma (4I)(3), we get that $[\bar{b}_3, \bar{f}, \bar{a}_2] = \bar{a}_1 \bar{b}_3, \bar{b}_2 \bar{f} \in \langle \bar{b}_1 \rangle$. Hence, $\bar{Y} \leq \bar{A}_1$ and using Lemmas (4I), (4J), we get that $\bar{Y} \leq \langle \bar{a}_1 \bar{b}_3, \bar{b}_1 \rangle$. Therefore, $\langle \bar{b}_1, \bar{t} \rangle \leq \bar{X} \leq \langle \bar{a}_1 \bar{b}_3, \bar{b}_1, \bar{t} \rangle$. That is, $\langle b_0, b_1, t \rangle \not< Z_3(S) \leq \langle a_1, b_2, b_0, b_1, t \rangle$. Hence $Z_3(Z_3(T)) = \langle b_0, b_1, t \rangle$.

Now let $U \in \text{Syl}_2(N(T))$. Then $t^U \leq \langle b_0, b_1, t \rangle$ by the above, and so $t^U = \langle b_0, b_1 \rangle t$. This shows that $|U: R| = 4$. Hence $U = T$ and $T \in \text{Syl}_2(G)$. The proof is complete.

Lemma (4N). $t \in G'$.

Proof. Let $h \in T - S$. Then $R \cap R^h \vartriangleleft T$ as $h^2 \in S \leq N(R)$ by Lemma (4M). Since $R = C_T(t)$ and $t^h \in \langle b_0 \rangle b_1 t$ by Lemmas (4E) and (4M),

$$R \cap R^h = C_R(t^h) = C_R(b_1) = \langle a_1, b_0, b_1, b_2, b_3, f, t \rangle.$$ 

Now $t \sim b_0 t \sim b_1 t$ by Lemma (4M), and since every involution of $L$ is conjugate in $L$ to $b_0$ or $b_1$, it follows that $t \sim x t$ for all $x \in I(L)$. Since $P^t = \langle A_1, f \rangle$ and $t^a = b_0 t$, we also have that $b_0 t \sim (f b_0) b_1 t = f t$. Hence $t \sim f t$. Also, $t^a \cap \langle a_1, b_0, b_1, b_2, b_3, f \rangle = \emptyset$ by Lemma (4F)(4). Therefore, we conclude that the subgroup generated by the products of two elements of $t^a \cap \langle a_1, b_0, b_1, b_2, b_3, f \rangle$ is trivial. This shows that $\langle a_1, b_0, b_1, b_2, b_3, f \rangle \vartriangleleft T$. Hence $\langle P, f \rangle \cap \langle P, f^h \rangle = \langle a_1, b_0, b_1, b_2, b_3, f \rangle$. Thus $N = \langle P, f \rangle \langle P, f^h \rangle$ is a normal...
Let $u$ be an extremal conjugate of $t$ in $T$. Assume that $u \in S$. Notice that $\langle b_0, t \rangle \triangleleft S$ and $S/\langle b_0, t \rangle \cong \langle P, f, g \rangle/\langle b_0 \rangle$ by Lemma (4K). Hence if $u \in R$, then $u \equiv g$ or $b_3 f g \mod B_1$, and so $|C_{S/B_1}(u)| = 4$ and $|C_{B_1/\langle b_0, t \rangle}(u)| = 8$ by Lemma (4J). Since $|C_T(u)| = 2^n$ by assumption, we get that $C_{\langle b_0, t \rangle}(u) = \langle b_0, t \rangle$. But then $u \in C_T(t) = R$, a contradiction. Hence $u \in R$ and so $u \in \langle P, f \rangle t \subseteq N t$ by Lemma (4F)(4).

Assume that $u \in S$. Then we may choose $h = u$. Now $B_1^h$ is an $E_{2n}$-subgroup of $S$, and $B_1^h \neq B_1$ since $S \in \text{Syl}_{2}(N(B_1))$ by Lemma (4G). Also, $t \in Z(S) \leq B_1^h$ by Lemma (4K)(4). Therefore, $B_1^h = \tilde{X} \langle \tilde{t} \rangle$ for some $E_{2n}$-subgroup $\tilde{X}$ of $\langle \tilde{P}, \tilde{f}, \tilde{g} \rangle$ different from $\tilde{A}_1$, by Lemma (4K)(2). Thus $\tilde{X} \tilde{A}_1/\tilde{A}_1$ is a nonidentity elementary abelian subgroup of $\langle \tilde{P}, \tilde{f}, \tilde{g} \rangle/\tilde{A}_1$, which centralizes the subgroup $\tilde{X} \cap \tilde{A}_1$ of $\tilde{A}_1$. We argue that $\tilde{X} \tilde{A}_1 = \langle \tilde{A}_1, \tilde{b}_3 \tilde{f}, \tilde{g} \rangle$. If not, then using Lemma (4I)(4), (5), (6), and Lemma (4J), we get that $\tilde{X} \tilde{A}_1 = \langle \tilde{A}_1, \tilde{g} \rangle$ or $\langle \tilde{A}_1, \tilde{b}_3 \tilde{f}, \tilde{g} \rangle$. Conjugating, we may assume the former. Then $\tilde{X} \cap \tilde{A}_1 = Z(\langle \tilde{A}_1, \tilde{g} \rangle) = \langle \tilde{a}, \tilde{b}_3 \tilde{f}, \tilde{a}_2, \tilde{b}_1 \rangle$ by Lemma (4J). But then $a_2 \in B_1^h \leq R^h$, so $a_2 \in R \cap R^h = \langle a_2, b_0, b_1, b_2, b_3, f, t \rangle$, which is a contradiction. Therefore, $\tilde{X} \tilde{A}_1 = \langle \tilde{A}_1, \tilde{b}_3 \tilde{f}, \tilde{g} \rangle$ and so $B_1^h = B_1^h \langle \tilde{B}_1, \tilde{b}_3 \tilde{f}, \tilde{g} \rangle$. This implies that $B_1 \cap B_1^h$ has index 4 in $B_1$, so that $|B_1 \cap B_1^h| = 2^4$. We also have that $B_1 \cap R^h = B_1 \cap (R \cap R^h) = \langle a_2, b_0, b_1, b_2, t \rangle$. Hence $|B_1 \cap R^h| = 2^4$. Now consider the following normal series of $T$.

$$B_1 \cap B_1^h \leq (B_1 \cap R^h)(B_1^h \cap R) \leq R \cap R^h \leq RR^h = S \leq T.$$  

The factors of this series have order 2 except for $(B_1 \cap R^h)(B_1^h \cap R)/B_1 \cap B_1^h$ and $RR^h/R \cap R^h$, which are fours groups. Therefore, the centralizer of $h$ in each factor has order 2. There are 4 factors and $|C_T(h)| = 2^n$ by the choice of $h$. Hence $h$ must centralize $B_1 \cap B_1^h$. But then, as $t \in Z_2(S) \leq B_1 \cap B_1^h$, $h \in C_T(t) \leq S$, which is a contradiction. Therefore, $u \in S$ and so $u \in N t$ as shown before.

We have shown that each extremal conjugate of $t$ in $T$ is contained in $N t$. Thus Lemma (1E) shows that $t \in G'$.

Lemma (4N) conflicts with our assumption. Therefore, we have proved Theorem (4A).

5. In this section, we shall make the following hypothesis.

**Hypothesis (5.1).** $t^{\langle L^g \rangle} = \{t, c_1 t, c_2 t, c_3 t, c_4 t, c_5 t\}$.

The purpose of this section is to prove the following.

**Theorem (5A).** Under Hypothesis (5.1), $r(\langle L^g \rangle) = 4$. 

The proof of this theorem is similar to that of Theorem (4A), although the arguments involved in this section are much more complicated than in § 4. We begin the proof by studying the permutation representation of \( N(B_2) \) on \( \Omega = t^{N(B_2)} \). Let

\[ n_i = t \text{ and } n_i = c_{i-1}t \]

for \( i \in \{2, 3, 4, 5, 6\} \), so that

\[ \Omega = \{ n_1, n_2, n_3, n_4, n_5, n_6 \} \]

**Lemma (5B).** \( N(B_2) N \equiv N(B_2)/C(B_2) \equiv \Sigma_6 \) or \( A_6 \).

**Proof.** First, observe that \( \langle \Omega \rangle = B_2 \). Hence \( C(\Omega) = C(B_2) \) and \( N(B_2)/C(B_2) \equiv \Sigma_6 \) or \( A_6 \) by Lemmas (2D) and (2H). By Hypothesis (5.1), \( |N(B_2) : N_c(B_2)| = 6 \). Since \( N(B_2)/C(B_2) \equiv \Sigma_6 \) or \( A_6 \) by Lemmas (2D) and (2H), it follows that \( |N(B_2)/C(B_2)| = 720 \) or 360. Thus \( N(B_2)/C(B_2) \) is a subgroup of the symmetric group on \( \Omega \) of index 1 or 2. Hence \( N(B_2)/C(B_2) \equiv \Sigma_6 \) or \( A_6 \).

Notice that Hypothesis (5.1) implies Hypothesis (3.1). Therefore, \( \langle t \rangle \in \text{Syl}_2(C(\Omega)) \) by Lemma (3B).

**Lemma (5C).** The following conditions hold.

1. \( N(A_2)/C(A_2) \equiv N(B_2)/C(B_2) \).
2. \( N(B_2) \cap C(A_2) = C(B_2) = B_2O(C) \).
3. \( B_2 \in \text{Syl}_2(C(A_2)) \).

**Proof.** Since \( \langle t \rangle \in \text{Syl}_2(C(\Omega)) \), Lemma (2H) shows that \( C(B_2) = B_2O(C) \). By Lemma (5B), \( N(B_2)/C(B_2) \) has no nonidentity normal 2-subgroups. Since \( N(B_2) \cap C(A_2)/C(B_2) \) is a normal 2-subgroup of \( N(B_2)/C(B_2) \) by Lemmas (3E) and (3F), it follows that \( N(B_2) \cap C(A_2) = C(B_2) \). This proves (3), since \( B_2 \in \text{Syl}_2(C(B_2)) \). Finally, (1) holds by a Frattini argument.

Now \( O(C(B_2))/O(C) \) by Lemma (5C)(2), so let bars denote images in \( N(B_2)/O(C) \). Then since \( C(B_2) = B_2O(C) \), \( N(B_2)/\bar{B}_2 \equiv \Sigma_6 \) or \( A_6 \) by Lemma (5B). Choose the subgroup \( \bar{M} \) of \( \bar{N}(B_2) \) such that \( \bar{B}_2 \leq \bar{M} \) and \( \bar{M}/\bar{B}_2 \equiv \bar{A}_6 \). Then since \( \bar{K}_2\bar{B}_2/\bar{B}_2 \equiv \bar{A}_6 \), \( \bar{K}_2\bar{B}_2 \leq \bar{M} \) and in particular, \( \bar{Q} \leq \bar{M} \). Now \( \bar{A}_6 \triangleleft N(\bar{B}_2) \) by Lemma (3E). Hence \( \bar{M}/\bar{A}_6 \) is an extension of \( Z_2 \) by \( \bar{A}_6 \), and it contains \( \bar{Q}/\bar{A}_6 \equiv E_8 \). Therefore, the extension splits, and there is a subgroup \( \bar{N} \) of \( \bar{M} \) such that \( \bar{A}_6 \leq \bar{N} \) and \( \bar{M}/\bar{A}_6 = \bar{N}/\bar{A}_6 \times \bar{B}_2/\bar{A}_6 \). As before, \( \bar{K}_2\bar{A}_6 \leq \bar{N} \), and so \( \bar{P} \leq \bar{N} \).

**Definition (5.1).** Let \( \bar{M} \) and \( \bar{N} \) be the preimages of \( \bar{M} \) and \( \bar{N} \),
respectively. Furthermore, let $Q \leq R \in \text{Syl}_2(C)$, $R \leq T \in \text{Syl}_2(N(B_2))$, $S = T \cap M$, and $U = S \cap N$.

Thus $U \lhd T$, $T = RU$, $R \cap U = P$, and $R \cap S = Q$ by the above remark. In particular, $T/U \cong R/P$. Notice also that $N(B_2)/C(B_2) \cong \Sigma_6$ if and only if $Q < R$, as $R \in \text{Syl}_2(N_c(B_2))$.

**Lemma (5D).** If $T/U$ is cyclic, then Theorem (5A) holds.

**Proof.** Suppose that $T/U$ is cyclic. Then $t^g \cap T \leq S$. Hence $t^g \cap R \leq S \cap R = Q$, so $B_2 = \langle t^g \cap B_2 \rangle$ is weakly closed in $R$ with respect to $G$ by Lemma (2A). Let $t^g \in B_2$. Then $B_2^{t^{-1}} \leq C$, so there is an element $c \in C$ such that $B_2^{c^{-1}} \leq R^c$. By the weak closure of $B_2$, $B_2^{c^{-1}} = B_2^c$ and $t^g = t^{gc} \in t^{N(B_2)}$. Therefore, $t^g \cap B_2 = t^{N(B_2)} = \Omega$.

Let $x \in t^g \cap (Q - B_2)$. Then $x \in B_2$ by Lemma (2A) and $x$ is conjugate to an element of $B_1 \cap B_2$ in $N_c(B_2)$ by Lemma (2E). Since $t^g \cap B_2 \cap B_2 = \Omega \cap B_2 = \{t, c_1 t\}$ and since $t$ and $c_1 t \in Z(N_c(B_2))$, $x = t$ or $c_1 t$ and so $x \in B_2$, which is a contradiction. Therefore, $t^g \cap Q = t^g \cap B_2$. This in turn implies that $t^g \cap S = t^g \cap B_2$, as $M/B_2$ has one conjugacy class of involutions by the definition of $M$. Thus $t^g \cap T = t^g \cap B_2 = \Omega$. Hence $N(T) \leq N(B_2)$ and so $T \in \text{Syl}_2(G)$. Also, $t^g \cap T \leq U$. Therefore, $t \in G'$ by Lemma (1E). Since $U \leq N' \leq G'$, we conclude that $U \in \text{Syl}_2(G')$.

Now $N(A_2)/C(A_2) \cong \Sigma_6$ or $A_6$ by Lemmas (5C) and (5B). As $N_{G'}(A_2)/C_{G'}(A_2) \cong A_6$ and $U \in \text{Syl}_2(N_{G'}(A_2))$, it follows that $N_{G'}(A_2)/C_{G'}(A_2) \cong A_6$. Also, since $B_2 \in \text{Syl}_2(C(A_2))$ and since $t \in G'$, $A_2 \in \text{Syl}_2(C(G'))$. Thus by [17, Theorem 3], $r(G') = 4$ and hence $r(\langle L^o \rangle) = 4$. The proof is complete.

In view of Lemma (5D), we shall assume from now on that $T/U$ is not cyclic. This implies that $T/U \cong E_4$. Let bars denote images in $N(B_2)/O(C)$. Then since $\bar{N}(B_2)/\bar{N} \cong \bar{T}/\bar{U}$, there is a subgroup $\bar{K}$ of $\bar{N}(B_2)$ such that $\bar{N} < \bar{K}$ and $\bar{N}(B_2)/\bar{A_2} = \bar{K}/\bar{A_2} \times \bar{B_2}/\bar{A_2}$.

**Definition (5.2).** Let $K$ be the preimage of $\bar{K}$ in $N(B_2)$ and set $V = T \cap K$.

Since $R/P \cong E_4$, we may assume without loss of generality that there is an involution $f \in R - Q$ whose action on $L$ is induced by the automorphism of $F_4'$ of order 2.

Now $A_2 \nmid R$, so $R$ acts on $A_2$ by conjugation. In the following lemma, we collect information on this action. For the proof, see Lemmas (2A) and (2F).
**Lemma (5E).** The following conditions hold.

1. \( b_0^{a_i} = b_0, b_0^{a_1} = b_1, b_0^{a_2} = b_0b_2, b_0^{a_3} = b_0b_3. \)
2. \( b_0^{a_2} = b_0, b_0^{a_3} = b_0b_1, b_0^{a_2} = b_2, b_0^{a_3} = b_0b_3. \)
3. \( b_0^f = b_0, b_1^f = b_1, b_2^f = b_1b_2, b_3^f = b_0. \)
4. \( C_{B_2}(a_i) = \langle b_0, b_1 \rangle. \)
5. \( C_{B_2}(a_2) = \langle b_0, b_2 \rangle. \)
6. \( C_{B_2}(f) = \langle b_0, b_1, b_3 \rangle. \)

Permutation representations of \( a_1, a_2, \) and \( f \) on \( \Omega \) can be computed by using Lemma (5E) and the expressions of \( c_i \)'s in terms of \( b_i \)'s given in § 2. We have that

\[
\begin{align*}
a_1^f &= (n_3, n_4)(n_5, n_6), & a_2^f &= (n_3, n_5)(n_4, n_6), & f^0 &= (n_3, n_6). \\
\end{align*}
\]

Therefore, we may assume without loss of generality that

\[
T^\alpha = \langle a^\alpha, f^\alpha, a_1^\alpha, a_2^\alpha \rangle,
\]

where

\[
a^\alpha = (n_1, n_2).
\]

That is, \( t^a = c_it, (c_it)^a = t, \) and \( (c_it)^a = c_it \) for \( i \in \{2, 3, 4, 5\}. \) Noticing that \( c_i = (c_it)_i, \) we get that \( c_i^a = c_i \) and \( c_i^a = c_i^a, \) for \( i \in \{2, 3, 4, 5\}. \)

Thus we can determine the action of \( a \) on \( B_2, \) using the relations \( b_0 = c_i, b_1 = c_0c_5, b_2 = c_0c_4, \) and \( b_3 = c_2. \) Furthermore, we can compute \([B_2, a] \) and \( C_{B_2}(a). \) Also, \( C_T(\Omega) = B_2 \) and \( a^\alpha \) is an involution which centralizes \( a_1^\alpha, a_2^\alpha, \) and \( f^\alpha. \)

Thus we have the following result.

**Lemma (5F).** There is an element \( a \in T - R \) which satisfies the following conditions.

1. \( a^2, [a_1, a], [a_2, a], \) and \( [f, a] \in B_2. \)
2. \( b_0^a = b_0, b_1^a = b_1, b_2^a = b_2, b_3^a = b_0b_5, t^a = b_0t. \)
3. \( [B_2, a] = \langle b_0 \rangle. \)
4. \( C_{B_2}(a) = \langle b_0, b_1, b_2, b_3t \rangle. \)

Our next result shows that \( T \) has the unique structure.

**Lemma (5G).**

1. We may choose \( a \) in Lemma (5F) and \( f \) so that \( a^2 = [a_1, a] = [a_2, a] = [f, a] = 1. \)

2. If \( P^*/A_2 \) is an \( E^- \)-subgroup of \( U/A_2 \) different from \( P/A_2, \) then \( \mathcal{E}^* (P^*) \) consists of two \( E_{16} \)-subgroups.

**Proof.** Observe first that \( V \cap R = \langle P, f \rangle \) or \( \langle P, ft \rangle. \) Replacing \( f \) by \( ft \) in the latter case, we may assume that \( f \in V. \)
Choose an element \( a \in T - R \) as in Lemma (5F), and let bars denote images in \( N(B_2)/C(B_2) \). Then \( \bar{T} = \langle \bar{a} \rangle \times \langle \bar{a}_1, \bar{a}_2, \bar{f} \rangle \equiv \mathbb{Z}_2 \times D_8 \) and \( Z(\bar{T}) = \langle \bar{a}, \bar{a}_1 \rangle \).

Now \( \bar{a}_1 \in Z(\bar{T}) \), so \( \langle a_1 \rangle A_2 \triangleleft V \). Also, \( C_{A_2}(a_1) = \langle b_0, b_1 \rangle \) and so \( I(a_1, A_2) = a_1^{t_2} \) by Lemma (1C). Thus \( V = C_\gamma(a_1)A_2 \), and consequently \( |C_\gamma(a_1)| = 64 \).

Now \( \langle a_1, a_2, f, b_0 \rangle \leq N(\langle a_1, a_2 \rangle) \cap C_\gamma(a_1) \). Suppose that equality holds here. Then \( C_\gamma(a_1) \cap C_\gamma(a_2) = C(a_2) \cap \langle a_1, a_2, f, b_0 \rangle = \langle a_1, a_2, b_0 \rangle \) and so \( |C_\gamma(a_1): C_\gamma(a_1) \cap C_\gamma(a_2)| = 8 \). This shows that \( |a_1^{C_\gamma(a_1)}| = 8 \).

Now \( \langle a_1, a_2, f, b_0 \rangle \leq N(\langle a_1, a_2 \rangle) \cap C_\gamma(a_1) \). Similarly, \( \langle a_1, a_2, f, b_0 \rangle \leq N(\langle a_1, a_2 \rangle) \cap C_\gamma(a_1) \). Hence \( a_1^{C_\gamma(a_1)} \leq a_2 \langle a_1, C_{A_2}(a_1) \rangle \), whereas \( I(a_2 \langle a_1, C_{A_2}(a_1) \rangle) = 4 \) as \( C(a_2) \cap \langle a_1, C_{A_2}(a_1) \rangle = \langle a_1, b_0 \rangle \) has order 4. This contradiction shows that \( \langle a_1, a_2, f, b_0 \rangle \neq N(\langle a_1, a_2 \rangle) \cap C_\gamma(a_1) \), so \( N(\langle a_1, a_2 \rangle) \cap C_\gamma(a_1) \) has index 2 in \( C_\gamma(a_1) \).

Now \( C_{A_2}(a_1) \leq N(\langle a_1, a_2 \rangle) \), so that by the above paragraph,

\[
C_\gamma(a_1) = (N(\langle a_1, a_2 \rangle) \cap C_\gamma(a_1))C_{A_2}(a_1).
\]

Thus \( V = N_\gamma(\langle a_1, a_2 \rangle)A_2 \) and so we may assume \( a \in N_\gamma(\langle a_1, a_2 \rangle) \).

Now \( \langle a_1, a_2, f, b_0 \rangle \leq N(\langle a_1, a_2 \rangle) \cap C_\gamma(a_1) \). Hence \( a_1^{C_\gamma(a_1)} \leq a_2 \langle a_1, C_{A_2}(a_1) \rangle \), whereas \( I(a_2 \langle a_1, C_{A_2}(a_1) \rangle) = 4 \) as \( C(a_2) \cap \langle a_1, C_{A_2}(a_1) \rangle = \langle a_1, b_0 \rangle \) has order 4. This contradiction shows that \( \langle a_1, a_2, f, b_0 \rangle \neq N(\langle a_1, a_2 \rangle) \cap C_\gamma(a_1) \), so \( N(\langle a_1, a_2 \rangle) \cap C_\gamma(a_1) \) has index 2 in \( C_\gamma(a_1) \).

Now \( C_{A_2}(a_1) \leq N(\langle a_1, a_2 \rangle) \), so that by the above paragraph,

\[
C_\gamma(a_1) = (N(\langle a_1, a_2 \rangle) \cap C_\gamma(a_1))C_{A_2}(a_1).
\]

Thus \( V = N_\gamma(\langle a_1, a_2 \rangle)A_2 \) and so we may assume \( a \in N_\gamma(\langle a_1, a_2 \rangle) \).

Then, since \( \langle \bar{a}, \langle \bar{a}_1, \bar{a}_2 \rangle \rangle = 1, \langle a, \langle a_1, a_2 \rangle \rangle = 1 \). Also, since \( \bar{a}_2 = (af)^2 = 1, a^2 \) and \( (af)^2 \in N_{A_2}(\langle a_1, a_2 \rangle) = \langle b_0 \rangle \). Using the relation \( t^a = b_0 t, \) we may deduce as follows:

\[
(af)^2 = (af)t^a = (af)^2((af)^{-1}t)(af)t
= (af)^2t^a = (af)^2tb_0
= (af)^2b_0.
\]

Also,

\[
(at)^2 = a^2t^at = a^2(b_0t)t = a^2b_0.
\]

If \( a^2 = b_0 \), let \( a_0 = at \). Then \( a_0 = 1 \) and \( a_0f)^2 = (af)^2b_0 \in \langle b_0 \rangle \) by the above. If \( (a_0f)^2 = b_0 \), let \( f_0 = ft \). Then \( (a_0f)^2 = (af)^2 = (a_0f)^2b_0 = 1 \). If \( a^2 = b_0 \), then \( (af)^2 = (af)f_0^2 = 1 \). Therefore, replacing \( a \) and \( f \) by \( at \) and \( ft \), if necessary, we may assume that \( a^2 = (af)^2 = 1 \). This proves (1).

Now \( (af)^2 = (n, n_2)(n_1, n_0) \) by definition, so \( af \in S \) and \( S = \langle a_1, a_2, af \rangle B_2 \). Since \( P^*B_2/B_2 \) is an \( E_1 \)-subgroup of \( S/B_2 \), different from \( P^*B_2/B_2 \) and since \( P^*B_2 = \langle a_1, a_2 \rangle B_2 \), it follows that \( P^*B_2 = \langle a_1, af \rangle B_2 \).

Hence if \( x \in P^* - A_2 \), then \( C_{A_2}(x) = C_{A_2}(a_1), C_{A_2}(af) \) or \( C_{A_2}(a_1, af) \), and so using Lemmas (5E) and (5F), we have that \( C_{A_2}(x) = \langle b_0, b_1 \rangle \). Now (1) shows that \( \langle a, a_1, a_2, f \rangle \) is a complement for \( B_2 \) in \( T \), so that \( B_2 \) has a complement \( Y \) in \( N(B_2) \) by Gaschütz’s theorem [19, Haupt- satz 17.4]. Then \( Y' \) is a complement for \( A_2 \) in \( N' \), and so there is a fours group \( X \) such that \( XA_2 = P^* \) and \( X \cap A_2 = 1 \). Since \( C_{A_2}(x) = \langle b_0, b_1 \rangle \)
\[ \langle b_0, b_i \rangle \text{ for } x \in X^\ast \], \text{[11, (1C)] shows that } \mathcal{G}^\circ(P^\circ) = \{A_2, X_0 \langle b_0, b_i \rangle \}. \text{ This proves (2).} \\

Now choose an element \( a \in T - R \) as in Lemma (5G). As remarke\(d in the proof of Lemma (5G)(2), T = \langle a, a_1, a_2, f \rangle B_2 \text{ and } \langle a, a_2, f \rangle \cap B_2 = 1. \]

**Lemma (5H).** The following conditions hold. 

(1) \( Z(T) = \langle b_0 \rangle. \)

(2) \( Z_2(T) = \langle a, b_0, b_1, t \rangle. \)

**Proof.** As \( Z(T) \subseteq C_T(t) = R, Z(T) \subseteq Z(R) = \langle b_0, t \rangle. \text{ As } t^a = b_0 t \text{ by Lemma (5F)(2), } Z(T) = \langle b_0 \rangle. \)

Now \( Z_2(T) \subseteq C_T(b_2) \langle b_0 \rangle \subseteq Z(T \mod B_2) = \langle a, a_1 \rangle B_2. \text{ Since } [a, B_2] = \langle b_0 \rangle \text{ by Lemma (5F)(3) and since } [a_1, B_2] = \langle b_0, b_1 \rangle \text{ by Lemma (5E)(1), we have that } \langle a \rangle \leq Z_2(T) \leq \langle a \rangle B_2. \text{ Hence if } X = B_2 \cap Z_2(T), \text{ then } Z_2(T) = \langle a \rangle X. \)

By definition \( X \leq Z_4(Q) = \langle b_0, b_1, b_2, t \rangle. \text{ Clearly, } b_0 \in X. \text{ We have that } [\langle a, a_1, a_2, f \rangle, b_1] = \langle b_0 \rangle \text{ by Lemmas (5E) and (5F). Also, } [\langle a, a_1, a_2, f \rangle, t] = \langle b_0 \rangle. \text{ Hence } b_1 \text{ and } t \in X. \text{ However, } b_2 \in X \text{ since } [f, b_2] = b_1 \text{ by Lemma (5E)(3). Therefore, } X = \langle b_0, b_1, t \rangle \text{ and so } Z_2(T) = \langle a, b_0, b_1, t \rangle. \]

**Lemma (5I).** The following conditions hold. 

(1) \( C_T(b_2) = \langle a a_2, a_1, f, B_2 \rangle. \)

(2) \( B_2 \text{ and } D = \langle a_1, f, b_0, b_1, t \rangle \text{ are } E_{32} \text{-subgroups of } C_T(b_2) \text{ and both are normal in } T. \)

(3) \( C_T(a) = \langle a, a_1, a_2, f, b_0, b_1, b_2, b_3, t \rangle. \)

(4) \( C_T(ab) = \langle a, a_1, f, b_0, b_1, b_2, b_3, t, a_2 t \rangle. \)

(5) \( E = \langle a, b_0, b_1, b_2, b_3 t \rangle \text{ and } F = \langle a, a_1, f, b_0, b_1 \rangle \text{ are } E_{32} \text{-subgroups of } C_T(a) \text{ and } C_T(ab), \text{ and both } E \text{ and } F \text{ are normal in } T. \)

**Proof.** Since \( B_2 \text{ is abelian, } C_T(b_2) = C_{(a_1, a_2, f)}(b_2)B_2. \text{ By Lemma (5E), } a_1 \text{ and } f \text{ centralize } b_2. \text{ Also, } (b_2 t)^{a_2} = (b_2 b_0 t)^{a_2} = b_2 b_0 b_2 t = b_2 \text{ by Lemmas (5E) and (5F). However, } a \notin C(b_2) \text{ by Lemma (5F)(2). Thus } C_{(a_1, a_2, f)}(b_2) = \langle a a_2, a_1, f \rangle \text{ and hence (1) follows.} \)

To prove (2), it is enough to show that \( a \in N(D) \text{ as } D = \langle C_{a_1}(f), f, t \rangle \setminus R \text{ by Lemma (2F). By Lemmas (5F) and (5G), } a \text{ centralizes } a_1, f, b_0, b_1. \text{ Also, } t^a = b_0 t. \text{ Thus } a \in N(D). \text{ (3) is a direct consequence of Lemmas (5G)(1) and (5F)(4).} \)

As a consequence of (3), we have that \( E \text{ is elementary of order } 32. \text{ Also, } F \text{ is elementary of order } 32 \text{ as } \langle a, a_1, f \rangle \text{ centralizes } \langle b_0, b_1 \rangle \text{ by Lemmas (5E) and (5F). Thus } E \text{ and } F \subseteq C_T(ab). \text{ Now } (ab)^{a^2} = (ab_0 b_1)^t = (ab_0)b_0 b_1 = ab_1 \text{ by Lemmas (5E) and (5F)(2). Hence} \)
\[ \langle E, F, a_x \rangle \leq C_T(ab) \] and as \( \langle E, F, a_x \rangle \) is maximal in \( T \) and \( ab \in Z(T) \) by Lemma (5H), we conclude that \( C_T(ab) = \langle a, a_x, f, b, b, b, t, a_x \rangle \).

Now \( \langle a, a_x, f \rangle \) centralizes \( a \) and normalizes \( \langle b, a, b, b, b, t \rangle \) by Lemmas (5E) and (5F). Also, \( [b, a] = \langle b, b \rangle \) and \( B \) centralizes \( \langle b, a, b, b, b, t \rangle \). Thus \( T = \langle a, a_x, f, E, B \rangle \) normalizes \( E \).

Similarly, we see that \( a_x \) normalizes \( \langle a, a_x, f \rangle \) and \( \langle b, b \rangle \). Furthermore, \( [\langle a, a_x, f \rangle, B] \leq \langle b, b \rangle \) and \( B \) centralizes \( \langle b, b \rangle \). Hence \( T = \langle a_x, F, B, B \rangle \) normalizes \( F \).

**Lemma (5J).** \( t^G \cap \langle A, t \rangle = t^F = \{t, b, t\} \) and \( t^G \cap B = t^{N(B)} \).

**Proof.** Suppose that \( t \sim b_t \). Since \( R \in Syl_2(C(t)) \), \( t \) is extremal in an \( S_2 \)-subgroup of \( G \) containing \( T \). Therefore, there is an element \( g \in G \) such that \( (b, t)^g = t \) and \( C_T(b, t)^g = R \). By Lemma (2F), \( B_2 \) and \( D \) are the only normal \( E_2 \)-subgroup of \( R \), so Lemma (5I) shows that \( [B_2, D]^g = [B_2, D] \). Since \( b_t \in t^{N(B)} \) by Hypothesis (5I), \( g \in N(B) \), therefore, \( D^g = B \).

Now \( T \leq N(C_T(b, t)) \cap N(D) \) by Lemma (5I), so \( T^g \leq N(B) \cap N(R) \). Also, \( T \leq N(B) \cap N(R) \). Hence there is an element \( h \in g(N(B) \cap N(R)) \) such that \( T^h = T \). Thus \( b_t \in t^H \) by Lemma (3J). Let \( A^h \leq T \cap O^*(N(B)) = U \) as \( O^*(N(B)) = N \). Suppose that \( A^h = A \). Then \( B^h = \langle A, b_t \rangle = \langle A, t \rangle \) or \( \langle A, b_t \rangle \), so \( h \in N(B) \leq N(Z(B)) \). However, \( Z(B) = \langle b_t \rangle \) and \( t^{-1} = b_t \in Z(B) \). This is a contradiction. Therefore, \( A^h \neq A \) and so \( A^h \leq P \) since \( A^h \leq P \) is the unique \( E_{10} \)-subgroup of \( P \). Hence \( A^h A_i \leq A \) is contained in the \( E_{10} \)-subgroup \( P^* / A \) of \( U / A \), different from \( P / A \), and so \( A_i \leq P^* \). However, \( |E^*(P^*)| = 2 \) by Lemma (5G), whereas \( |E^*(A)| > 2 \). This is a contradiction. Therefore, \( t \sim b_t \) and then \( t^G \cap B = t^{N(B)} \) by Lemma (2D). Now \( t^G \cap A = \emptyset \) by Lemma (3C). Also, (2E) shows that involutions in \( A_t - \{t, b_t\} \) are conjugate to \( b_t \). Thus \( t^G \cap A = \{t, b_t\} \). Since \( b_t = t^a \) and \( R = C_T(t) \) is unique \( t \) in \( T \), we conclude that \( t^G \cap A = \{t, b_t\} = t^F \).

**Lemma (5K).** Let \( T \in Syl_2(N) \). Then the following holds.

1. \( |T : T| \leq 2 \).
2. \( \text{If } g \in T - T, \text{ then } \langle b, b, b, t \rangle^g = \langle a, b, b, b \rangle, B^g = F, F^g = B, D^g = E, \text{ and } E^g = D \).
3. \( \text{If } T \leq T, \text{ then there is an element } g \in T - T \) such that \( g^a \in \langle b, b \rangle \).
4. \( \text{If } T \leq T, \text{ then there is an element } g \in T - T \) such that \( t^a = a \text{ or } ab \).
Proof. First of all, \( Z_2(T) = \langle a, b_0, b_1, t \rangle \) and \( C_{(b_0, b_1, t)}(a) = \langle b_0, b_1 \rangle \) by Lemmas (5F) and (5H). Hence
\[
\mathcal{G}^*(Z_2(T)) = \{ \langle a, b_0, b_1 \rangle, \langle b_0, b_1, t \rangle \}
\]
and
\[
\langle b_0, b_1 \rangle = Z(Z_2(T)) \triangleleft T_1.
\]
Assume that \( T < T_1 \) and let \( g \in T_1 - T \). By Lemma (5J),
\[
t^g \cap \langle b_0, b_1, t \rangle = \{ t, b_0t \}.
\]
On the other hand, \( |t^g| = |T(g): R| \geq 4 \). Hence we must have that \( \langle b_0, b_1, t \rangle \triangleleft T \) by Lemma (5H). Therefore, \( g \in N(\langle b_0, b_1, t \rangle) \). Since \( g \) acts on \( \mathcal{G}^*(Z_2(T)) \), we conclude that
\[
\langle b_0, b_1 \rangle = \langle a, b_0, b_1 \rangle.
\]
As a consequence of this, we have that \( |t^g \cap \langle a, b_0, b_1 \rangle| = 2 \) and moreover \( t^g \cap \langle a, b_0, b_1 \rangle \leq a \langle b_0, b_1 \rangle \) since \( \langle b_0, b_1 \rangle \triangleleft T_1 \). Now \( a^g = ab_0 \) and \( (ab_0)^{g^z} = ab_0 b_1 \) by Lemmas (5E) and (5F). Hence
\[
t^g \cap \langle a, b_0, b_1 \rangle = \{ a, ab_0 \} \text{ or } \{ ab_1, ab_0 b_1 \}.
\]
This proves (4), and we may assume that \( t^g = a \) or \( ab_1 \) in proving the remaining part of (2) since \( B_2, D, E, \) and \( F \triangleleft T \).

Now we have shown that \( t^g \cap Z_2(T) = \{ t, b_0t, a, ab_0 \} \text{ or } \{ t, b_0t, ab_1, ab_0 b_1 \} \). Therefore, \( |T_1: R| = |t^T_1| \leq 4 \) and \( |T_1: T| \leq 2 \).

Let \( g \in T_1 - T \) and suppose \( t^g = a \) or \( ab_1 \). By Lemma (2F), \( B_2 \) and \( D \) are the only normal \( E_{2z} \)-subgroups of \( C_T(t) = R \). Also, \( E \) and \( F \) are normal \( E_{2z} \)-subgroups of \( C_T(a) = C_T(ab_1) \) by Lemma (5L). Hence \( \{ B_2, D \}^g = \{ E, F \} \). Now \( \langle a, B_2 \rangle \) is conjugate to \( \langle f, B_2 \rangle \) in \( N(B_2) \) since \( a^g = (n_1, n_2) \) and \( f^g = (n_3, n_6) \). Since \( \mathcal{G}^*(\langle a, B_2 \rangle) = \{ E, B_2 \} \) by Lemma (5F)(4) and since \( \mathcal{G}^*(\langle f, B_2 \rangle) = \langle C_{2z}(f), f, t, B_2 \rangle \), it follows that \( E \) is conjugate to \( \langle C_{2z}(f), f, t \rangle \) in \( N(B_2) \). Thus \( B_2^g \neq E \) by Lemma (3H) and so \( B_2^g = F \) and \( D^g = E \). This proves (2) as \( g^g \in T \leq N(B_2) \cap N(D) \).

Now \( \langle b_0, b_1 \rangle \triangleleft T_1 \) and \( \langle b_0, b_1 \rangle \not\triangleleft Z(T) \), so \( C_T(\langle b_0, b_1 \rangle) \) is a subgroup of \( C_{T_1}(\langle b_0, b_1 \rangle) \) of index 2. Furthermore, \( C_T(\langle b_0, b_1 \rangle) = B_2 F \) and \( B_2 \cap F = \langle b_0, b_1 \rangle \). The assertion (3) now follows from Lemma (1B) applied to \( C_{T_1}(\langle b_0, b_1 \rangle)/\langle b_0, b_1 \rangle \).

**Lemma (5L).** If \( T < T_1 \in \text{Syl}_2(N(T)) \), then the following conditions hold.

1. \( Z(T_1) = \langle b_0 \rangle \).
2. \( Z_2(T_1) = \langle b_0, b_1, at \rangle \).
FINITE GROUPS WITH A STANDARD SUBGROUP

(3) $Z_3(T_1) = \langle a, a, b_3, b_0, b_1, t \rangle$.

Proof. Since $Z(T_1) \leq C(t) \cap T_1 = R \leq T$, $Z(T_1) \leq Z(T) = \langle b_o \rangle$ by Lemma (5H). Hence $Z(T_1) = \langle b_o \rangle$, and consequently, $Z_3(T_1) \leq N_{T_1}(B_3) = T$. Since $Z(T_1) = Z(T)$, $Z_3(T_1) \leq Z_3(T) = \langle a, b_o, b_i, t \rangle$ by Lemma (5H). Now Lemma (5K)(2) shows that $T_1$ normalizes $\langle b_o, b_i \rangle$, so $\langle b_o, b_i \rangle \leq Z_3(T_1)$. Furthermore, if $g \in T_1 - T$, then $g$ interchanges $\langle a, b_o, b_i \rangle$ and $\langle b_o, b_i, t \rangle$. Hence $\langle b_o, b_i \rangle \leq Z_3(T_1) \leq \langle b_o, b_i, at \rangle$. We show that $at \in Z_3(T_1)$. We may assume that $t^g = a$ or $ab_i$ by Lemma (5K)(4). If $t^g = a$, then $a^g = t$ or $b_0 t$ since $g^2 \in T$ and $t^g = \{ t, b_o t \}$. Hence $(at)^g = atb_0$ or $at$ by Lemma (5F)(2). If $t^g = ab_i$, then $(ab_i)^g = t$ or $b_0 t$, so $(ab_i)^g = (ab_i) b_0$ or $ab_i t$. In either case, $at \in Z_3(T_1)$. Therefore, $Z_3(T_1) = \langle b_o, b_i, at \rangle$.

It remains to prove (3). Suppose first that $Z_3(T_1) \leq T$. Then we may choose $g \in Z_3(T_1) - T$. However, since $g$ normalizes $Z_3(T_1)B_3 = \langle a, B_3 \rangle$ and since $g^2(\langle a, B_3 \rangle) = \{ E, B_3 \}$ by Lemma (5F), we must have that $B_3^g = E$, contrary to Lemma (5K)(2). Thus $Z_3(T_1) \leq T$.

Let bars denote images in $T_1/\langle b_o, b_i \rangle$. Then $\overline{FB_3}$ is a normal $E_{4r}$-subgroup of $\overline{T_1}$ by Lemma (5K)(2) and $\overline{T_4} = \overline{FB_3} \langle \overline{a_2}, \overline{g} \rangle$. We choose $\overline{a}, \overline{f}, \overline{a}, \overline{b_2}, \overline{b_3}, \overline{t}$ as a basis of $\overline{FB_3}$ and represent $\overline{a}_2$ and $\overline{g}$ by $6 \times 6$ matrices with respect to this basis. Using Lemmas (5E) and (5F), we see that $\overline{a}_2$ has the following matrix form.

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}
\]

Therefore, $Z(\overline{T}) = C_{\overline{FB_3}}(\overline{a}_2) = \langle \overline{a}, \overline{a}, \overline{b_2}, \overline{t} \rangle$. Then by Lemma (5K)(2), $\overline{g}$ interchanges $\langle \overline{a}, \overline{a} \rangle$ and $\langle \overline{b_2}, \overline{t} \rangle$ as $\langle \overline{a}, \overline{a} \rangle = Z(\overline{T}) \cap \overline{F}$ and $\langle \overline{b_2}, \overline{t} \rangle = Z(\overline{T}) \cap \overline{B}_2$. Also, $\overline{g}$ interchanges $\langle \overline{a}, \overline{f} \rangle$ and $\langle \overline{b_2}, \overline{b_3} \overline{f} \rangle$ as $\langle \overline{a}, \overline{f} \rangle = \overline{F} \cap \overline{D}$ and $\langle \overline{b_2}, \overline{b_3} \overline{f} \rangle = \overline{E} \cap \overline{B}_2$. Thus $\overline{g}$ interchanges $\langle \overline{a}, \overline{f} \rangle$ and $\langle \overline{b_2}, \overline{b_3} \overline{f} \rangle$, and also interchanges $\langle \overline{a}_1, \overline{af} \rangle$ and $\langle \overline{b_2}, \overline{b_3} \overline{f} \rangle$. Since $\overline{g}$ also interchanges $\langle \overline{t} \rangle$ and $\langle \overline{a} \rangle$ by Lemma (5K)(2), we get that the matrix of $\overline{g}$ has the following shape.

\[
\begin{pmatrix}
1 & \alpha & 1 & 1 \\
1 & \beta & 1 & 1 \\
1 & \end{pmatrix}
\]
By Lemma (5K)(3), we may assume from the outset that $g^2 = 1$. This implies that the square of the above matrix is the unit matrix. Hence $\alpha = \beta$ and $\bar{g}$ has the following matrix form.

$$
\begin{pmatrix}
1 & 1 & 1 \\
\alpha & 1 & 1 \\
1 & \alpha & 1 \\
1 & 1 & 1
\end{pmatrix}
$$

Now an element $\bar{x}$ of $\overline{FB}_2$ is represented by a sextuplet $(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$. Using matrix forms of $\bar{a}_2$ and $\bar{g}$, we see that $[\bar{x}, \bar{a}_2]$ and $[\bar{x}, \bar{g}]$ are represented by the sextuplets $(\beta_2, 0, \beta_5, 0, 0)$ and $(\beta_1 + \beta_4 + \alpha \beta_5, \beta_2 + \beta_3, \beta_3 + \beta_2 + \beta_3, \beta_1 + \alpha \beta_2 + \beta_4, \beta_2 + \beta_5, \beta_2 + \beta_3 + \beta_5)$, respectively. This shows first that $[\overline{FB}_2, \bar{a}_2] = \langle \beta_2, \beta_5 \rangle \not\subset \langle \alpha \bar{t} \rangle$. Therefore, $Z_2(T_1) \leq \overline{FB}_2$. Next, both $[\bar{x}, \bar{a}_2]$ and $[\bar{x}, \bar{g}]$ are contained in $\langle \alpha \bar{t} \rangle$ if and only if the following equations hold.

$$
\beta_2 = \beta_5 = 0, \quad \beta_1 + \beta_4 + \alpha \beta_5 = 0, \quad \beta_2 + \beta_5 = 0,
\beta_3 + \beta_5 + \beta_6 = \beta_2 + \beta_3 + \beta_5, \quad \beta_1 + \alpha \beta_2 + \beta_4 = 0.
$$

These are satisfied if and only if $\beta_1 = \beta_4$ and $\beta_2 = \beta_5 = 0$. This implies that $Z_2(T_1) = \langle \alpha \bar{a}_2, \bar{a}, \bar{t} \rangle$. Hence (3) follows.

In the course of the proof of Lemma (5L), we have proved the following.

**Lemma (5M).** Let $T_1 \in \text{Syl}_2(N(T))$ and let $g$ be an element of $T_1 - T$ such that $g^2 \in \langle b_0, b_1 \rangle$. Then $g$ acts on $\overline{FB}_2 = FB_2/\langle b_0, b_1 \rangle$ in the following fashion.

$$
\begin{align*}
\bar{a}_i^g &= \bar{b}_i, \quad \bar{f}^g = \bar{b}_2^2 \bar{b}_3 \bar{t}, \quad \bar{a}^g = \bar{t}, \\
\bar{b}_2^g &= \bar{a}_1, \quad \bar{b}_3^g = \bar{a}_1 \bar{f} \bar{a}, \quad \bar{t}^g = \bar{a}.
\end{align*}
$$

Here, $\alpha = 0$ or 1.

**Lemma (5N).** $N(T)$ contains an $S_2$-subgroup of $G$.

**Proof.** Let $T_1 \in \text{Syl}_2(N(T))$. If $T = T_1$, then $T \in \text{Syl}_2(G)$. Therefore, assume that $T < T_1$. Then by Lemmas (5L), (5E), and (5F),

$$
\begin{align*}
Z_2(T_1) &= \langle a, a_i b_2, b_0, b_1, t \rangle \\
&= \langle b_1 \rangle \times \langle a, t \rangle \ast \langle a_i b_2 \rangle \\
&\cong Z_2 \times D_8 * Z_4.
\end{align*}
$$
Therefore, \( Z_s(T_1) \) has exactly 3 abelian maximal subgroups

\[
Y_1 = \langle b_1, t, a, b_2 \rangle , \\
Y_2 = \langle b_1, a, a_2 b_2 \rangle , \\
Y_3 = \langle b_1, at, a, b_2 \rangle .
\]

Let \( X \in \text{Syl}_2(N(T_1)) \). Since \( Y_3 \) contains \( Z_2(T_1) = \langle b_0, b_1, at \rangle \) while \( Y_1 \) and \( Y_2 \) do not, \( X \) acts on \( \{ Y_1, Y_2 \} \). Since \( t^6 \cap Y_1 = \{ t, b_0 t \} = t^7 \)
by Lemma (5J), \( N_X(Y_1) \leq N_X(\{ t, b_0 t \}) = T \). Thus \( |X: T| \leq 2 \) and so \( X = T_1 \). This shows \( T_1 \in \text{Syl}_2(G) \).

Now let \( T_1 \) be an \( S_2 \)-subgroup of \( G \) containing \( T \).

**Lemma (50)**. The following conditions hold.

1. \( W = \langle a, a_1, a_2, b_0, b_1, b_2, t \rangle = \langle A_1, a, t \rangle \) is a normal subgroup of \( T_1 \).
2. \( W \) is an extra-special group of order \( 2^n \), and \( Z(W) = \langle b_0 \rangle \).
3. \( T/W = \langle \langle f, b_0, W \rangle/W \cong E_4 \text{ if } T = T_1, \rangle \)
   \( \langle f, g, W \rangle/W \cong D_8 \text{ if } g \in T_1 - T. \)

**Proof.** First of all, \( |T_1: T| \leq 2 \) by Lemmas (5K) and (5N). Next, using Lemmas (5E) and (5F), we have that \( \mathcal{O}^*(T/B_2) = \{ B_2/F, \langle a_2, b_2, t \rangle F/F \} \). Since \( T_1 \) permutes \( B_2 \) and \( F \) and since \( B_2 F < T_1 \) by Lemmas (5I) and (5K), it follows that \( T_1 \) permutes \( \langle a, a_1, a_2 \rangle B_2 \) and \( \langle a_2, b_2, t \rangle F \). Hence \( T_1 \) normalizes their intersection. Since \( \langle a_2, b_2, t \rangle F = \langle a, a_1, a_2 \rangle \langle b_0, b_1, b_2, t \rangle \), the intersection is equal to \( \langle a, a_1, a_2 \rangle \langle b_0, b_1, b_2, t \rangle = W \). Hence (1) holds.

Now \( W = \langle a_1, b_2 \rangle \langle a_2, b_1 \rangle \langle a, t \rangle \cong D_8 \times D_8 \times D_8 \) and \( Z(W) = \langle b_0 \rangle \). We have that \( T = \langle f, b_0, W \rangle \), so \( T/W \cong E_4 \). Assume that \( T < T_1 \). Then by Lemma (5K), there is an element \( g \in T_1 \) such that \( T_1 = \langle g \rangle T \) and \( g^2 \in \langle b_0, b_1 \rangle \cong W \). Lemma (5M) shows that \( f^g \in b_1 W \). Thus \( T_1 = \langle f, g, W \rangle \) and \( T_1/W \cong D_8 \). The proof is complete.

Now let bars denote images in \( C(b_0)/\langle b_0 \rangle \). Then \( T_1 \) acts on \( \bar{W} \) by Lemma (50). In the following two lemmas, we collect information on this action. Notice that we may choose \( \bar{a}_1, \bar{b}_2, \bar{a}_2, \bar{b}_1, \bar{a}, \bar{t} \) as a basis of \( \bar{W} \).

**Lemma (5P)**. The following conditions hold.

1. \( \bar{a}_2 b_2 = \bar{a}_1 b_1, \bar{b}_2 b_2 = \bar{b}_1 b_1, \bar{a}_2 b_2 = \bar{b}_1 \bar{a}_1, \bar{b}_1 b_2 = \bar{b}_1, \bar{a}_2 = \bar{a}, \bar{t}_2 = \bar{t} \).
2. \( \bar{a}_1 = \bar{a}, \bar{b}_2 = \bar{b}_1, \bar{a}_2 = \bar{a}_1, \bar{b}_1 = \bar{b}_1, \bar{a}_1 = \bar{a}, \bar{t} = \bar{t} \).
3. \( \bar{a}_2 b_2 = \bar{a}_1 b_1, \bar{a}_2 b_2 = \bar{a}_1 \bar{b}_2 \bar{a}_2 b_1, \bar{b}_1 b_2 = \bar{b}_1, \bar{a}_2 = \bar{a}, \bar{t}_2 = \bar{t} \).
4. \( C_{\bar{W}}(b_0) = \langle \bar{b}_2, \bar{b}_1, \bar{a}, \bar{t} \rangle \).
(5) \( C_W(f) = \langle \bar{a}_1, \bar{b}_1, \bar{a}, \bar{t} \rangle \).

(6) \( C_W(fb_3) = \langle \bar{a}_1, \bar{b}_2, \bar{b}_1, \bar{a}, \bar{t} \rangle \).

Proof. (1), (2), and (3) follow from relations listed in Lemmas (2A) and (2F) together with Lemmas (5F) and (5G)(1). (4), (5), and (6) are consequences of (1), (2), and (3), respectively.

**Lemma (5Q).** If \( T < T_1 \), then there is an element \( g \in T_1 - T \) which satisfies the following conditions.

1. \( g^a \in \langle A, at \rangle \).
2. \( \bar{a}_1^a = \bar{b}_2, \bar{b}_2^a = \bar{a}_1, \bar{b}_1^a = \bar{a}_2(\bar{b}_1, \bar{a}_1), \bar{b}_2^a = \bar{b}_1, \bar{a}^a = \bar{b}^a \bar{t}, \bar{b}_1^a = \bar{b}^a \bar{t}, \bar{a}_2^a = \bar{b}^a \bar{t} \), \( \bar{a}_2 = \bar{b}_1 \; \) where \( \alpha = 0 \) or \( 1 \).
3. \( C_W(g) = \langle \bar{a}_1, \bar{b}_2, \bar{b}_1, \bar{a}_2, \bar{a}, \bar{t} \rangle if \; \alpha = 0 \),
   \( \langle \bar{a}_1, \bar{b}_2, \bar{b}_1, \bar{a}_2, \bar{a}, \bar{t} \rangle if \; \alpha = 1 \).

Proof. Choose \( \bar{a}_1, \bar{b}_2, \bar{a}_2, \bar{b}_1, \bar{a}, \bar{t} \) as a basis of \( \bar{W} \). Lemma (5P) shows that \( b_3, f \), and \( fb_3 \) have the following matrix forms with respect to this basis, respectively.

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

Choosing a suitable element \( g \in T_1 - T \), we determine the matrix of \( g \). We choose \( g \) so that \( g^a \in \langle b_3, b_1 \rangle \) by Lemma (5K)(3). From Lemmas (5L) and (5M), we get that \( \langle a, b_0, b_1 \rangle = \langle b_0, b_1 \rangle, \langle b_0, b_1 \rangle = \langle b_0, b_1 \rangle \), and \( \langle a, b_0, b_1 \rangle = \langle b_0, b_1, t \rangle \). Hence \( g \) has the following matrix form.

\[
\begin{pmatrix}
1 & \alpha \\
\gamma_1 & \gamma_2 \\
\gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\
1 & 1 \\
\delta & 1 \\
\varepsilon & 1
\end{pmatrix}
\]

Clearly, \( \gamma_3 = 1 \). Since \( g^a \in W \), the square of this matrix should be the unit matrix. Hence we have that \( \alpha = \beta, \delta = \varepsilon, \gamma_1 = \gamma_2, \gamma_5 = \gamma_6, \) and so, changing notation, we see that \( g \) has the following
matrix form.

\[
\begin{pmatrix}
1 & \alpha \\
1 & \alpha \\
\beta & \beta & 1 & \gamma & \delta & \delta \\
1 & \\
\varepsilon & 1 \\
\varepsilon & 1
\end{pmatrix}
\]

By Lemma (5M), \(gbg \in fW\). This implies that

\[
\begin{pmatrix}
1 & \alpha \\
1 & \alpha \\
\beta & \beta & 1 & \gamma & \delta & \delta \\
1 & \\
\varepsilon & 1 \\
\varepsilon & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
\beta & \beta & 1 & \gamma & \delta & \delta \\
1 & \\
\varepsilon & 1 \\
\varepsilon & 1
\end{pmatrix}
\begin{pmatrix}
1 & \alpha \\
1 & \alpha \\
\beta & \beta & 1 & \gamma & \delta & \delta \\
1 & \\
\varepsilon & 1 \\
\varepsilon & 1
\end{pmatrix}
\]

is equal to the matrix of \(f\). Hence we have that \(\alpha = \beta\). Now \(gbg\) has the following matrix form.

\[
\begin{pmatrix}
1 & \alpha + 1 \\
1 & \alpha + 1 \\
\alpha + 1 & \alpha + 1 & 1 & \gamma + 1 & \delta & \delta \\
1 & \\
\varepsilon & 1 \\
\varepsilon & 1
\end{pmatrix}
\]

Hence, replacing \(g\) by \(gbg\), if \(\alpha = 1\), we may assume that \(\alpha = 0\). Thus the matrix of \(g\) has the following shape.

\[
\begin{pmatrix}
1 \\
1 \\
1 & \gamma & \delta & \delta \\
1 & \\
\varepsilon & 1 \\
\varepsilon & 1
\end{pmatrix}
\]

This in turn implies that \(a_{x}^{2} \in a_{x}b_{x}a_{x}^{-2}b_{x}^{-1}\) and so \(1 = (a_{x}^{2})^{2} = (a_{x}b_{x}^{-1})^{2}(a_{x}^{-1}b_{x}^{-1})^{2}\). Hence we have that \(\gamma = \delta\). Finally, \(\mathcal{W}\) becomes a nonsingular symplectic space over \(\mathbb{F}_{2}\) with respect to the bilinear form \((\bar{x}, \bar{y}) = \lambda\), where \([x, y] = b_{x}^{-1}b_{y}\), \(\lambda \in \{0, 1\}\), and the basis we have chosen is a symplectic basis. Furthermore, \(g\) induces a symplectic transforma-
tion on $\bar{W}$. This implies that the matrix of $g$ is invariant under the transpose-inverse mapping followed by conjugation by the matrix

$$
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}.
$$

Hence we have that $\gamma = \varepsilon$. Thus, changing notation, we conclude that $g$ has the following matrix form.

$$
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & \alpha & \alpha & \alpha \\
1 & 1 & \alpha & 1 \\
1 & \alpha & 1
\end{pmatrix}
$$

This implies that $g$ satisfies (2).

Now let $W_0 = \langle A, at \rangle$. We have chosen $g$ so that $g^2 \in \langle b_0, b_1 \rangle \leq W_0$, and we may have replaced $g$ by $gb_3$. However, Lemma (5M) shows that $(f b_3)^g \in \langle a_1 b_0, b_0, b_1, at \rangle f b_3 \leq W_0 f b_3$ and so $(gb_3)^g = g^2 (f b_3)^g f b_3 \in W_0$. Therefore, the property that $g^2 \in W_0$ is preserved. Thus $g$ satisfies (1). Since (3) is a consequence of (2), we have proved the lemma.

**Lemma (5R).** $W$ is weakly closed in $T_1$ with respect to $G$.

**Proof.** Assume that $T_1$ contains a conjugate $X$ of $W$ different from $W$. Since $|XW: W| \leq |T_1: W| \leq 2^4$, $|X \cap W| \geq 2^4$. If $Z(X) \not\subseteq W$, then $(X \cap W)^2 \leq W \cap Z(X) = 1$ and $(X \cap W)Z(X)$ is elementary abelian of order at least $2^5$. However, this is impossible as $X$ is extra-special of order $2^7$. Therefore, $Z(X) \subseteq W$. Then $X^2 = Z(X) \subseteq W$, so $XW/W$ is elementary abelian. Hence $|XW: W| \leq 2^2$ by Lemma (5O), and $|X \cap W| \geq 2^4$. Thus, $W' = (X \cap W)' = X'$ and so $X$ centralizes $X \cap W/W'$. Since $|X \cap W/W'| \geq 2^4$ and since no element of $T_1 - W$ centralizes a hyperplane of $W/W'$ by Lemmas (5P) and (5Q), we have that $|X \cap W/W'| = 2^4$ and $|XW/W| = 2^2$. However, $XW = \langle f, b_3, W \rangle$ or $\langle fb_3, g, W \rangle$ by Lemma (5O) and so $|C_{W/W}(X)| < 2^4$ by Lemmas (5P) and (5Q). Here we choose $g$ so that $g^2 \in W$. This is a contradiction proving the lemma.
LEMMA (5S). \( t \in G' \).

**Proof.** Define

\[
W_0 = \langle A, at \rangle,
\]

and

\[
T_0 = \begin{cases} 
\langle af, b, W_0 \rangle & \text{if } T = T_1, \\
\langle af, b, g, W_0 \rangle & \text{if } g \in T_1 - T.
\end{cases}
\]

We choose \( g \) as in Lemma (5Q). Lemmas (5P) and (5Q) show that \( f \) and \( b, g \) normalize \( A \) and \( \langle at \rangle \), and that \( g \) normalizes \( W_0 \). Hence \( W_0 \leq T_1 \). Using Lemmas (5E) and (5F), we get that \( \langle afb, b, W_0 \rangle = \langle af, b, W_0 \rangle \) has order 2⁴. By the choice of \( g \) and Lemma (5M), \( \langle af, b, W_0 \rangle = \langle af, b, W_0 \rangle \) has order 2⁴. Hence \( g \) normalizes \( \langle af, b, W_0 \rangle \) and \( \langle af, b, g, W_0 \rangle \). In particular, \( |\langle af, b, g, W_0 \rangle| = 2^6 \). Hence \( T_0 \) is a maximal subgroup of \( T_1 \) in either case.

Assume that \( t \in G' \). Then \( T_0 \) contains an extremal conjugate \( u \) of \( t \) in \( T_1 \) by Lemma (1E). We may assume that \( u^s = t \) and \( C_{\Gamma_1}(u)^s = C_{\Gamma_1}(t) = R \) for some \( x \in G \).

Suppose \( u \in W_0 \). Since \( u \in Z(W) = \langle b, g \rangle \), \( |C_{\Gamma_1}(u)| = 2^6 \) by Lemma (1D), and so \( |C_{\Gamma_1}(u):C_{\Gamma_1}(u)| = 2^6 \). Hence \( C_{\Gamma_1}(u)^s \leq \langle b, g \rangle \). Since \( C_{\Gamma_1}(u)^s = R \) and since \( R'' = \langle b, g \rangle \), it follows that \( x \in C(b, g) \). Now \( W/W_0 \) is weakly closed in \( C(b, g)/\langle b, g \rangle = C(1, 0) \) by Lemma (5R), so there exists an element \( y \in N(W) \) such that \( x = y \). Then \( t^x = u \) or \( ub, g \), and so \( C_{\Gamma_1}(u)^s = C_{\Gamma_1}(u) \). Now \( |C_{\Gamma_1}(u):C_{\Gamma_1}(u)| = 2^6 \), so \( \langle fb, g, W_0 \rangle \leq C_{\Gamma_1}(u) \). Hence \( u \in C_{\Gamma_1}(u)^s = C_{\Gamma_1}(u) \). Thus \( u \in \langle b, g \rangle \langle b, at \rangle \). Also, \( u \in A, at \) as \( t^6 \cap A_1 = \emptyset \). Since \( u \neq 1 \), we conclude that \( u = a, b, at \). Now \( a, b, at = (a, b, at)^s \), and \( a, b, at = (a, b, at)^s \). Therefore, \( a, b, at \leq u^6 \cap C_{\Gamma_1}(u) \). But now \( t^6 \cap C_{\Gamma_1}(u) = t^6 \cap A_1, t = \{ t, b, t \} \) by Lemma (5J), so \( t^6 \cap C_{\Gamma_1}(u) = u^6 \cap C_{\Gamma_1}(u) \) contains only two elements. This contradiction shows that \( u \neq W_0 \).

Suppose \( u \in T_1 - \langle fb, W_0 \rangle \). Then \( C_{\Gamma_1}(u) \leq T_1 \) or \( \langle fb, g, W_0 \rangle \), so \( |C_{\Gamma_1}(u):C_{\Gamma_1}(u)| \leq 2^6 \). Also, \( uW \) is conjugate to \( fW, bW, gW \) in \( T_1 \), so \( |C_{\Gamma_1}(u)| \leq 2^6 \) by Lemmas (5P) and (5Q). But then \( |C_{\Gamma_1}(u)| \leq 2^6 \) and \( |C_{\Gamma_1}(u)| \leq 2^6 \), which is a contradiction. Therefore, \( u \in \langle fb, W_0 \rangle \cap T_0 = \langle afb, W_0 \rangle \) and then \( u \in afb, W_0 \).

Now \( (afb)^2 = b, afb \) is an involution which normalizes \( A_1 \) and \( \langle at \rangle \). Moreover, \( C_{\Gamma_1}(afb) = \langle b, b, b \rangle \) by Lemma (5O), hence Lemma (1C) shows that \( \bar{u} \) is conjugate to \( afb, afb \) under \( A_1 \). Since \( u^s = 1 \), we have that \( u \) is conjugate in \( T_1 \) to an element of \( afb, at \). Notice that \( afb, afb, t = afb, t \) by (5F) and (5G). So we assume that \( u \in afb, t \). Then \( C_{\Gamma_1}(u) = C_{\Gamma_1}(afb, t) \). Now \( C_{\Gamma_1}(afb, t) = \langle afb, afb, t \rangle \). Therefore, \( C_{\Gamma_1}(afb, t) = \langle afb, afb, t \rangle \).
C_w(f b_t) = \langle \bar{a}, \tilde{b}, \bar{b}, \bar{a}, \tilde{t} \rangle$ by Lemma (5P), and so $C_w(f b_t) \leq \langle a, b_2, b_t, a, t \rangle$. Equality does not hold here, since $(f b_t)^a b_2 = (f b_2 b_t)^{b_2} = f b_2 b_t b_2 a, t)$. Therefore, $|C_w(f b_t)| \leq 2^4$ and since $|C_{\tau_1}(f b_t): C_w(f b_t)| \leq 2^3$, it follows that $|C_{\tau_1}(f b_t)| \leq 2^7$. This is a contradiction because $C_{\tau_1}(f b_t) = C_{\tau_1}(u)$ has order $2^8$. Therefore, $t \notin G'$.

Now we conclude the proof of Theorem (5A). Let $X = \langle L^o \rangle$ and let bars denote images in $G/O(G)$. Since $|G|_2 \leq 2^{10}$ and $t \notin G'$, we have that $|\bar{X}|_2 \leq 2^9$. Hence by Lemma (1H), $\bar{X}$ is a simple group and $C_G(\bar{X}) = 1$. Now $N(A_2)/C(A_2) \cong \Sigma_6$ or $A_6$ by Lemmas (5B) and (5C). Since $O^2(N) = \langle P \rangle \leq N_X(A_2)$, it follows that $N_X(A_2)/C_X(A_2) \cong \Sigma_6$ or $A_6$. Also, since $B_2 \in \text{Syl}_2(C(A_2))$ and since $t \notin X$, we get that $A_2 \in \text{Syl}_2(C_X(A_2))$. Assume that $N_X(A_2)/C_X(A_2) \cong \Sigma_6$. Then since $|\bar{X}|_2 \leq 2^9$, [26] shows that $\bar{X}$ is isomorphic to the Higman-Sims simple group. However, the centralizer of an involution in the automorphism group of the Higman-Sims group does not have a component isomorphic to $PSL(4, 2)$ (see [2]). Hence $N_X(A_2)/C_X(A_2) \cong A_6$, and so $r(X) = 4$ by [17, Theorem 3].

6. In this section, we consider the following situation.

**Hypothesis (6.1).** $t^{\gamma(B_2)} = A_2 t$.

Notice that this implies Hypothesis (3.1). Hence $\langle t \rangle \in \text{Syl}_2(C_0(L))$ by Lemma (3B). We prove the following theorem.

**Theorem (6A).** Under Hypothesis (6.1), $\langle L^o \rangle \cong PSL(4, 4)$ or $PSU(4, 2) \times PSU(4, 2)$, or else Case (3) of the main theorem occurs.

We begin the proof by studying the structure of $N(B_2)$.

**Definition (6.1).** Let $D_2 = O^2(N(B_2))$.

**Lemma (6B).** The following conditions hold.

1. $N(B_2) = N_0(B_2) D_2$ and $N_0(B_2) \cap D_2 = B_2$.
2. $D_2/B_2$ is elementary abelian and commutation by $t$ induces an $N_0(B_2)$-isomorphism $D_2/B_2 \to A_2$.
3. $Z(D_2) = D_2^t = A_2$.

**Proof.** By Hypothesis (6.1), $|N(B_2) : N_0(B_2)| = 16$. As $N_0(B_2)/C(B_2) \cong A_5$ or $\Sigma_6$, we have that $|N(B_2)/C(B_2)| = 2^6 \cdot 3 \cdot 5$ or $2^7 \cdot 3 \cdot 5$. Then a theorem of [4] shows that $N(B_2)/C(B_2)$ is not simple; so let $C(B_2) < X \lhd N(B_2), X \neq N(B_2)$. Recall from Lemma (3G) that $N(B_2)/C(B_2)$ is a primitive permutation group on $\Omega = A_4 \Gamma$. Hence we have
\[ N(B_2) = N_\varphi(B_2)X. \] Furthermore, either \( N_\varphi(B_2) \cap X/C(B_2) \cong A_4 \) or 1. Assume the former. Then \( N_\varphi(B_2)/C(B_2) \cong \Sigma_5 \) as \( X \not\cong N(B_2) \), and so \( |N(B_2)/C(B_2)_2| = 2^7 \). Hence \( N(B_2)/C(B_2) \) can not be embedded in \( GL(4,2) \). Thus Lemma (3E) forces \( C(B_2) < C(A_2) \cap N(B_2) \lhd N(B_2) \), and so \( C(A_2) \cap N(B_2)/C(B_2) \) is a nontrivial normal 2-subgroup of \( N(B_2)/C(B_2) \) by Lemma (3F). Therefore, we can always choose \( X \) so that \( N_\varphi(B_2) \cap X = C(B_2) \). Let us fix such \( X \), and let bars denote images in \( N(B_2)/C(B_2) \). Then \( X^\circ \) is the regular normal subgroup of \( N(B_2) \) and so \( X \) is a self-centralizing elementary abelian subgroup of order 16. Let \( Y = C(O(C)) \cap N(B_2) \). Then as \( C(B_2) = B_2 \times O(C) \), \( O(C) \lhd N(B_2) \) and \( Y \lhd \bar{N}(B_2) \). Moreover, \( \bar{Y} \neq 1 \) as \( \bar{K}_2 \leq \bar{Y} \). Hence we have \( \bar{X} \cap \bar{Y} \neq 1 \), and so \( \bar{X} \leq \bar{Y} \). This implies that \( X = C_x(O(C))O(C) \). Thus \( X \) is 2-closed and, as \( O_2(N_\varphi(B_2)) = B_2 \), the statement (1) follows.

Now \( A_2 \lhd D_2 \) by Lemma (3E), so \( A_2 \cap Z(D_2) \neq 1 \). As \( K_2 \) acts irreducibly on \( A_2 \), it follows that \( A_2 \leq Z(D_2) \). Also, \( Z(D_2) \leq C_{D_2}(t) = B_2 \). Therefore, \( Z(D_2) = A_2 \). Consequently, (2) holds. Moreover, \( A_2 \cap D_2^i \neq 1 \) and so \( A_2 \leq D_2^i \leq B_2 \). Suppose that \( D_2^i = B_2 \). Then \( D_2/A_2 \) has a cyclic subgroup \( X/A_2 \) of order 4. As \( A_2 = Z(D_2) \), \( X \) is abelian. But this contradicts \( C_{D_2}(t) = B_2 \). Therefore, \( D_2^i = A_2 \).

**Definition (6.2).** Let \( Q_2 = QD_2 \), \( Q_1 = N_{Q_2}(Q) \), and \( F = N_{Q_2}(Q_1) \). Let \( V = \langle Z, t \rangle \), \( D_1 = O_2(N(B_1)) \), and \( D_0 = C_{D_1}(A_1) \).

**Remark.** We have \( Q_1/B_2 = Q/B_2 \times N_{D_2/B_2}(Q/B_2) \) and the \( N_\varphi(B_2) \)-isomorphism \( D_2/B_2 \rightarrow A_2 \) maps \( N_{D_2/B_2}(Q/B_2) \) onto \( C_{A_2}(Q) = Z(P) \). Hence \( |N_{D_2/B_2}(Q/B_2)| = 2 \) and \( |Q_1/Q| = 2 \). Also, \( F \) is the product of \( Q \) and the group of elements \( x \) of \( D_1 \) such that \( [Q, x] \leq N_{D_2}(Q) \). Commutation by \( t \) maps the latter group onto the group of elements \( y \in A_2 \) such that \( [Q, y] \leq Z(P) \), which is equal to \( A_1 \cap A_2 \). Thus we have \( |F/B_2| = 32 \).

**Lemma (6C).** The following conditions hold.

1. \( N(B_2) \leq N(A_2) \).
2. \( N(B_2) = N(V) \).
3. \( N(B_2)/B_2 = N_\varphi(B_1)/B_1 \times D_1/B_1 \).
4. \( QD_1 = Q_1 \).
5. \( D_1 = B_1D_0 \) and \( B_1 \cap D_0 = V \).
6. \( D_0 \cong D_2 \).
7. \( D_0 \leq D_2 \).
8. \( [N_{Z}(A_1), D_0] = 1 \).

**Proof.** Every involution of \( A_1t \) is conjugate to an element of
A_t under L, and so it is conjugate to t by Hypothesis (6.1). As 
\( t^x \cap A_1 = \emptyset \) by Lemma (3C) and as \( A_1 = \Omega_1(A_i) \), it follows that \( A_1 = \langle ab \mid a, b \in t^x \cap B_i \rangle \). Hence (1) follows.

Now \( |Q_1 \cap D_2 : B_2| = 2 \) by Lemma (6B) and so \( Q_1 \cap D_2 = B_2 \langle Q_1 \cap D_1 \cap C(HO(C)) \rangle \). Let \( x \in Q_1 \cap D_1 \cap C(HO(C)) - B_2 \). Then \( x \in \mathcal{N}(B_i) \) by Lemma (3J). In particular, \( N_c(B_i) < \mathcal{N}(B_i) \). Now, \( \mathcal{N}(B_i) = \mathcal{N}(V) \) as \( Z(B_i) = V \), and \( N_c(B_i) = N_c(V) = O_4(N_L(V)) = A_i \). Moreover, \( |N(V) : N_c(V)| \leq 2 \) as \( t^{N(V)} \leq \{ t, \sigma, t \} \). Hence \( \mathcal{N}(B_i) = \mathcal{N}(V) = \langle N_c(B_i), x \rangle \). In particular, (2) holds.

Now \( B_iC(B_i) = B_i \times O(C) \) by Lemma (2G). Hence \( O(C) \triangleleft \mathcal{N}(B_i) \) and \( X = C_{N(B_i)}(O(C))O(C) \) is a normal subgroup of \( \mathcal{N}(B_i) \) containing \( B_iO(C) \). Let bars denote images in \( \mathcal{N}(B_i) \). Then \( \mathcal{N}(B_i) = \langle N_c(B_i), \bar{x} \rangle \), it follows that \( \bar{H} < \mathcal{N}(B_i) \). Hence \( \bar{Y} = C_{\bar{H}}(\bar{H}) \) is a normal subgroup of \( \mathcal{N}(B_i) \). Now, \( \bar{x} \in \bar{Y} \) by the choice of \( x \), and so \( \bar{Y} = \langle \bar{Y} \cap N_c(B_i), \bar{x} \rangle \). As \( \mathcal{N}(A_i) = K_i \times H \leq \bar{Y} \cap N_c(B_i) \leq C(H) \cap N_c(B_i) = N_L(A_i) \), it follows that \( \bar{Y} = (K_i \times H) \langle \bar{x} \rangle \). Now \( K_i \cong \Sigma_3 \). Hence \( K_i = O_3(K_i \times H) \langle \bar{Y} \rangle \), and so, as \( \text{Aut}(\Sigma_3) \cong \Sigma_3 \), it follows that \( \bar{Y} = K_i \times H \times \bar{K} \) for some subgroup \( \bar{K} \) of order 2. Clearly, \( \bar{K} = O_3(\bar{Y}) \triangleleft \mathcal{N}(B_i) \). Now let \( K \) denote the preimage of \( \bar{K} \) in \( \mathcal{N}(B_i) \). Then as \( O(C) \leq K \leq X \), \( K = C_{\bar{H}}(O(C))O(C) \) and thus \( K \) is 2-closed. As \( O_3(N_c(B_i)) = B_i \) by Lemma (2G), (3) holds.

As a consequence of (3) we have \( D_1 \leq N(Q) \), so \( D_1 \leq N(B_i) \) by Lemma (3J). Hence \( D_1 \) normalizes \( Q_2 = QD_2 \). Also, \( B_1 \cap B_2 < B_i < D_1 \) is a series of \( H \)-invariant normal subgroups of \( D_1 \). As \( H \) acts irreducibly on \( B_i/B_1 \cap B_2 \) by Lemma (2B), it follows that \( D_1 \) centralizes \( B_i/B_1 < B_2 \). Noticing that \( B_i/B_1 \cap B_2 \cong Q_2/D_2 \), we conclude that \( D_1 \) centralizes \( Q_2/D_2 \). However, \( \mathcal{N}(B_i)/D_2O(C) \cong A_3 \) or \( \Sigma_5 \) by Lemma (6B) and, in particular, an \( S_5 \)-subgroup of \( \mathcal{N}(B_i)/D_2 \) is either \( E_6 \) or \( D_6 \). Thus we have \( D_1 \leq Q_2 \), and as \( D_1 \leq N(Q) \) and \( |Q_1 : Q_2| = 2 \), (4) follows.

To prove the remaining assertions, set \( D = C_{D_2}(H) \). Then as \( H \) centralizes \( D_2/B_1 \) and as \( C_{B_1}(H) = V \), we have \( D_1 = B_2D \) and \( B_1 \cap D = V \). Consequently, \( |D| = 8 \) and as \( C_D(t) = C_{B_1}(H) = V \), we see that \( D \cong D_0 \). Now \( D \leq Q_2 \) by (4) and \( H \) acts regularly on \( Q_2/D_2 \) as \( Q_2/D_2 \cong Q/B_2 \) as \( H \)-modules. Therefore, \( D \leq D_1 \), and then \( D \leq D_1 : s_1 \leq N(D) \) by the definition of \( D \). Thus by Lemma (6B), \( D \) centralizes \( \langle A_2, A_3^3, H \rangle = N_L(A_i) \). In particular, \( [A_i, D] = 1 \) and hence it follows that \( D = D_0 \). Thus all parts of the lemma hold.

**Lemma (6D).** \( D_2 \) has a maximal subgroup \( E_2 \) which is either elementary abelian or homocyclic of exponent 4 and is inverted by \( t \).

**Proof.** Let \( \Gamma = \{ e, c_2, c_5, e, c_5 \} \). We may choose elements \( d_i \in
$D_i, i \in \{1, 2, 3, 4, 5\}$, such that $[d, t] = c_i$ by Lemma (6B)(2). Let $D_i = D_i/B_i$ and $\Delta = \{d, d, d, d, d\}$. Now $\Gamma$ is the set of central involutions of $L$ contained in $A_i$, so $N_{C}(B)$ acts transitively on $\Gamma$. Hence $N(B_2)$ acts transitively on $\Delta$ by Lemma (6B). We may choose each $d_i$ to be an involution. Indeed, we can choose $d_i \in I(D_0)$ by Lemma (6C), and then choose conjugates $d_2, d_3, d_4, d_5$ of $d_1$ under $N_{C}(B_2)$. Then $\langle d_i, A \rangle$ is elementary abelian since $A_2 = Z(D_i)$, and moreover, $C_{(d, A)}(t) = A_2$. Hence $\varphi^*(\langle d_i, B_2 \rangle) = \langle d_i, A \rangle$, $B_2$, and so if $D_2 = D_2/A_2$, then $\langle d_i, d_2, \ldots, d_5 \rangle$ is $N(B_2)$-invariant. Now $c_1, c_2 \cdots c_5 = 1$, so $\tilde{d}_i \tilde{d}_j \cdots \tilde{d}_5 = 1$. Thus there are two cases: $\tilde{d}_i \tilde{d}_j \cdots \tilde{d}_5 = 1$ or $\tilde{t}$. As $A_2 = \langle c_1, c_2, \ldots, c_5 \rangle$, $\tilde{D}_2 = \langle \tilde{d}_i, \tilde{d}_2, \ldots, \tilde{d}_5 \rangle$ and so $\tilde{D}_2 = \langle \tilde{d}_i, \tilde{d}_2, \ldots, \tilde{d}_5, \tilde{t} \rangle$. Hence if $\tilde{d}_i \tilde{d}_j \cdots \tilde{d}_5 = 1$, then we may choose $\tilde{d}_i, \tilde{d}_2, \ldots, \tilde{d}_5, \tilde{t}$ as a basis of $\tilde{D}_2$. If $\tilde{d}_i \tilde{d}_j \cdots \tilde{d}_5 = \tilde{t}$, then we may choose $\tilde{d}_i, \tilde{d}_2, \ldots, \tilde{d}_5$ as a basis of $\tilde{D}_2$. In either case, the basis of $\tilde{D}_2$ we have chosen is $N(B_2)$-invariant. Hence if we define $\tilde{E}_2$ to be the subgroup of $\tilde{D}_2$ generated by the elements that are the products of even number of the basis elements, then $\tilde{E}_2$ is an $N(B_2)$-invariant maximal subgroup of $\tilde{D}_2$ and $\tilde{B}_2 \cap \tilde{E}_2 = 1$.

Let $E_2$ be the preimage of $\tilde{E}_2$ in $D_2$. Then $E_2/A_2 \cong A_2$ as $K$-modules by Lemma (6B)(2), so $E_2$ is abelian by Theorem 1 of [24].

If $\tilde{d}_i \tilde{d}_j \cdots \tilde{d}_5 = 1$, then $\tilde{d}_i = (\tilde{d}_i \tilde{t}) (\tilde{d}_i \tilde{t}) (\tilde{d}_i \tilde{t}) (\tilde{d}_i \tilde{t}) \in \tilde{E}_2$ by the definition of $\tilde{E}_2$, and so $E_2$ is generated by involutions. If $\tilde{d}_i \tilde{d}_j \cdots \tilde{d}_5 = \tilde{t}$, then $\tilde{d}_i \tilde{d}_j \cdots \tilde{d}_5 \tilde{t} \in \tilde{E}_2$. As $(\tilde{d}_i \tilde{t})^5 = [d_i, t] = c_i$, $E_2$ has a basis consisting of elements of order 4 inverted by $t$. The proof is complete.

**Definition (6.3).** Let $W = D_0 \cap E_2$.

Since $D_i = E_2 \langle t \rangle$ and $t \in D_0 \leq D_2$, we have $D_0 = W \langle t \rangle$ and $W \cong Z_4$ or $E_4$. Also, $WA_2 = Q_1 \cap E_2$. Indeed, $A_2 W \subseteq Q_1 \cap E_2$ by definition, $|Q_1 \cap E_2 : A_2| = 2$ by a remark following Definition (6.2), and $W \not\subseteq A_2$ as $W \langle t \rangle = D_0 \leq B_2 = A_2 \langle t \rangle$ by Lemma (6C).

**Lemma (6E).** The following conditions hold.

1. $N(B_2) \leq N(D_2) \leq N(D_1) \leq N(A_1 W) \leq N(W)$.
2. $Q_2 \cap N(D_1) = F$.
3. If $N(B_2) = N(D_2)$, let $D = O_2(N(D_2))$. Then $N(D_1) = N(B_1) D, N(B_2) \cap D = D_1, D/D_1$ is elementary abelian, and $D/D_1 \cong A_1 / Z$ as $N(B_2)$-modules.
4. If $N(B_1) < N(D_0)$, then the following hold.
   1. $C(D_1 / W) = D_0 O(C)$.
   2. $N(D_0) / D_0 O(C) \cong \Sigma_5$.
   3. $N(D_0) / D_0 O(C) \cong \Sigma_5$ wreath $Z_2$.
   4. $W \cong Z_4$.
   5. $C = C_0(L)$. 

**Proof.**
Proof. By definition, \( D_0 = C_{D_0}(A_i) \triangleleft N(A_i) \cap N(D_i) \). As \( N(B_i) \leq N(A_i) \cap N(D_i) \) by Lemma (6C), \( N(B_i) \leq N(D_0) \). Recall also from Lemma (6C) that \( N(B_i) = N(V) \) and that \( D_0 \cong D_i \). These show

(a) \[ |N(D_0) : N(B_i)| \leq 2 \]

as \( V \) is one of the two \( E_i \)-subgroups of \( D_i \). In particular, \( N(B_i) \triangleleft N(D_0) \) and so, as \( D_i = O_{s}(N(B_i)) \), we have \( N(D_0) \leq N(D_i) \). As \( A_i = C_{D_i}(D_0) \), we also have that

(b) \[ N(D_0) \leq N(A_i) \]}

We argue that \( N(D_0) \leq N(W) \) and \( V \not\sim W \). If \( W \cong Z \), this is obvious. If \( W \cong E_4 \), then \( E_4 \cong E_{200} \) by Lemma (6D) and so \( t^{o} \cap W = \emptyset \) as \( m(C) = 5 \). Thus \( V \not\sim W \) and consequently \( N(D_0) \leq N(W) \). Furthermore, if \( N(B_i) < N(D_0) \), then \( W \cong Z \) as otherwise \( V \sim W \) in \( N(D_0) \), a contradiction. As \( C_{D_1}(W) = A_1W \), it follows that \( N(D_1) \cap N(W) \leq N(A_1W) \). Finally, \( N(A_1W) \leq N(W) \) as \( Z(A_1W) = W \). Thus we have proved the following.

(c) \[ N(B_i) \leq N(D_0) \leq N(D_i) \cap N(W) \leq N(A_1W) \leq N(W) \]}

Let \( X = N(D_i) \cap N(W) \) and \( a = |X : N(D_0)| \). We shall determine the value of \( a \) and prove that \( X = N(D_i) \). The statement (1) will, then, follow from (c). First, we shall obtain two expressions for \( |X : N(Q)| \). It follows from the structure of \( N_{c}(B_i) \), Lemma (3J), and Lemma (6C) that \( |N(B_i) : N(Q)| = 3 \). Hence

(d) \[ |X : N(Q)| = 3|N(D_0) : N(B_i)|a \]}

Now \( Q_1 = QD_1 = QD_0 = P*D_0 \) by Lemma (6C), so \( Z = Z(Q_1) \) and \( \mathcal{L}^{*}(Q_1/Z) = \{A_1D_0/Z, A_2D_0/Z\} \). Thus \( N(Q_1) \) normalizes \( A_1D_0 = D_1 \) and, in particular, \( F \leq N(D_1) \). Also, \( F \leq N(W) \) as \( Q_2 = B_1E_2 \) normalizes \( W \). Therefore, \( F \leq X \). More precisely, we have that \( F = Q_2 \cap X \) as \( Q_2 \cap N(D_1) \) normalizes \( Q_1 = D_1B_2 \). The statement (2) will follow from this once we prove \( X = N(D_i) \). By Lemma (3J) and the definition of \( D_i, i \in \{1, 2\} \), \( N(Q) \leq N(B_i) \leq N(D_i) \). Hence \( N(Q) \leq N(Q_1) \) and then \( N(Q) \leq N(F') \). Furthermore,

(e) \[ N(D_0) \cap F = Q_i \]}

as \( N(D_0) \cap F \) normalizes \( Q = A_iB_2 \) by (b). In particular, \( N(Q) \cap F = Q_i \). Thus setting \( b = |X : N(Q)F| \), we have another expression:

(f) \[ |X : N(Q)| = 4b \]}

Now let bars denote images in \( X/W \). Then, as \( \langle \bar{t} \rangle = \bar{D}_0 \), \( C(\bar{t}) = N(D_0) \) and
Also, as \( \bar{D}_i = \langle \bar{t} \rangle \times \bar{A}_i \) and \( \bar{A}_i \triangleleft \bar{X} \),
\[
|\bar{t}^{\bar{X}}| = 1 + |\bar{t}^{\bar{X}} \cap \bar{t} \bar{A}_i^*| .
\]

To determine the second term, consider the action of \( C(\bar{t}) = \overline{N(D_0)} \) on \( \bar{A}_i^* = (A_iW/W)^* \). By (b), \( A_iW/W \cong A_i/Z \) as \( N(D_0) \)-modules. We know that under the action of \( N_i(A_i) \), which is contained in \( N(D_0) \), \((A_i/Z)^* \) decomposes into two orbits of lengths 9 and 6, one corresponding to the involutions of \( A_i - Z \) and the other corresponding to the elements of order 4 of \( A_i \) (see Lemma (2C)). Therefore, under the action of \( C(\bar{t}) \), \( \bar{A}_i^* \) decomposes into two orbits of lengths 9 and 6. Thus
\[
|\bar{t}^{\bar{X}} \cap \bar{t} \bar{A}_i^*| = 0, 6, 9 \text{ or } 15 ,
\]
and hence
\[(g) \quad a = 1, 7, 10 \text{ or } 16 .
\]

Now recall that \( t^\alpha \cap A_i = \emptyset \). This yields that \( t^{N(D_0)} \leq I(D_i - A_i) \), so
\[
|t^{N(D_0)}| \leq 52
\]
as \( D_i \cong D_6 * D_8 * D_8 \) and \( A_i \cong D_6 * D_8 \). On the other hand,
\[
|t^{N(D_0)}| = |N(D_i) : X| \cdot |X : N_c(B_i)|
\]
as \( N(D_i) \cap C = N_c(B_i) \), so
\[
|t^{N(D_0)}| = \begin{cases} 
2 |N(D_i) : X| a & \text{if } N(B_i) = N(D_0) , \\
4 |N(D_i) : X| a & \text{if } N(B_i) < N(D_0) .
\end{cases}
\]

Therefore,
\[(h) \quad |N(D_i) : X| a \leq \begin{cases} 
26 & \text{if } N(B_i) = N(D_0) , \\
13 & \text{if } N(B_i) < N(D_0) .
\end{cases}
\]

Now assume that \( N(B_i) = N(D_0) \). Then \( 3a = 4b \) by (d) and (f). Thus \( a = 16 \) by (g), and then \( N(D_i) = X \) by (h). Assume next that \( N(B_i) < N(D_0) \). Then \( 3a = 2b \) by (a), (d), and (f). Also, \( a \leq 13 \) by (h). Therefore, \( a = 10 \) by (g) and then \( N(D_i) = X \) by (h). Thus \( a = 10 \) or 16 and \( N(D_i) = X \) in either case. Statements (1) and (2) follow from this as remarked before.

Now \( \langle \bar{t}^{\bar{X}} \rangle = \bar{D}_i \) in either case and so \( \bar{X} = \bar{X}/C(\bar{D}_i) \) is a permutation group on \( Q = \bar{X}^\alpha \). Furthermore, \( \bar{X}^\alpha \) is primitive in either case. We shall determine the structure of \( \bar{X}^\alpha \). By Lemma (6C), \( D_i \leq \)
Also, $N(B_1) = D_i N_c(B_1)$, and $C_c(B_1/Z) = B_i O(C)$ by Lemma (2G). Hence
\[
C(D_i/W) \cap N(B_1) = D_i (C(D_i/W) \cap N_c(B_1)) = D_i (C(B_1/Z) \cap N_c(B_1)) = D_i (B_i O(C)) = D_i O(C).
\]

Notice that $[D_i, O(C)] = 1$ as $O(C)$ stabilizes the series $1 \leq B_i \leq D_i$.

Assume that $N(B_1) = N(D_o)$. Then $|\Omega| = 16$ and $C_{\tilde{\Omega}}(\tilde{t}) = \overline{N(D_o)} = N(B_1)$, and consequently, $C(\tilde{D}_1) = \overline{D_i O(C)}$ by the above. Thus $|\tilde{X}| = 16$ and $C_{\tilde{\Omega}}(\tilde{t}) = N(B_1)/B_i O(C) \cong \Sigma_3 \times \Sigma_3$ or $\Sigma_3 \times \Sigma_3$ by Lemma (2C) and Lemma (2G). This shows that $\tilde{X}$ is a $[2, 3]$-group that has no nonidentity normal 3-subgroup. Then by Burnside’s theorem [12, Theorem 4.3.3], $O_{3}(\tilde{X}) \neq 1$ and so $\tilde{X}$ has a regular normal subgroup $\tilde{Y}$. As $1 \neq \tilde{K}_1 \leq C_{\tilde{\Omega}}(C) \triangleleft \tilde{X}$ and $\tilde{Y}$ is a self-centralizing minimal normal subgroup of $\tilde{X}$, it follows that $\tilde{Y} \leq C_{\tilde{\Omega}}(O(C))$. This implies that the preimage $Y$ of $\tilde{Y}$ in $X$ is written as $Y = C_{\bar{\Omega}}(O(C)) O(C)$. Hence $Y$ is 2-closed and if $D \in \text{Sy}_1(Y)$, then $D = O_{3}(N(D_o))$, $N(D_1) = N(B_1)/D$, $N(B_1) \cap D = D_1$, and $D/D_1$ is elementary. Furthermore, the irreducible action of $N(B_1)$ on $\tilde{A}_1$ yields that $\tilde{A}_1 = Z(\tilde{D})$ and so commutation by $\tilde{t}$ induces an $N(B_1)$-isomorphism $\tilde{D}/\tilde{D}_1 \to \tilde{A}_1$. Thus (3) holds.

Assume, therefore, that $N(B_1) < N(D_o)$. Recall that $W \cong Z_4$ in this case. The $\tilde{X}^2$ is a 2-transitive group of degree 10, and the point-stabilizer $C_{\tilde{\Omega}}(\tilde{t}) = \overline{N(D_o)}$ has a normal subgroup $O_{3}(\overline{N(B_1)}) = O_{3}((K_1)^H \overline{H}$ which is isomorphic to $Z_3 \times Z_3$ and is regular on $\Omega - \{\tilde{t}\}$ (see Lemma (2C)). A theorem of [18] now shows that $PSL(2, 9) \twoheadrightarrow \tilde{X} \twoheadrightarrow PGL(2, 9)$.

Now $|\tilde{X}| = 10$, $|N(D_o)| = 2$, and $N(B_1)/D_i O(C) \equiv \Sigma_3 \times \Sigma_3$ or $\Sigma_3 \times \Sigma_3$. Furthermore, $C(D_i/W) \cap N(B_1) = D_i O(C)$ as remarked before. Therefore, $|\tilde{X}|_2 \leq 16$ and equality holds only when $C(D_i/W) = D_i O(C)$ and $N(B_1)/D_i O(C) \equiv \Sigma_3 \times \Sigma_3$. We argue that $F/D_i$ is elementary. Indeed, $F/D_i \cong F \cap E_2/D_i \cap E_2$. By Lemmas (6B) and (6D), the mapping which associates with each element of $E_2$ its square induces an $N_c(B_2)$-isomorphism $E_2/A_2 \to A_2$, and it maps $F \cap E_2$ onto $A_1 \cap A_2$ by the definition of $F$. Thus $(F \cap E_2)^2 = A_1 \cap A_2$ and consequently, $F/D_i$ is elementary. This implies that $m(\tilde{X}) \geq 3$ as $F \cap C(D_i/W) = F \cap N(D_o) \cap C(D_i/W) = Q_1 \cap C(D_i/W) = D_1$ by (e). Thus $\tilde{X} = \Sigma_3$ is the only possibility. In particular, $|\tilde{X}|_2 = 16$ and hence $C(D_i/W) = D_i O(C)$ and $N(B_1)/D_i O(C) \equiv \Sigma_3 \times \Sigma_3$. This occurs only if $C \neq LC_c(L)$ (see Lemmas (2C) and (2G)). Furthermore, $N(D_o)/D_i O(C) = C(D_i/W) \cap N(B_1) = D_i O(C)$.
Cy(t) \cong \Sigma_3 \text{ wreath } Z_2 \text{ by the structure of } \Sigma_3. \text{ Thus all parts of the lemma hold.}

**Lemma (6F).** If \( N(B_i) < N(D_0) \), then Case (3) of the main theorem occurs.

**Proof.** We shall apply Lemma (1R) with \( C(W), W, A_1W/W \), and \( t \) in place of \( \hat{G}, \hat{Z}, A \), and \( t \), respectively. Recall from Lemma (6E) that

\[
N(D_i) \leq N(A_1W) \leq N(W) .
\]

\( N(D_i) \cap C(A_1W/W)/C(D_i/W) \) is a normal 2-subgroup of \( N(D_i)/C(D_i/W) \) and so by Lemma (6E),

(a) \[ N(D_i) \cap C(A_1W/W) = D_iO(C) . \]

As a consequence, we have that

(b) \[ D_i \in \text{Syl}_2(C(A_1W/W)) . \]

Moreover,

(c) \[ N(A_1W) = N(D_i)C(A_1W/W) \]

by a Frattini argument, and hence

(d) \[ N(A_1W)/C(A_1W/W) \cong \Sigma_3 \]

by (a) and Lemma (6E). Now \( C \not= LC_G(L) \) by Lemma (6E)(4.5), so there is an element \( f \in N_G(Q) - Q \) such that \( f^2 \in Q \). Then \( f \in N(B_i) \cap N(B_2) \) by Lemma (3J) and so \( f \) normalizes \( Q_2 = D_1D_2 \) and \( Q_2 \langle f \rangle \) has order \( 2^2 \). Also, \( f \in N(D_i) \leq N(W) \) and \( Q_2 = D_1E_2 \leq N(W) \). Thus \( Q_2 \langle f \rangle \leq N(W) \). Furthermore,

\[
N(A_1W) \cap Q_2 \langle f \rangle = (N(A_1W) \cap Q_2) \langle f \rangle = F\langle f \rangle
\]

as \( N(A_1W) \cap Q_2 \) normalizes \( A_1WB_2 = Q_i \). Now \( |F\langle f \rangle| = 2^2 \). Thus, \( F\langle f \rangle \in \text{Syl}_2(N(A_1W)) \) by (b) and (d), and hence

(e) \[ |N(W):N(A_1W)| \text{ is even} . \]

Now \( W \cong Z_i \) by Lemma (6E) and \( t \in C(W) \), so

\[ N(W) = C(W)\langle t \rangle . \]

It is now clear that (d), (b), and (e) imply the conditions (1), (2), and (3) of Lemma (1R), respectively.

Now notice that \( \langle t, W \rangle = D_0 \), and recall from Lemma (6E) that

\[ N(D_0) \leq N(D_i) \text{ and } N(D_0)/D_iO(C) \cong \Sigma_3 \text{ wreath } Z_2 . \]
Thus

\((f)\)

\[A_1W \leq N(D_0) \leq N(A_1W),\]

and using (a), we have

\((g)\)

\[N(D_0)C(A_1W/W)/C(A_1W/W) \cong \Sigma_3 \text{ wreath } Z_2.\]

Noticing that \(\langle t, A_1W \rangle = D_1\), we can now derive conditions (5), (6), and (7) of Lemma (1R) from (f), (g), and (c), respectively. We know that conditions (4) and (8) are satisfied. Thus by Lemma (1R), \(C(W)\) has a quasisimple characteristic subgroup \(K\) containing \(W\) such that

\((h)\)

\[C(K) = WO(C(W))\]

and either \(K/O(K) \cong SU(4,3)\) or \(K/Z(K)\) has an \(S_2\)-subgroup of type \(PSL(6, q)\), \(q \equiv 3 \mod 4\). Now \(N(W) \leq C(Z)\), \(K \triangleleft N(W)\), and \(W/Z \in \text{Syl}_2(C(K/Z))\) by (h). Thus \(K/Z\) is a standard subgroup of \(C(Z)/Z\). The fours group \(D_0/Z\) acts on \(X = O(C(Z))\). Let \(x \in N(D_0) - N(B_1)\). Then \(V^x \neq V\) as \(N(V) = N(B_1)\) and so \(X = \langle N_x(V), N_x(V^x), N_x(W) \rangle \leq O(N(W))\). Hence \([K, X] = 1\). We have proved that Case (3) of the main theorem occurs.

In view of Lemma (6F), we assume from now on that \(G\) satisfies the following.

**Hypothesis (6.2).** \(N(B_1) = N(D_0)\).

Furthermore, we make the following definition.

**Definition (6.4).** Let \(D = O_1(N(D_0))\) and \(R_1 = Q_1D\).

Then by Lemma (6E)(3), \(N(D_0) = N(B_1)D_1\), \(N(B_1) \cap D = D_1\), \(D/D_1\) is elementary, and \(D/D_1 \cong A_4/Z\) as \(N(B_1)\)-modules.

**Lemma (6G).** The following conditions hold.

1. \(R_1 \cap Q_2 = F\).
2. \(R_1 \leq N(Q_2)\).
3. \(E_2\) is elementary abelian.
4. \(N(D_0) = N(B_1) \leq N(E_2)\).
5. \(N(Q_2) \leq N(E_2)\).

**Proof.** By Lemma (6E)(2), \(N(D_1) \cap Q_2 = F\). Hence (1) will follow once we show \(F \leq R_1\). To see this, notice first that \(|N(D_1)/D_1| \leq 4\) by Lemmas (6C)(3) and (6E)(3). Next, \(F \leq N(R_1)\) as \(F \leq N(Q_1) \cap N(D_1)\). Hence \(Q_1 \leq R_1 \cap F \leq F\). As \(H\) acts irreducibly on \(F/Q_1\) by
Lemma (6B) and $H \leq N(R_i \cap F)$, we have that $F = R_i \cap F$, proving (1).

Now Lemma (6E)(3) in particular implies that $|N_{R_i}(Q_i)/D_i| = 8$, so $F = N_{R_i}(Q_i)$ and consequently, $F \lhd R_i$ by Lemma (1C).

We show that $F \cap E_2$ is the only $A_{128}$-subgroup of $F$. Suppose $X$ is an $A_{128}$-subgroup of $F$. If $X \lhd F \cap E_2$, then $F \cap E_2$ is an abelian maximal subgroup of $F \cap D_2$ and as $Z(F \cap D_2) \leq B_2$, it follows that $X = F \cap E_2$. Assume, therefore, that $X \not\lhd F \cap D_2$. Then $F \neq X(F \cap E_2)$. For otherwise, $Y = X \cap F \cap E_2$ has order 16 and $Y \leq Z(F)$. However, $Z(F) \leq Z(C_F(t)) = Z(Q) = V$, a contradiction. Thus $|Y| \geq 32$ and so if $x \in X - D_2$, then $|C_{E_2}(x)| \geq 32$. However, on the other hand, Lemma (6B) shows that $|C_{E_2/A_2}(x)| = 4 = |C_{A_2}(x)|$ if $x \in Q_2 - D_2$. This contradiction shows that $F \cap E_2$ is the only $A_{128}$-subgroup of $F$.

A similar argument shows that $E_2$ is the only $A_{56}$-subgroup of $Q_2$. Therefore, $N(F) \lhd N(F \cap E_2)$ and $N(Q_2) \leq N(E_2)$.

Now $R_i \leq N(F) \leq N(F \cap E_2)$. However, $R_i \not\leq N(A_2)$ as $N_{R_i}(A_2) \leq N_{R_i}(A_2D_i) = N_{R_i}(Q_i) = F$. These and Lemma (6D) imply that $F \cap E_2$ is elementary abelian, and hence (3) follows. The statement (4) now follows from Lemma (1C). By the same lemma, $C(F/F \cap E_2) \leq N(F \cap D_2) \leq N(B_2)$. Also, $Q_2 \leq C(F/F \cap E_2)$ as $Q_2/F \cap E_2 = F/F \cap E_2 \times E_2/F \cap E_2$ and $F/F \cap E_2 \cong Q/A_2$. Therefore, $Q_2$ is the only $S_2$-subgroup of $C(F/F \cap E_2)$ by the structure of $N(B_2)/B_2$ discussed in Lemma (6B). Thus $Q_2 \lhd N(F)$ as $C(F/F \cap E_2) \lhd N(F)$. In particular, $R_i \leq N(Q_2)$. The proof is complete.

**Definition (6.5).** Let $T = R_iQ_2$, $S = C_T(W)$, and $E_i = C_D(W)$.

Because of Lemma (6G)(2), $T$ is a subgroup.

**Lemma (6H).** The following conditions hold.

1. $T \leq N(E_2)$.
2. $T = S\langle t \rangle$.
3. $D = E_2\langle t \rangle$.
4. $W^{*2} = (E_2 \cap E_2)^{*2} = ((E_1 \cap E_2) \cap (E_1 \cap E_2)^{*1})^{*2}$ is a complement for $E_1$ in $S$.
5. $(E_1 \cap E_2) \cap (E_1 \cap E_2)^{*1}$ is a complement for $E_2$ in $S$.
6. $E_2/W$ is elementary abelian.
7. $N(Q) \leq N(S)$.

**Proof.** The assertion (1) follows from Lemma (6G)(5). By Lemma (6E)(1), $R_i \leq N(D_i) \leq N(W)$. Also, $Q_2 = D_iE_2$ normalizes $W$. Therefore, $T \leq N(W)$ and hence (2) and (3) follow.

Let $X = E_2 \cap E_2^{*1}$. Then as $B_2 \cap B_2^{*1} = V$ and $N(V) = N(B_2)$ by
Lemma (6C)(2), we have that $X \leq N_Q(B_j) = Q_i$. Thus $X \leq Q_i \cap Q_i^1 = D_i$. By Lemma (6C), $D_1 \cap D_2 = (B_1 \cap D_2)D_0 = (B_1 \cap B_2)D_0$ and then $X \leq (B_1 \cap B_2)D_0 \cap ((B_1 \cap B_2)D_0)^t = D_0$. Thus $X \leq D_0 \cap E_2 = W$. As $W = W^t \leq X$ by Lemma (6C)(8), we conclude that $W = E_2 \cap E_2^t = (E_1 \cap E_2) \cap (E_2 \cap E_2)^t$. Furthermore, as $|E_i| = 2^{10}$ by (3), we have $E_1 = (E_1 \cap E_2)(E_1 \cap E_2)^t$ by order consideration. As $E_1 \cap E_2 \not< E_1$ by (1), (6) holds by Lemma (6G)(3).

Now by Lemma (6B), commutation by $t$ induces an $N_Q(B_j)$-iso-
morphism $E_1/A_2 \rightarrow A_2$, which maps $WA_2/A_2$ onto $Z$ and $F \cap E_2/A_2$ onto $A_1 \cap A_2$. Hence $(F \cap E_2) \cap W^{s_2} = A_2$ as $(A_1 \cap A_2) \cap Z = 1$. Notice that $E_1 \cap E_2 \leq F \cap E_2$ by Lemma (6G)(1) and that $E_1 \cap A_2 = A_1 \cap A_2$ by Lemmas (6C)(3) and (6E)(3). Therefore, $E_1 \cap W^{s_2} \leq (A_1 \cap A_2) \cap Z = 1$. As $|S: E_1| = 4$ by (2) and (3), we conclude that $W^{s_2}$ is a complement for $E_1$ in $S$, proving (4). In particular, $S = E_1E_2$.

As a consequence of (4), we have that $(E_1 \cap E_2)^{s_2} = ((E_1 \cap E_2) \cap (E_1 \cap E_2)^{s_2}) \times W^{s_2}$ and so

$$(E_1 \cap E_2)^{s_1} = ((E_1 \cap E_2) \cap (E_1 \cap E_2)^{s_2})^{s_1} \times W.$$  

Hence

$$S = E_1E_2$$

$$= (E_1 \cap E_2)(E_1 \cap E_2)^t E_2$$

$$= (E_1 \cap E_2)^t E_2$$

$$= ((E_1 \cap E_2) \cap (E_1 \cap E_2)^s)^t W E_2$$

$$= ((E_1 \cap E_2) \cap (E_1 \cap E_2)^{s_2})^{s_1} E_2.$$  

Furthermore,

$$((E_1 \cap E_2) \cap (E_1 \cap E_2)^{s_2})^{s_1} \cap E_2$$

$$\leq (E_1 \cap E_2) \cap (E_1 \cap E_2)^{s_2}$$

$$= W.$$  

Therefore (5) holds.

Finally, $N(Q) \leq N(B_i) \cap N(B_j)$ by Lemma (3J). Hence subgroups used to define $S$ are all normalized by $N(Q)$ (see Definitions (6.1)-(6.5)). Thus $N(Q) \leq N(S)$.

DEFINITION (6.6). Let $K = K_2E_2$ and $L_2 = \langle K^{N(E_2)} \rangle$.

LEMMA (6I). The following conditions hold.

1. $L_2/E_2 \cong SL(2, 4) \times SL(2, 4)$ and $t$ interchanges two compo-

teus of $L_2/E_2$.

2. $S \in Syl_2(L_2)$.  

(3) \( O(N(E_2) \mod E_2) = C(L_2/E_2) \).
(4) \( C(E_2) \leq O(N(E_2) \mod E_2) \).
(5) \( Z(S) = W \).

Proof. Let bars denote images in \( N(E_2)/E_2 \). Then by Lemma (6G)(4) and Lemma (6B), \( C(\bar{t}) = \bar{N}(B_2) = \bar{N}_c(B_2) \). Therefore, \( \bar{K} \leq C(\bar{t}) \) and \( \langle \bar{t} \rangle \in \text{Syl}_1(C(K) \cap C(\bar{t})) \). Furthermore, \( \bar{S} \) is an \( E_2 \)-subgroup of \( \bar{N}(E_2) \) and is invariant under \( N(Q_2) \cap N(B_2) = N(Q)E_2 \) by Lemma (6H). Thus (2) and (3) hold and either (1) holds or \( L_2/E_2 \cong SL(2, 16) \) by Lemma (6I). As a consequence, we have that \( C(E_2) \cap L_2 = E_2 \) since \( K \cong C(A_4) \). Thus (4) follows from (3). Hence \( Z(S) \leq N_{E_2}(P) \leq Q_4 \cap E_2 = A_2W \), and then \( Z(S) \leq Z(PW) = W \). As \( W \) centralizes \( S = E_2E_2 \) by Lemma (6I)(4), (5), (5) holds.

Now \( \bar{P} \in \text{Syl}_2(\bar{K}) \), \( \bar{P} \leq \bar{S} \in \text{Syl}_2(\bar{L}_2) \), and \( C_{\bar{E}_2}(\bar{S}) \cong Z(S) = W \). Furthermore, \( A_2 \) is a \( \bar{K} \)-invariant subgroup of \( \bar{E}_2 \) and \( C_{\bar{A}_2}(\bar{P}) = Z < W \). Thus \( \bar{L}_2 \cong SL(2, 16) \) by Lemma (1K). The proof is complete.

In view of Lemma (6I), we make the following definition.

**Definition (6.7).** Let \( L_2/E_2 = M_2/E_2 \times M_2/E_2 \) with \( M_2/E_2 \cong SL(2, 4) \), and set \( S_2 = S \cap M_2 \).

**Lemma (6J).** Assume that \( C_{\bar{E}_2}(M_2) = 1 \). Then \( \langle L^\circ \rangle \cong PSL(4, 4) \).

Proof. Let \( N = N(E_2) \) and let bars denote images in \( N/C(E_2) \). Our aim is to use Lemma (1L) to \( E_2 \) and \( \bar{N} \). By Lemma (6I)(3), \( E_2 \) is elementary abelian of order 256. By Lemma (6I)(4), \( C(E_2) = E_2O(N) \) and so Definition (6.7) and Lemma (6I)(3) imply that \( \bar{N} \) satisfies the conditions (1) and (2) of Hypothesis (1.1). Also, \( C_{\bar{E}_2}(\bar{S}_2\bar{S}_2) = C_{\bar{E}_2}(\bar{S}) = Z(S) = W \) by Lemma (6I)(5), so \( \bar{N} \) satisfies the condition (3) of Hypothesis (1.1) as well. Our assumption implies that \( C_{\bar{E}_2}(\bar{M}_2) = 1 \), so that \( \bar{N} \) satisfies the condition (4) of Lemma (1L). Now \( \bar{K} = C_{\bar{E}_2}(\bar{t}) = \{txt\bar{x} \in \bar{L}_2 \} \) and \( \bar{H} \) is a complement for \( \bar{P} = C_{\bar{S}}(\bar{t}) \) in \( N_{\bar{L}}(\bar{P}) \) as \( \bar{K} = \bar{K}_2 \). Hence \( \bar{H} = \{htht\bar{h} \in \bar{H}^* \} \) for some complement \( \bar{H}^* \) for \( \bar{S}_2 \) in \( N_{\bar{B}_2}(\bar{S}) \). Since \( [W, H] = 1 \) by Lemma (6C)(8), \( \bar{N} \) satisfies the condition (5) Lemma (1L) as well. Thus we can apply Lemma (1L) to determine the structure of \( \bar{N} \) and the action of \( \bar{N} \) on \( E_2 \). As for the structure of \( \bar{N} \), we have

\[
\langle L^*, t^* \rangle \leftarrow \bar{N} \leftarrow \langle L^*, t^*, f^*, D^* \rangle.
\]

In this embedding, \( \bar{L}_2, \bar{M}_2, \bar{S} \), and \( \bar{t} \) correspond to \( L^*, M^*, R^*R^{*+}, \) and \( t^* \), respectively.

Let \( S_0 = ((E_1 \cap E_2) \cap (E_1 \cap E_2)^*)^2 \). Then by Lemma (6H)(5) \( S_0 \langle t \rangle = S_0 \langle t \rangle \) is a complement for \( E_2 \) in \( T \). Since \( S \in \text{Syl}_2(L_2) \) by Lemma
(6I)(2), \( T \in \text{Syl}_2(\langle L_2, t \rangle) \) and hence \( E_2 \) has a complement in \( \langle L_2, t \rangle \) by Gaschütz's theorem [19, Hauptsatz 17.4]. Therefore, the structure of \( \langle L_2, t \rangle \) is uniquely determined by Lemma (1L). There is an isomorphism

\[
\sigma: \langle L_2, t \rangle \rightarrow \langle L^*E^*, t^* \rangle .
\]

Here \( L_2^o = L^*E^*, \ (tE_2)^o = t^*E^* \), and \( \sigma \) maps \( S \) onto the group \( S^* \) of matrices

\[
\begin{pmatrix}
1 & a & 1 \\
b & c & 1 \\
d & e & f
\end{pmatrix}
\]

with entries in \( F_4 \). We know that each \( S^* \) and \( S^*/Z(S^*) \) has precisely one \( E_{256} \)-subgroup, \( E_i^* \) and \( E_i^*/Z(S^*) \). Since \( E_2 \) and \( E_i/W \) are elementary and \( Z(S) = W \) (see Lemmas (6G)-(6I)), it follows that \( E_i \) and \( E_i^* \) are characteristic subgroups of \( S \) and that \( E_i^* = E_i^* \) for \( i \in \{1, 2\} \).

Now consider the case where \( N \) does not contain an element that corresponds to \( f^* \). Then \( T = \langle S, t \rangle \in \text{Syl}_2(N) \). Since \( \langle S, t \rangle^o = \langle S^*, t^* \rangle \), we see that \( E_2 \) is the only \( E_{256} \)-subgroup of \( T \). Hence \( N(T) \leq N \), which implies that \( T \in \text{Syl}_2(G) \). Next, since \( S^o = S^* \) and \( I(S^o) = I(E_i^*) \cup I(E_i^*) \), we have \( I(S) = I(E_i) \cup I(E_i) \). Hence if \( x \in t^o \cap S \), then \( x \in E_i \) for some \( i \in \{1, 2\} \). Since \( |C_{E_i}(x)| \geq 256 \) by Lemma (1D) and \( |C_{E_i}(x)| \leq 256 \), we have \( C_{E_i}(x) \in \text{Syl}_2(C(x)) \). But class of \( C_{E_i}(x) \) \( \leq 2 \) and class of \( P = 3 \), a contradiction. Therefore, \( t^o \cap S = \emptyset \). Then \( t \in G' \) by Lemma (1E), and since \( L_i^o = L^*E^* \) is perfect, \( S \in \text{Syl}_2(G') \). We now appeal to [22.4] to conclude that \( O^o(G'/O(G')) \cong O^o(X) \) for some parabolic subgroup \( X \) of \( PSL(4, 4) \). By Lemma (1H), \( L(G) = \langle L^o \rangle \) and \( [\langle L^o \rangle, O(G)] = 1 \). Therefore, \( \langle L^o \rangle \cong PSL(4, 4) \).

Assume, therefore, that \( \tilde{N} \) contains an element \( \tilde{f} \) that corresponds to \( f^* \). Let \( f' \) be a preimage of \( \tilde{f} \) in \( N \). Since \( \tilde{f} \in N(T) \), we may choose \( f' \in N_x(T) \). Then as \( \tilde{f} \in C(\tilde{t}) \) and \( \langle \tilde{t} \rangle = \tilde{D}_2 \), \( f' \in N(D_2) \) by Lemma (6G)(4). Also, since \( \tilde{f} \) normalizes \( Q_2 = C(\tilde{t}) \), \( f' \in N(Q_2) \). Recall that \( N(B_2) = N_C(B_2)E_2 \) and \( N_C(B_2) \cap E_2 = A_2 \). Hence we may choose \( f' \in N_C(B_2) \). Then \( f' \) normalizes \( Q_2 \cap C = Q \), but \( f' \in Q \). Thus \( f' \in C - L C_L(L) \). Also, we may choose \( f' \) so that \( f'^o \in E_2 \). Then \( f'^o \in D \cap E_2 = A_2 \leq L \). Therefore, \( L \langle f' \rangle \cong \text{Aut}(L) \). We can now choose \( f \in I(Lf') \) so that the action of \( f \) on \( L \) is induced by the involutive automorphism of \( F_i \). Then \( f \in C(s_i) \cap C(s_2) \) and \( f \in N(S) \) by Lemma (6H)(7), hence \( f \in N(S) \). Thus, \( \langle S, t, f \rangle \) is a complement for \( E_2 \) in \( \langle S, t \rangle \). As \( \langle S, t, f \rangle \in \text{Syl}_2(\langle L_2, t, f \rangle) \), \( E_2 \) has a complement in \( \langle L_2, t, f \rangle \) by Gaschütz's theorem, and the structure
of \( \langle L_2, t, f \rangle \) is uniquely determined by Lemma (1L). Notice that \( f \in P f^h \) for some \( h \in H \), hence \( f \in N_H(M_2) \). Hence by Lemma (1L), there is an isomorphism

\[ \sigma: \langle L_2, t, f \rangle \rightarrow \langle L^*, E^*, t^*, f^* \rangle \]

such that \( L^*_2 = L^*E^* \), \( S^* = S^* \), \( (tE_2)^\sigma = t^*E^* \), and \( (fE_2)^\sigma = f^*E^* \). As \( I(t^*E^*) = t^*E^* \), we may assume that \( t^* = t^* \). Replacing \( f \) by \( f^{-1} \), we may also assume that \( f^* = f^* \). Thus \( f \) is an involution of \( C \) normalizing \( P = C_S(t) \).

Now let \( X = C(tf) \), \( Y = C_L(f) \), and \( M = C_{L_2}(tf) \). As \( C(f) \cap N_L(A_2) = C(f) \cap C(t) \cap L_2 \cong C(f^*) \cap C(t^*) \cap L^*E^* \), \( C(f) \cap N_L(A_2) \) is an extension of \( E_8 \) by \( SL(2, 2) \). Thus \( f \) acts on \( L \) as a field automorphism by Lemma (2K)(4), hence \( Y \cong Sp(4, 2) \). Also, \( M \cong C_{L^*E^*}(t^*f^*) \) is isomorphic to the commutator subgroup of a maximal parabolic subgroup of \( Sp(4, 4) \), and as \( x^t = x^t \) for \( x \in M \), the action of \( t \) on \( M \) is induced by a field automorphism of \( Sp(4, 4) \). As \( C \) is a semi-direct product of \( \langle L, t, f \rangle \) and \( O(C) \), we have

\[ C_X(t) = C(f) \cap C(t) = \langle Y, t, f, C_{o(C)}(f) \rangle. \]

We argue that \( t \not\sim f \). Indeed, \( C_{L_2}(f') \langle f' \rangle \cong C_{L^*E^*}(f^*) \langle f^* \rangle \) is an extension of an elementary abelian group of order \( 32 \) by \( SL(2, 2) \times SL(2, 2) \), while \( C \) does not contain such a group by Lemma (3J). Let bars denote images in \( X/\langle tf \rangle \). Then \( \bar{t} \in I(\bar{X}) \) and since \( t \not\sim f \),

\[ C_{\bar{X}}(\bar{t}) = N_{\bar{X}}(\langle t, tf \rangle) = C_X(t). \]

Therefore,

\[ C_{\bar{X}}(\bar{t}) = \bar{Y} \times \langle \bar{t} \rangle \times O(C_{\bar{X}}(\bar{t})) \]

with \( \bar{Y} \cong Sp(4, 2) \). We can now apply Lemma (1P) to conclude that \( E(\bar{X}) \cong Sp(4, 4) \) and \( C_{\bar{X}}(E(\bar{X})) = O(\bar{X}) \). Consequently, \( |X| \leq 2^{11} \). As the Schur multiplier of \( Sp(4, 4) \) is trivial, it follows that \( E(X) \cong Sp(4, 4) \) and \( C_X(E(X)) = \langle tf, O(X) \rangle \). Thus \( E(X) \) is a standard subgroup of \( G \) and \( C(E(X)) \) has a cyclic \( S_2 \)-subgroup. Also, as \( |G: X| \) is even, \( tf \in Z^*(G) \) and so \( E(X)O(G) \not\subseteq G \) by Lemma (1H). Appealing to [11], we conclude that \( \langle E(X)^\sigma \rangle \cong PSU(4, 4), \ PSU(5, 4), \ PSL(4, 4), \ PSL(5, 4), \ PSp(4, 16) \) or \( Sp(4, 4) \times Sp(4, 4) \). Since \( C(t) \) has a component of type \( PSU(4, 2) \), we must have that \( \langle E(X)^\sigma \rangle \cong PSL(4, 4) \) (see [3, §19]). Thus by Lemma (1H), \( \langle L^\sigma \rangle \cong PSL(4, 4) \). The proof is complete.

In view of Lemma (6J), we now study the following situation.

**Hypothesis (6.3).** \( C_{L_2}(M_2) \neq 1 \).
LEMMA (6K). $L_2 = N_2 \times N_2^*$, where $N_2^*$ is isomorphic to the semidirect product of the natural $A_5$-module by $A_5$.

Proof. By Lemma (6H)(5) and Gaschütz’s theorem, $E_2$ has a complement $N$ in $L_2(t)$. As in the proof of Lemma (6J), $E_2$ and $N$ satisfy Hypothesis (1.1) and $C_{E_2}(S_5S_2^*) = W$. Also, $C_{E_2}(M_2) \neq 1$ by our hypothesis. As $W \cap W^{*_2} = 1$ by Lemma (6H)(4), the assertion follows from Lemma (1M).

DEFINITION (6.8). Let $R = S \cap N_2$, $F_2 = O_2(N_2)$, and $U = Z(R)$. Let $F_1/U$ be an element of $G^*(R/U)$ different from $F_2/U$.

REMARK. $N_2 \cong K_2A_2$ and $R \in \text{Syl}_2(N_2)$, hence $R \cong P$. Thus $G^*(R/U) = \{F_1/U, F_2/U\}$ and $F_1$ is extra-special of order $32$. Also, $W = U \times U^t$ by Lemma (6I).

LEMMA (6L). For $i \in \{1, 2\}$, the following holds.

1. $E_i = F_1 \times F_1^i$.
2. $s_i \in N(F_i)$.

Proof. For $i = 2$, the assertion is obvious, so consider the case $i = 1$. As $S/W = R/W \times R/W/W$ and $RW/W \cong R/U$, we have $G^*(S/W) = \{F_1F_1^i/W, F_2F_2^i/W, F_1F_1^i/W, F_1F_2^i/W\}$. Therefore, $F_1F_1^i/W$ is the only member of $G^*(S/W)$ of order greater than or equal to $2^a$. As $E_1/W$ is elementary of order $2^a$ by Lemma (6H), (1) holds.

Now $s_1 \in C(W) \leq C(U)$ by Lemma (6C)(8), and hence $s_1$ acts on $Z(E_1/U) = U^tF_1/U$. Now $K_2A_2 = C_{L_2}(t) = \{xx^t | x \in N_2\}$ and $H$ is a complement for $P = C_2(t)$ in $N_{K_2A_2}(P)$, so $H = \{xx^t | x \in H^*\}$ for some complement $H^*$ for $R$ in $N_{K_2}(R)$. As $H^*$ acts fixed-point-freely on $F_1/U$ by the structure of $N_2$, so also does $H$. Hence it follows that $[U^tF_1/U, H] = F_1/U$ since $H$ centralizes $U^t$ by Lemma (6C)(8). Therefore, $s_1 \in N(F_1)$.

DEFINITION (6.9). Let $L_1 = \langle S, S^* \rangle$, $N_1 = \langle R, R^* \rangle$, $G_0 = \langle L_1, L_2 \rangle$, and $G_1 = \langle N_1, N_2 \rangle$. Notice that $N_2 = \langle R, R^* \rangle$.

LEMMA (6M). $G_0$ is a central product of $G_1$ and $G_1^i$.

Proof. It is clear that $G_0 = \langle G_1, G_1^i \rangle$, so we shall prove $[G_1, G_1^i] = 1$. The structure of $N(E_2)/E_2$ shows $S \cap S^{*_2} = E_2$ (see Lemma (6I)). In particular, $E_1 \cap E_2^{*_2} \leq E_2$ so $(E_1 \cap E_2^{*_2})^t_1$ is a complement for $E_2$
in $S$ by Lemma (6H)(5). Thus

$$S = E_2(E_1 \cap E_t^{s_{11}}).$$

Now, $E_1^{s_1} = F_1^{s_1}F_1^{s_2}$ and $E_1^{s_2} = F_1^{s_2}F_1^{s_3}$ by Lemma (6L). As $F_1^{s_1}, F_1^{s_2} \leq N_2^{s_1}$ and $L_1^{s_1} = N_2^{s_1} \times N_2^{s_2}$, we have that

$$(E_1 \cap E_1^{s_2})^{s_1} = (F_1 \cap F_1^{s_2})^{s_1} \times (F_1 \cap F_1^{s_2})^{s_2}.$$  

Also, $E_2 = F_2 \times F_1^{s}$. As $F_2^{s} \leq F_2^{s} \times F_2^{s}$, the above factorization of $S$ yields that

$$R = F_2(F_1 \cap F_1^{s_2})^{s_1}.$$  

This shows that $R = F_2 F_1$ and $R^{s_2} = F_2(F_1 \cap F_1^{s_2})^{s_1}$ as $s_1 \in N(F_1)$ by Lemma (6L). Hence if $X = \langle F_1, (F_1 \cap F_1^{s_2})^{s_1} \rangle$, then $N_2 = F_2 X$ and so $F_2 \cap F_1 \leq F_2 \cap X \trianglelefteq N_2$. As $N_2$ acts irreducibly on $F_2, F_2 \cap X = F_2$. Thus

$$N_2 = \langle F_1, (F_1 \cap F_1^{s_2})^{s_1} \rangle.$$  

Now

$$[F_1^{s}, F_1^{s_2}] \leq [N_2, N_1^{s_2}] = 1.$$  

Since $s_1 \in N(F_1)$,

$$[F_1, (F_1 \cap F_1^{s_2})^{s_1}] \leq [F_1^{s_1}, F_1^{s_2}] \leq [N_2, N_1^{s_2}] = 1.$$  

Conjugating this by $s_1t$, we have

$$[(F_1 \cap F_1^{s_2})^{s_1}, F_1^{s_1}] = 1.$$  

Also, since $(s_2s_1)^2 = (s_2s_1)$,

$$[(F_1 \cap F_1^{s_2})^{s_1}, (F_1 \cap F_1^{s_2})^{s_1}] \leq [F_1^{s_1}, F_1^{s_2}] = [F_1^{s_1}, F_1^{s_2}] = 1.$$  

Since $N_2^{s_1} = \langle F_1, (F_1 \cap F_1^{s_2})^{s_1} \rangle$ and $N_1^t = \langle F_1^{t}, (F_1 \cap F_1^{s_2})^{s_1} \rangle$, we conclude that

(1)  

$$[N_2^{s_1}, N_1^t] = 1.$$  

In particular, $[R^{s_1}, R^t] = 1$, and since $[R, R^t] = 1$ and $N_1 = \langle R, R^{s_1} \rangle$, it follows that

(2)  

$$[N_2, N_1^{s_2}] = 1.$$  

Also, $[R^{s_1}, N_2] \leq [N_2, N_1^{s_1}] = 1$. As $[R^{t}, N_2] \leq [N_2, N_2] = 1$, it follows that
The equations (1), (2), and (3) show $[G_i, G_i^t] = 1$, as desired.

**Lemma (6N).** The following conditions hold.

1. $G_i \cong PSU(4, 2)$.
2. $G_0 = G_i \times G_i^t$.
3. $L = C_{G_0}(t) = \{xx^t | x \in G_i\}$.
4. $C(G_0) = O(N(G_0))$.
5. $R \in \text{Syl}_2(G_i)$.

**Proof.** By Lemma (6K), $N_2$ is perfect. Therefore, $R \leq N_2 \leq G_i^t$ and then $R_2 \leq (G_i^t)^{r_1} = G_i^t$ as $s_1 \in G_0 \leq N(G_1)$. Thus $N_1 = \langle R, R_2 \rangle \leq G_i^t$ and $G_i = G_i^t$. Let $L_0 = \{xx^t | x \in G_i\}$ and $Z_0 = G_i \cap G_i^t$. Then, as $G_0 = G_i \times G_i^t$ by Lemma (6M), it follows that $C_{G_0}(t) = L_0C_{Z_0}(t)$. By the same reason, the mapping $x \to xx^t$ is a homomorphism from $G_i$ onto $L_0$ with the kernel contained in $Z(G_i)$. In particular, $L_0$ is perfect by the first paragraph and so $C_{G_0}(t) = C_{G_0}(t)^\omega = L_0$. On the other hand, $L = \langle P, s_1, s_2 \rangle \leq C_{G_0}(t)$ and so $C_{G_0}(t)^\omega = L$ as $C_\omega = L$. Thus $L = L_0$, and consequently $G_i/Z(G_i) \cong PSU(4, 2)$.

Now $C(G_0) < C(L) \cap N(G_0)$ as $L \leq G_0$. Since $\langle t \rangle \in \text{Syl}_2(C(L) \cap N(G_0))$ and $t \not\in C(G_0)$, it follows that $C(G_0)$ has odd order. This proves (4) as $G_0$ is semisimple. Now $Z(G_i)$ has odd order, so as the Schur multiplier of $PSU(4, 2)$ has order 2, we have that $Z(G_i) = 1$. Hence (1), (2), and (3) follow. Finally, (5) is obvious by (1).

**Lemma (60).** If $t \in N(G_0)^\varphi$ for $g \in G$, then $g \in N(G_0)$.

**Proof.** We first show that $N(Q) \leq N(G_0)$. By Lemma (3J), $N(Q) \leq N(B_1)$, hence $N(Q) = D_1N_0(Q) = A_1WN_0(Q)$ (see Lemma (6C) and a remark after Definition (6.3)). $A_1W$ and $N_L(P) \leq L_0 \leq G_0$, and $N_0(Q) = \langle N_L(P), t, O(C) \rangle$ or $\langle N_L(P), t, O(C), f \rangle$, where $f$ is an element of $C$ acting on $L$ as a field automorphism. Thus it is enough to show $t, O(C)$, and $f \in N(G_0)$. Clearly, $t, O(C)$, and $f$ normalize $Q$ and centralize $s_i, s_\varphi$. By Lemma (6H)(7), $N(Q) \leq N(S)$. Hence $t, O(C)$, and $f$ normalize $L_i = \langle S, S^t \rangle$ for $i \in \{1, 2\}$, and hence normalize $G_0 = \langle L_{1, 2} \rangle$. Thus $N(Q) \leq N(G_0)$.

Now assume that $t \in N(G_0)^\varphi$. Then $t$ acts, by conjugation, on the set $\{G_i^t, G_i^{t\varphi}\}$. Suppose that $t$ normalizes $G_i^t$ and $G_i^{t\varphi}$. Then both $G_i^t \cap C(t)$ and $G_i^{t\varphi} \cap C(t)$ have 2-rank at least 3 by Lemmas (2E) and (2K), so $m(G_0^t \cap C(t)) \geq 6$. This is a contradiction because $m(C) = 5$ by Lemma (3J). Therefore, $t$ interchanges $G_i^t$ and $G_i^{t\varphi}$. As a consequence, we have $L = G_0^t \cap C(t) = \{xx^t | x \in G_i^t\}$ since $G_0^t = G_i^t \times G_i^{t\varphi}$.
Hence if $Y \in \text{Syl}_2(G)$, then $\hat{P} = \{yy' \mid y \in Y\}$ is an $S_2$-subgroup of $L$. As $Q$ and $\langle \hat{P}, t \rangle$ are conjugate by an element of $L \leq G_o$, $N(\langle \hat{P}, t \rangle) \leq N(G_o)$ by the first paragraph. Let $z \in Z(Y)^t$. Then as $z^2 = 1$, $z^{-1}tz = ztzt^{-1}t \in \hat{P}t$, so that $z \in N(\langle \hat{P}, t \rangle)$. As $z \in L$, we conclude that $L < N(G_o) \cap G_o^g$. Then [1, Lemma 2.5] shows that $G_o^g = N(G_o)^\infty = G_o$. The proof is complete.

**Definition (6.10).** Let $T \leq S_1 \in \text{Syl}_2(N(G_o))$, $S_0 = N_{S_1}(G_i)$, and $R_o = C_{S_0}(G_i)$. Notice that $S_o = N_{S_1}(G_i^1)$ by Lemma (6N), and that $R \leq R_o$ and $S \leq S_o$.

**Lemma (6P).** $S_1 \in \text{Syl}_2(G)$.

**Proof.** Let $g \in N(S_1)$. Then $t^g \in S_1 \leq N(G_o)$, so that $g \in N(G_o)$ by Lemma (6O). Thus $N(S_1) \leq N(G_o)$, and the assertion follows.

**Lemma (6Q).** $S \in \text{Syl}_2(G^\infty)$.

**Proof.** There are three cases to consider:
1. $R_o \neq R$.
2. $R_o = R$ but $S_o \neq S$.

Let $N = N(G_o)$. Then Lemma (6N)(4) shows that $R_o \cap R_o^t = 1$ and that $C_N(G_i^1)/O(N) \leq \text{Aut}(G_i)$. Hence $R_oS \cap R_oS = S$ and $|R_oS/S| = |R_o/R| \leq 2$ as $S \cap R_o = R_o$. Also, $N_{N}(G_i)/C_N(G_i) \leq \text{Aut}(G_i)$, hence $|S_o/R_oS| \leq 2$. Therefore in Case 1, $|R_oS/S| = |R_o/R| = 2$ and $S_o/S = R_oS/S \times R_oS/S$. Similarly, $|S_o/S| = 2$ in Case 2.

Suppose $t^g \in N_N(G_i)$. Then $t^g \in N$ and so $g \in N$ by Lemma (6O). But then $t^g \in N_N(G_i)$ as $N_N(G_i) \triangleleft N$, a contradiction. Therefore, $t^g \cap S_o = \emptyset$.

In Case 3, $T = S_1 \in \text{Syl}_2(G)$ by Lemma (6P) and $t^g \cap S = \emptyset$ by the above. Therefore, $t \in G'$ by Lemma (1E). Since

$$S \leq G_o \leq G^\infty,$$

it follows that $S \in \text{Syl}_2(G^\infty)$. Therefore, we assume that

$$S < S_o.$$

Then $S < N_{S_0}(T)$. Also, $N_{S_0}(T) = C_{S_0}(t)S$ as $I(T - S) = t^S$ by Lemma (1B). Thus $C_{S_0}(t) > C_S(t) = P$. As $t \not\in C_{S_0}(t)$, $C_{S_0}(t)$ is isomorphic to an $S_r$-subgroup of $\text{Aut}(L)$. Therefore, we can choose an involution $a \in C_{S_0}(t) - S$.

We compute $|C_{S_1}(x)|$ for $x \in I(N_{S_0}(T) - S)$. In Case 1, $S_o = R_o \times$
so that $x = yz$ with $y \in I(R_0 - R)$ and $z \in I(R'_0 - R')$. Hence $C_{S_0}(x) = C_{R_0}(y) \times C_{R'_0}(z)$. As $y$ induces an outer automorphism on $G_1$, $|C_{R_0}(y)| \leq 32$, and similarly $|C_{R'_0}(z)| \leq 32$ (see Lemma (2E)). Thus $|C_{S_0}(x)| \leq 1024$ and $|C_{S_1}(x)| \leq 2048$. In Case 2, $x$ induces outer automorphisms on $G_1$ and $G'_1$, so $|C_{S_0}(x)| \leq 512$ and $|C_{S_1}(x)| \leq 1024$.

We show that

$$a^g \cap (R_0 S \cup R'_0 S) = \emptyset.$$ 

Suppose that $a^g \in R_0 S \cup R'_0 S$ for some $g \in G$. Choose $a^g$ so that $|C_{S_1}(a^g)|$ is maximal. As $R_0 S = R_0 \times R'$, we may write $a^g = uv$ with $u \in R_0$ and $v \in R'$. Assume Case 1. Then conjugating in $N(G_0)$, we may assume that $|C_{R_0}(u)| \geq 32$ and that $|C_{R'_0}(v)| \geq 64$ (see Lemmas (2E) and (2K)), so $|C_{S_0}(a^g)| \geq 2048$. Similarly in Case 2, we may assume that $|C_{R_0}(u)|$ and $|C_{R'_0}(v)| \geq 32$, so that $|C_{S}(a^g)| \geq 1024$. Thus in any case, we may assume that $a^g$ is an extremal conjugate of $a$ in $S_1$. Then we may also assume that $C_{S_1}(a)^g \subseteq S_1$, since $S_1 \in Syl_2(G)$. Then $t^g \in S_1 \subseteq N$, and Lemma (60) yields that $g \in N$. But now $a \in X = G_1 C_{S_0}(G_1) \cup G'_1 C_{S_0}(G'_1)$ and $a^g \in X$, which is a contradiction because $X$ is a normal subset of $N(G_0)$. Thus we have proved that $a^g \cap (R_0 S \cup R'_0 S) = \emptyset$.

Consider Case 1. Then $S_1/S \cong D_8$, and $S_0/S$ and $\langle t, a, S \rangle/S$ are the four subgroups of $S_1/S$. Since $S_1 \in Syl_2(G)$ and since $a^g \cap S_0 \leq aS$ and $t^g \cap S_0 = \emptyset$, Lemma (1G) shows that $S \in Syl_2(G^\omega)$.

Therefore, assume that Case 2 holds. We show

$$\langle ta \rangle^g \cap S = \emptyset.$$ 

Suppose $b \in \langle ta \rangle^g \cap S$. As before, we may choose $b$ so that $|C_{S_1}(b)| \geq 1024$. Since $|C_{S_1}(x)| \leq 1024$ for $x \in I(S_0 - S)$ and since $|C_{S_1}(y)| \leq 256$ for any $y \in I(S_1 - S_0)$, we may assume that $b$ is an extremal conjugate of $ta$ in $S_1$. Then we may assume $b = (ta)^g$ and $C_{S_1}(ta)^g \subseteq S_1$ for some $g \in G$. But then Lemma (60) yields a contradiction just as before. Therefore, $\langle ta \rangle^g \cap S = \emptyset$. Since $t^g \cap \langle a, S \rangle = \emptyset$ and $a^g \cap S = \emptyset$, Lemma (1F) shows that $S \in Syl_2(G^\omega)$. The proof is complete.

**Lemma (6R).** $\langle L^o \rangle \cong PSU(4, 2) \times PSU(4, 2)$.

**Proof.** We argue that $R$ is strongly involution closed in $S$ with respect to $G^\omega$ (see [25]). By way of contradiction, let $x \in I(R)$ and assume $x^g \in S - R$ with $g \in G^\omega$. By conjugating in $G_0$, we may choose $x \in F_2$ and $x^g \in F_2 \times F'_1 - F_2$. Since $E_2$ is the unique $E_{265}$-
subgroup of $S$ and $S \in \text{Syl}_2(G^\infty)$ by Lemma (6Q), we may also choose $g \in N(E_2) \cap G^\infty$. Now $Y = N(E_2) \cap G^\infty$ acts, by conjugation, on $\{F_2, F'_2\}$ since $F_2 = O_3(N_2)$. Hence $|Y : N_Y(F_2)| \leq 2$. Since $S \in \text{Syl}_2(Y)$ by Lemma (6Q) and since $S \leq N(F_2)$, it follows that $Y \leq N(F_2)$. Thus $g \in N(F_2)$. But then $x^g \in F_2$, which is a contradiction proving the assertion.

We can now apply Corollary 2 of [25] to get that
\[
[\langle I(R)^{G^\infty} \rangle, \langle I(R^i)^{G^\infty} \rangle] \leq O(G^\infty).
\]
Set $X = \langle I(R)^{G^\infty} \rangle$ and let bars denote images in $G/O(G)$. Then $[\bar{X}, \bar{X}^i] = 1$ so $F^*(\bar{G})$ can not be simple. Thus Lemma (1H) shows $\langle L^\sigma \rangle \cong PSU(4, 2) \times PSU(4, 2)$.

Lemma (6R) completes the proof of Theorem (6A). The main theorem follows from Lemmas (3H), (3G), Theorems (4A), (5A), and (6A).

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