

# Pacific Journal of Mathematics

**A CONVOLUTION RELATED TO GOLOMB'S ROOT  
FUNCTION**

E. E. GUERIN

## A CONVOLUTION RELATED TO GOLOMB'S ROOT FUNCTION

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The root function  $\gamma(n)$  is defined by Golomb for  $n > 1$  as the number of distinct representations  $n = a^b$  with positive integers  $a$  and  $b$ . In this paper we define a convolution  $\nabla$  such that  $\gamma$  is the  $\nabla$ -analog of the (Dirichlet) divisor function  $\tau$ . The structure of the ring of arithmetic functions under addition and  $\nabla$  is discussed. We compute and interpret  $\nabla$ -analogs of the Moebius function and Euler's  $\phi$ -function. Formulas and an algorithm for computing the number of distinct representations of an integer  $n \geq 2$  in the form  $n = a_1^{a_1} \cdots a_k^{a_k}$ , with  $a_i$  a positive integer,  $i = 1, \dots, k$ , are given.

1. Introduction. Let  $Z$  denote the set of positive integers, let  $A$  denote the set of arithmetic functions (complex-valued functions with domain  $Z$ ), and let  $F$  denote the set of elements of  $Z$  which are not  $k$ th powers of any positive integer for  $k > 1 (k \in Z)$ . Note that  $1 \notin F$ . The divisor function  $\tau$  can be defined as  $\tau = \nu_0 * \nu_0$ , where  $\nu_0 \in A$ ,  $\nu_0(n) = 1$  for all  $n \in Z$ , and  $*$  is the Dirichlet convolution defined for  $\alpha, \beta \in A$  by  $(\alpha * \beta)(n) = \sum_{d|n} \alpha(d)\beta(n/d)$ .

Any integer  $n \geq 2$  having canonical form  $n = p_1^{e_1} \cdots p_r^{e_r}$  is uniquely expressible as  $n = m^g$ , where  $g = g.c.d. (e_1, \dots, e_r)$  and  $m \in F$ . Golomb [1] defines the root function  $\gamma(n)$  for  $n \in Z, n > 1$ , as the number of distinct representations  $n = a^b$  with  $a, b \in Z$ ; and he notes that  $\gamma(n) = \tau(g)$  for  $n = m^g, m \in F, g \in Z$ . We let  $\gamma(1) = 1$ .

For  $\alpha, \beta \in A, n = m^g$ , with  $m \in F, g \in Z$ , we define the  $G$ -convolution ("Golomb" convolution),  $\nabla$ , by

$$(1.1) \quad (\alpha \nabla \beta)(n) = \sum_{d|n} \alpha(m^d)\beta(m^{g/d}).$$

We define  $(\alpha \nabla \beta)(1) = 1$ . This  $G$ -convolution is not of the Narkiewicz type [2, 4].

In § 2, we show that  $\{A, +, \nabla\}$  (where  $(\alpha + \beta)(n) = \alpha(n) + \beta(n), n \in Z$ ) is a commutative ring with unity and we characterize the units and the divisors of zero. We define a  $G$ -multiplicative function and note that the set of  $G$ -multiplicative units in  $\{A, +, \nabla\}$  forms an Abelian group under the operation  $\nabla$ .

We choose to define  $\nabla$  as in (1.1) because then  $(\nu_0 \nabla \nu_0)(n)$  equals  $\gamma(n)$ , the number of distinct representations of  $n$  as  $a^b, a, b \in Z$ ;

this is an analog of  $\tau(n) = (\nu_0 * \nu_0)(n)$  which is the number of distinct representations of  $n$  as  $a \cdot b$ ,  $a, b \in Z$ . In § 3,  $\mathcal{V}$ -analogs of the Moebius function  $\mu$ , the sum of divisors function  $\sigma$ , and Euler's  $\phi$ -function are computed and interpreted.

In § 4, we state formulas and an algorithm for computing the number of distinct representations of an integer  $n \geq 2$  in the form

$$(1.2) \quad n = \alpha_1^{a_1} \cdots \alpha_k^{a_k}$$

with  $a_i \in Z$ ,  $i = 1, \dots, k$ .

2. The ring  $\{A, +, \mathcal{V}\}$ . First we state some properties related to the  $G$ -convolution.

**THEOREM 2.1.** (i) *The system  $\{A, +, \mathcal{V}\}$  is a commutative ring with unity  $\varepsilon_{\mathcal{V}}$  (where  $\varepsilon_{\mathcal{V}}(n) = 1$  if  $n=1$  or  $n \in F$ ,  $\varepsilon_{\mathcal{V}}(n)=0$  otherwise).*

(ii)  *$\alpha$  is a unit in  $\{A, +, \mathcal{V}\}$  if and only if  $\alpha(1) \neq 0$  and  $\alpha(m) \neq 0$  for all  $m \in F$ .*

(iii) *A nonzero arithmetic function  $\alpha$  is a nonzerodivisor in  $\{A, +, \mathcal{V}\}$  if and only if  $\alpha(1) \neq 0$  and for each  $m \in F$  there is a positive integer  $g$  such that  $\alpha(m^g) \neq 0$ .*

*Proof.* (i) The associativity of  $\mathcal{V}$  follows from (1.1) and the associativity of the Dirichlet convolution  $*$ . The commutativity of  $\mathcal{V}$  and the distributivity of  $\mathcal{V}$  over  $+$  follow directly from the definition of the  $G$ -convolution. If  $n = m^g$ ,  $g \in Z$ ,  $m \in F$ , then  $(\varepsilon_{\mathcal{V}} \mathcal{V} \alpha)(n) = \sum_{d|g} \varepsilon_{\mathcal{V}}(m^d) \alpha(m^{g/d}) = \alpha(m^g) = \alpha(n)$ ;  $(\varepsilon_{\mathcal{V}} \mathcal{V} \alpha)(1) = \alpha(1)$ . Therefore,  $\varepsilon_{\mathcal{V}}$  is the unity element in  $\{A, +, \mathcal{V}\}$ .

(ii) An element  $\beta$  in  $A$  such that  $\alpha \mathcal{V} \beta = \varepsilon_{\mathcal{V}}$  is defined if and only if  $\alpha(1)\beta(1)=1$ ,  $\alpha(m)\beta(m)=1$  for  $m \in F$ , and  $\sum_{d|g} \alpha(m^d)\beta(m^{g/d})=0$  for  $m \in F, g \in Z, g > 1$ . Thus,  $\alpha(1) \neq 0$ ,  $\alpha(m) \neq 0$  for  $m \in F$ , if and only if  $\alpha$  is a unit in  $\{A, +, \mathcal{V}\}$ .

(iii) If  $\alpha(1) = 0$ , define  $\beta \in A$  by  $\beta(1) = 1$ ,  $\beta(n) = 0$  if  $n > 1$ . Then  $(\alpha \mathcal{V} \beta)(n) = 0$  for every  $n \in Z$  and  $\alpha$  is a divisor of zero. If there exists an  $m \in F$  such that  $\alpha(m^g) = 0$  for every  $g \in Z$ , define  $\beta \in A$  by  $\beta(m) = 1$ ,  $\beta(n) = 0$  for  $n \in Z, n \neq m$ . Then  $(\alpha \mathcal{V} \beta)(n) = 0$  for all  $n \in Z$  and  $\alpha$  is a divisor of zero.

Assume that  $\alpha$  is a zero divisor in  $\{A, +, \mathcal{V}\}$ . Then there is some  $\beta \in A, \beta \neq \bar{0}$  (where  $\bar{0}(n)=0$  for all  $n \in Z$ ), such that  $\alpha \mathcal{V} \beta = \bar{0}$ .

(1) If  $\beta(1) \neq 0$  then  $\alpha \mathcal{V} \beta = \bar{0}$  implies that  $\alpha(1)\beta(1) = 0$  and that  $\alpha(1) = 0$ . (2) If  $\beta(1) = 0$ , let  $n$  be the smallest positive integer such that  $\beta(n) \neq 0$ ; if  $n = m^v, m \in F, v \in Z$ , we show that  $\alpha(m^w) = 0$  for all  $w \in Z$ . First,  $(\alpha \mathcal{V} \beta)(m^v) = \sum_{d|v} \alpha(m^d)\beta(m^{v/d}) = 0$  implies that

$\alpha(m)\beta(m^v) = 0$  and that  $\alpha(m) = 0$ . And  $(\alpha\mathcal{V}\beta)(m^{2^v}) = 0$  implies that  $\alpha(m)\beta(m^{2^v}) + \alpha(m^2)\beta(m^v) = 0$  and so  $\alpha(m^2) = 0$ . Assume that  $\alpha(m^t) = 0, 1 \leq t < r$ . Then  $(\alpha\mathcal{V}\beta)(m^{r^v}) = \sum_{d|rv} \alpha(m^d)\beta(m^{r^v/d}) = 0$  implies that  $\alpha(m^r)\beta(m^v) = 0$  and  $\alpha(m^r) = 0$ . Therefore,  $\alpha(m^w) = 0$  for all  $w \in \mathbb{Z}$  by induction. This completes the proof of the theorem.

We define  $\alpha \in A$  to be  $G$ -multiplicative if  $\alpha(1) = 1$ , and whenever  $(a, b) = 1$  and  $m \in F, \alpha(m^{ab}) = \alpha(m^a)\alpha(m^b)$ .

**THEOREM 2.2.** *The set of  $G$ -multiplicative functions which are units in  $\{A, +, \mathcal{V}\}$  form an abelian group under  $\mathcal{V}$ .*

*Proof.* If  $\alpha$  and  $\beta$  are  $G$ -multiplicative, then  $\alpha\mathcal{V}\beta$  is also; the proof is similar to that of the multiplicativity of  $\alpha*\beta$  given that  $\alpha$  and  $\beta$  are multiplicative [3, p. 93]. It is then easy to verify the required group properties.

3. The functions  $\sigma_r, \mu_r, \phi_r$ . As noted earlier,  $\gamma = \nu_0\mathcal{V}\nu_0$  is the  $\mathcal{V}$ -analog of  $\tau = \nu_0*\nu_0$ . For example,  $\gamma(64) = \gamma(2^6) = \tau(6) = 4$ , and 64 can be represented in the form  $a^b$  for  $a, b \in \mathbb{Z}$  in four ways:  $(2^1)^6 = 2^6, (2^2)^3 = 4^3, (2^3)^2 = 8^2$ , and  $(2^6)^1 = 64^1$ .

If we define  $\sigma_r$  by  $\sigma_r = \nu_0\mathcal{V}\nu_1$ , then for  $n = m^g, m \in F, g \in \mathbb{Z}, \sigma_r(n) = \sum_{a|g} m^a$ . So  $\sigma_r(n)$  is the sum of the  $a$ 's such that  $a^b = n$ , whereas  $\sigma(n) = (\nu_0*\nu_1)(n)$  is the sum of the  $a$ 's such that  $a \cdot b = n(a, b \in \mathbb{Z})$ .

An analog  $\mu_r$  of the Moebius function  $\mu$  (where  $\mu$  satisfies  $\nu_0*\mu = \varepsilon$  with  $\varepsilon(1) = 1, \varepsilon(n) = 0$  otherwise) is defined by  $\nu_0\mathcal{V}\mu_r = \varepsilon_r$ . Then  $\mu_r(n) = 1$  if  $n = 1, \mu_r(n) = \mu(g)$  if  $n = m^g, m \in F, g \in \mathbb{Z}$ .

Euler's  $\phi$ -function, which satisfies  $\phi = \mu*\nu_1$  (where  $\nu_1(n) = n$  for all  $n \in \mathbb{Z}$ ), has an analog  $\phi_r$  with  $\phi_r(1) = 1, \phi_r(n) = (\mu_r\mathcal{V}\nu_1)(n) = \sum_{d|g} \mu(d)m^{g/d}$  for  $n = m^g, m \in F, g \in \mathbb{Z}$ . Thus,  $\phi_r(m) = m$  for  $m \notin F$  and  $\phi_r(m^p) = m^p - m$  for  $m \in F, p$  prime. If  $n = m^g, m \in F, g \in \mathbb{Z}$ , then  $\phi_r(n)$  is  $n$  minus the number of positive integers less than or equal to  $n$  which are expressible as  $r^d, r \in \mathbb{Z}, d|g, d > 1$ . Here,  $n$  and  $r^d$  have a common power  $d > 1$  (since  $n = a^d$  with  $a = m^{g/d}$ ); this corresponds, in the computation of  $\phi(n)$ , to nonrelativity-prime  $n$  and  $m$  having a common divisor  $d > 1$ . To illustrate,  $\phi_r(64) = 2^6 - 2^3 - 2^2 + 2^1 = 64 - 10 = 54$ . The ten integers of the form  $r^d, r \in \mathbb{Z}, d|6, d > 1, r^d \leq 64$ , are

$$1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2 = 4^3 = 2^6, 2^3, 3^3.$$

And, for example,  $3^2$  and  $n = 8^2$  have common power 2, while  $2^3$  and  $n = 4^3$  have common power 3.

It can be verified that  $\gamma, \varepsilon_r, \nu_0$ , and  $\mu_r$  are  $G$ -multiplicative functions whereas  $\nu_1, \sigma_r$ , and  $\phi_r$  are not.

If  $n = m^g, m \in F, g \in Z$ , then  $\sigma_r(n) = 2n$  has no solutions. But if we define a  $G$ -perfect number  $n = m^g, m \in F, g \in Z$ , as one such that  $\prod_{d|g} m^d = n^2$ , then  $n$  is  $G$ -perfect if and only if  $g$  is perfect if and only if  $(\nu_0 * \nu_1)(g) = 2g$ .

4. **Power representations of  $n$ .** If  $n = m^g, m \in F, g \in Z$ , define  $\rho \in A$  by  $\rho(n) = g$ ; define  $\rho(1) = 1$ . Then  $\gamma(n) = \tau(\rho(n)) = (\nu_0 \mathcal{F} \nu_0)(n) = ((\nu_0 * \nu_0) \circ \rho)(n)$  (where  $(\alpha \circ \beta)(n) = \alpha(\beta(n))$ ). We note that  $\mu_r(n) = \mu(\rho(n))$  and  $\varepsilon_r(n) = \varepsilon(\rho(n))$ .

Let  $R_k(n)$  denote the number of distinct representations of  $n = m^g, m \in F, g \in Z$ , in the form given in (1.2). (Assume that  $R_k(1) = 1$  for all  $k \in Z$ .) We have the following formulas.

$$R_1(n) = 1.$$

$$R_2(n) = \gamma(n) = \tau(\rho(n)) = (\nu_0 \mathcal{F} \nu_0)(n).$$

$$\begin{aligned} R_3(n) &= \sum_{d|g} \gamma(d) = \sum_{d|\rho(n)} \tau(\rho(d)) = (\nu_0 * (\tau \circ \rho))(\rho(n)) \\ &= ((\nu_0 * (\nu_0 \mathcal{F} \nu_0)) \circ \rho)(n). \end{aligned}$$

$$\begin{aligned} R_4(n) &= \sum_{d|g} \sum_{r|\rho(d)} \gamma(r) = \sum_{d|\rho(n)} \sum_{r|\rho(d)} \tau(\rho(r)) = (\nu_0 * ((\nu_0 * (\tau \circ \rho)) \circ \rho))(\rho(n)) \\ &= ((\nu_0 * ((\nu_0 * (\nu_0 \mathcal{F} \nu_0)) \circ \rho)) \circ \rho)(n). \end{aligned}$$

Similar formulas can be written for  $R_k(n)$  for any  $k \in Z$ .

If  $n > 1$ , then  $R_k(n)$  can be computed as follows. List  $d_1$  such that  $d_1|g$ , list  $\rho(d_1)$ , list  $d_2$  such that  $d_2|\rho(d_1)$ , list  $\rho(d_2), \dots$ , list  $d_{k-2}$  such that  $d_{k-2}|\rho(d_{k-3})$ , list  $\rho(d_{k-2})$ ; and  $R_k(n)$  is the sum of the number of divisors of the entries in the final list.

For example, if  $n = 20^{400}, g = \rho(n) = 2^4 \cdot 5^2$ . For  $d_1|g, d_2|\rho(d_1), d_3|\rho(d_2)$ , we have these lists.

$$\begin{aligned} d_1 &= 1, 2, 4, 8, 16, 1 \cdot 5, 2 \cdot 5, 4 \cdot 5, 8 \cdot 5, 16 \cdot 5, 1 \cdot 5^2, 2 \cdot 5^2, 4 \cdot 5^2, 8 \cdot 5^2, 16 \cdot 5^2 \\ \rho(d_1) &= 1, 1, 2, 3, 4, \quad 1, 1, 1, 1, 1, 2, \quad 1, \quad 2, \quad 1, 2 \\ d_2 &= 1, 1, 1, 2, 1, 3, 1, 2, 4, \quad 1, 1, 1, 1, 1, 1, 2, 1, 1, 2, 1, 1, 2 \\ \rho(d_2) &= 1, 1, 1, 1, 1, 1, 1, 1, 2, \quad 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 \\ d_3 &= 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 \\ \rho(d_3) &= 1, 1 \end{aligned}$$

Then  $R_3(20^{400}) = 2\tau(1) + \tau(2) + \tau(3) + \tau(4) + 5\tau(1) + \tau(2) + \tau(1) + \tau(2) + \tau(1) + \tau(2) = 22$ . And  $R_4(20^{400}) = 23, R_5(20^{400}) = 23$ ; in fact,  $R_k(20^{400}) = 23$  for  $k \geq 4$ . There are four representations of  $n = 20^{400}$  in the form given in (1.2) for  $k = 4$  which correspond to  $d_1 = 16$  (since  $\tau(1) + \tau(1) + \tau(2) = 4$ ). They are

$$a^{16^1}, \quad a^{4^2}, \quad a^{2^4}, \quad a^{2^2},$$

where  $a = 335, 544, 320, 000, 000, 000, 000, 000, 000, 000$  (which is  $20^{25}$  in expanded form). In only one of these representations is  $a_i \neq 1, i = 1, \dots, 4$ . In general, the number of distinct representations of  $n = m^g, m \in F, g \in Z$ , in the form given in (1.2) with the additional requirement that  $a_i \neq 1, i = 1, \dots, k$ , is the sum of the number of divisors less one of the entries in the final list (for  $\rho(d_{k-2})$ ).

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