MAPS AND $h$-NORMAL SPACES

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Further consequences of hard sets are explored in this paper, and some new relations between a space $X$ and its extension $\delta X$ are shown. A generalization of perfect maps, called $\delta$-perfect maps, is introduced. It is found that among the $WZ$-maps, these are precisely the ones which pull hard sets back to hard sets. Applications to $\delta X$ are made. Maps which carry hard sets to closed sets and maps which carry hard sets to hard sets are considered, and it is seen that the image of a realcompact space under a closed map is realcompact if and only if the map carries hard sets to hard sets.

The last part of the paper introduces a generalization of normality, called $h$-normal, in which disjoint hard sets are completely separated. It is found that $X$ is $h$-normal whenever $\nu X$ is normal. The hereditary and productive properties of $h$-normal spaces are investigated, and the $h$-normal spaces are characterized in terms of $\delta$-perfect $WZ$-maps. Finally as an analogue of closed maps on normal spaces, a necessary and sufficient condition is found that the image of an $h$-normal space under a $\delta$-perfect $WZ$-map be $h$-normal.

1. Introduction. All spaces discussed in this paper are assumed Tychonov (completely regular and Hausdorff) and the word map means a continuous surjection. The notation of [2] is used throughout. In particular, $\beta X$ is the Stone-Čech compactification and $\nu X$ is the Hewitt realcompactification of $X$.

The following facts concerning hard sets will be used here. They are found in [8] and [9].

**Definition 1.** For any space $X$, let $cl_{\beta X}(\nu X - X) = K(= K_X)$. A set $H \subseteq X$ is called hard (in $X$) if $H$ is closed as a subset of $X \cup K$. (A characterization of hard sets internal to $X$ is given in [8].) Let $\delta X$ be the subspace of $\beta X$ given by $\delta X = \beta X - (K - X)$. Thus $X \subseteq \delta X \subseteq \beta X$.

**Proposition 2.** A subset $H$ of space $X$ is hard if and only if there is a compact subset of $\delta X$ whose restriction to $X$ is $H$.

**Proposition 3.** Every compact set in $X$ is hard, but every hard set is compact if and only if $X = \delta X$. (Note every pseudocompact space is of this type.)

**Proposition 4.** Every hard set of $X$ is closed, but every closed
set is hard if and only if $X$ is realcompact.

**Proposition 5.** A closed subset of a hard set is hard.

It follows immediately from the definition that $X$ is realcompact if and only if $\delta X$ is compact. We conclude this section with some new results.

**Lemma 6.** The set of points at which $\delta X$ fails to be locally compact is precisely the set of points at which $X$ fails to be locally realcompact.

**Proof.** Let $R(\delta X)$ be the set of points at which $\delta X$ fails to be locally compact. By ([5], 2.10), the set of points at which $X$ fails to be locally realcompact is $X \cap K$. But $\beta(\delta X) - \delta X = \beta X - \delta X = K - X$. Thus $cl_{\beta X}(\beta X - \delta X) = cl_{\beta X}(K - X) = K$. So $R(\delta X) = \delta X \cap cl_{\beta X}(\beta \delta X - \delta X) = \delta X \cap K = X \cap K$.

**Corollary 7.** $X$ is locally realcompact if and only if $\delta X$ is locally compact. 

**Corollary 8.** Let $X$ be locally realcompact. The hard zero sets form a base for the hard sets. 

**Proof.** In $\delta X$ as in any locally compact space, the compact zero sets form a base for the compact sets.

**Corollary 9.** $X$ is locally realcompact if and only if every hard set of $X$ is contained in the interior of a regular-hard (i.e., hard and regular-closed) set of $X$.

**Proof.** Let $H$ be a hard set of $X$. Then $cl_{\delta X} H$ is compact in the locally compact space $\delta X$, so it is contained in the interior of a regular compact set $B$ of $\delta X$. Restrict $B$ to $X$.

**Theorem 10.** For any $X$, $\delta X$ is the union of the $\beta X$-closures of the hard sets of $X$.

**Proof.** Let $p \in \delta X - X$. By Lemma 6, there is a compact set $F$ such that $x \in \text{int}_{\delta X}(F) \subseteq F \subseteq \delta X$. Let $G = X \cap \text{int}_{\delta X}(F)$, then $cl_{\delta X}(G) = cl_{\delta X} \text{int}_{\delta X}(F)$. Let $H = cl_{X}(G)$, so $H$ is a hard set of $X$ and $p \in cl_{\delta X}(H) = cl_{\delta X}(H)$.

II. $\delta$-perfect maps. Let $f: X \to Y$ be any map and $f_{\beta}: \beta X \to \beta Y$
be its Stone extension. Henriksen and Isbell [3] have studied those maps, now called *perfect*, which are closed and pull compact sets back to compact sets.

**Proposition 11.** A map \( f: X \to Y \) is perfect if and only if for each \( y \in Y \), \( f_\beta(y) \subseteq X \).

**Proof.** This follows from the characterization in [3] that \( f \) is perfect if and only if \( f_\beta[\beta X - X] = \beta Y - Y \).

**Definition 12.** The map \( f: X \to Y \) is \( \delta \)-perfect if for each \( y \in Y \), \( f_\delta(y) \subseteq \delta X \).

Clearly every perfect map is \( \delta \)-perfect. Yet if \( Y \) is compact and \( X \) is realcompact and not compact, there are no perfect maps from \( X \) onto \( Y \), but every map is \( \delta \)-perfect since \( \delta X = \beta X \).

**Lemma 13.** A map \( f: X \to Y \) is \( \delta \)-perfect if and only if \( f_\delta[\delta Y] \subseteq \delta X \).

**Proof.** One direction is trivial. For the other, note \( \nu X \subseteq f_\delta[\nu Y] = f_\delta[Y] \cup f_\delta[\nu Y - Y] \). By hypothesis, \( f_\delta[y] \cap (\nu X - X) = \emptyset \). Thus \( \nu X - X \subseteq f_\delta[\nu Y - Y] \subseteq f_\delta[K_Y] \) which is a compact set. Whence \( cl_{\beta X}(\nu X - X) \subseteq f_\delta[K_Y] \), so \( X \cup K_X \subseteq f_\delta[Y \cup K_Y] \). Therefore \( f_\delta[\delta Y] \subseteq \delta X \).

**Corollary 14.** The composition of \( \delta \)-perfect maps is \( \delta \)-perfect.

In [4], Isiwata introduced the concept of a \( WZ \)-map as a map \( f: X \to Y \) such that for each \( y \in Y \), \( f_\gamma(y) = cl_\beta f^{-}(y) \). He showed that every \( Z \)-map (i.e., a map which carries zero sets to closed sets) is a \( WZ \)-map. Clearly every closed map is a \( Z \)-map, and every perfect map is a \( WZ \)-map. We shall see (Lemma 19 and Corollary 21) that \( \delta \)-perfect maps and \( WZ \)-maps are independent concepts; but those maps which are both \( \delta \)-perfect and \( WZ \) are of particular interest.

**Lemma 15.** A map \( f: X \to Y \) is a \( \delta \)-perfect \( WZ \)-map if and only if for all \( y \in Y \), \( f_\delta(y) = cl_{\delta X} f^{-}(y) \).

**Proof.** \( cl_{\delta X} f^{-}(y) \subseteq cl_{\delta X} f^{-}(y) \subseteq f_\delta(y) \).

In [8], we showed that a perfect map pulls hard sets back to hard sets. More generally,
THEOREM 16. Let \( f: X \to Y \) be a map. Each of the following conditions implies the next one.

(a) \( f \) is \( \delta \)-perfect.
(b) \( f \) pulls hard sets back to hard sets.
(c) \( f \) pulls points back to hard sets.

Moreover if \( f \) is a WZ-map, they are all equivalent.

Proof. (a) implies (b) since a set \( H \) in \( Y \) is hard if and only if \( cl_{\delta Y}H \subseteq \delta Y \). Thus \( f^{-1}_{\delta}[cl_{\delta Y}H] \) is compact and contained in \( \delta X \), whence \( X \cap f^{-1}_{\delta}[cl_{\delta Y}H] \) is hard in \( X \) and contains the closed set \( f^{-1}[H] \). But a closed subset of a hard set is hard. (b) implies (c) since every compact set is hard.

Finally, suppose \( f \) is a WZ-map satisfying (c). Then for every \( y \in Y \), \( f^{-1}_{\delta}(y) = cl_{\delta x}f^{-1}(y) = cl_{\delta x}f^{-1}(y) \). Whence by Lemma 15, \( f \) is \( \delta \)-perfect.

COROLLARY 17. If \( X = \delta X \) and \( f: X \to Y \) is a \( \delta \)-perfect map, then \( Y = \delta Y \).

Proof. Let \( H \) be a hard set in \( Y \). Then \( f^{-1}[H] \) is hard, hence compact in \( \delta X \). Thus \( H = f^{-1}\circ f^{-1}[H] \) is compact in \( Y \). Therefore, by Proposition 3, \( Y = \delta Y \).

COROLLARY 18. If \( X \) is compact and \( X \times Y = \delta(X \times Y) \), then \( Y = \delta Y \).

Zenor [11] constructed a useful map: let \( A \) be a closed subset of space \( X \) and define \( \varphi_A \) to be the natural function taking \( X \) onto \( Y = X/A \). Topologize \( Y \) with the finest completely regular topology making \( \varphi_A \) continuous. Zenor shows that \( \varphi_A \) is always a WZ-map.

LEMMA 19. \( \varphi_A \) is \( \delta \)-perfect if and only if \( A \) is hard in \( X \).

Proof. By Theorem 16, \( \varphi_A \) is \( \delta \)-perfect if and only if the pre-image of every point is hard. The pre-image of every point other than \( \varphi_A(A) \) is itself, and compact sets are always hard. But \( A = \varphi_A^{-1}(\varphi_A(A)) \).

THEOREM 20. A space \( X \) is realcompact if and only if every map on \( X \) (to a Tychonov space) is \( \delta \)-perfect.

Proof. We have already observed one direction. Conversely, let \( A \) be an arbitrary nonempty closed set of \( X \). The Zenor's map
φ_A: X → Y = X/A is δ-perfect. Whence by Lemma 19, A is hard. The result follows from Proposition 4.

**Corollary 21.** Any nonclosed map on a normal realcompact space is δ-perfect and not WZ.

*Proof.* Isiwata ([4], 1.3) has shown that every WZ-map on a normal space is closed.

**Theorem 22.** X = δX if and only if every δ-perfect WZ-map on X (to a Tychonov space) is perfect.

*Proof.* (If). Let A be an arbitrary hard set of X and φ_A: X → Y = X/A be the Zenor map. By Lemma 19, φ_A is a δ-perfect WZ-map, so it is perfect. Hence the pre-image of every point is compact. In particular, the pre-image A of the point φ_A(A) is compact. But X = δX precisely when every hard set is compact. (Only if). For each y ∈ Y, f^A(y) = cl_δ_X f^-^A(y) = cl_δ_X f^-^A(y) = f^-^A(y) ⊆ X.

**Definition 23.** A map f: X → Y is an H-map if the image of each hard set in X is a closed set of Y. If f carries hard sets to hard sets, we shall call f a hard map.

Clearly closed maps and hard maps are H-maps. If X is realcompact, then every closed set is hard, so every H-map on a realcompact space is closed. If X = δX, then every hard set is compact, so every map on X is a hard map. Isiwata ([4], 3.6) has constructed an example of a map on a pseudocompact space which is not a WZ-map. Thus an H-map need not be WZ. However,

**Lemma 24.** If f: X → Y is a δ-perfect H-map, then f is a WZ-map.

*Proof.* Let y ∈ Y. Since f^A(y) ⊆ δX, we see that cl_δ_X f^-^A(y) = cl_δ_X f^-^A(y).

Suppose x ∈ f^A(y) - cl_δ_X f^-^A(y). Since x ∈ δX - X by Lemma 6 there is a δX-open set N such that x ∈ N ⊆ cl_δ_X N ⊆ δX - cl_δ_X f^-^A(y). Let M = cl_δ_X (N ∩ X). Since X is dense in δX, cl_δ_X (M) = cl_δ_X (N), and M is a nonempty hard set of X disjoint from f^-^A(y). Thus y is not in f(M), and since f is an H-map, f(M) = cl_δ_Y f(M). But y = f^A(x) ∈ f^A [cl_δ_X M] ∩ Y = cl_δ_Y [f^A(M)] ∩ Y = cl_δ_Y [f(M)] ∩ Y = cl_δ_Y f(M) = f(M), contradiction.

**Lemma 25.** Let f: X → Y be a hard map. Then δX ⊆ f^A [δY].

*Proof.* By Theorem 10, δX = ∪ {cl_δ_X H: H is hard in X}. For
each hard set $H$ of $X$, $f[H]$ is hard in $Y$ and $cl_{βX}H \subseteq f_β^{-1}[cl_{βY}f(H)]$.

**Theorem 26.** $f: X \to Y$ is a hard map if and only if $f$ is an $H$-map and $δX \subseteq f_β[δY]$.

*Proof.* Every hard map is an $H$-map, so one direction follows from Lemma 25. Conversely, let $H$ be a hard set of $X$. Then $f_β[cl_{βX}H] \subseteq δY$. But $cl_{βX}H$ is compact, so $f_β[cl_{βX}H]$ is compact. Since $f(H) \subseteq cl_{βY}f(H) \subseteq f_β(cl_{βX}H)$, we have $Y \cap cl_{βY}f(H) = cl_Yf(H)$ is hard in $Y$. Since $f$ is an $H$-map, $f(H) = cl_Yf(H)$.

**Corollary 27.** Let $f: X \to Y$ be a $δ$-perfect $H$-map. Then $f$ is a hard map if and only if $δX = f_β[δY]$.

*Proof.* Theorem 26 and Lemma 13.

**Corollary 28.** Let $X$ be realcompact and $f: X \to Y$ be a closed map. Then $Y$ is realcompact if and only if $f$ is a hard map.

*Proof.* Since $X$ is realcompact, $δX = βX$ and $f_β[δX] = βY$. Every map on a realcompact space is $δ$-perfect, so by Corollary 27, $f$ is a hard map if and only if $βY = δY$, i.e., $Y$ is realcompact.

In a private communication, John Mack states that he has investigated a class of maps $f: X \to Y$, which he calls $R$-perfect maps, satisfying the condition that the graph of $f$, $E(f)$, is closed in $(υX) \times Y$. Since these results are not reproduced elsewhere, the author has Mack's permission to include them here.

**Lemma 29 (Mack).** Let $f: X \to Y$ be a map and $f_\nu: υX \to υY$ be its Hewitt extension. The following are equivalent:

(a) $f$ is $R$-perfect.
(b) $E(f) = E(f_\nu) \cap (υX \times Y)$.
(c) $f_\nu(υY - Y) = υX - X$.

*Proof.* (a) implies (b). For any map, $E(f_\nu)$ is the closure of $E(f)$ in $υX \times υY$. So if $f$ is $R$-perfect, then $E(f)$ is the intersection of $υX \times Y$ with the $υX \times υY$-closure of $E(f)$, which is $(υX \times Y) \cap E(f_\nu)$. (b) implies (c). By (b), we have $f_\nu(Y) = f^{-1}(Y) = X$, whence $f_\nu(υY - Y) = υX - X$.

(c) implies (a) $f_\nu(υY - Y) = υX - X$ implies $E(f) = (υX \times Y) \cap E(f_\nu)$, which is the intersection of $υX \times Y$ with the $υX \times υY$-closure of $E(f)$. Thus $E(f)$ is closed in $υX \times Y$. 

THEOREM 30 (Mack). Let \( f: X \rightarrow Y \) be an R-perfect map.

(a) If \( F \subseteq Y \) is realcompact, then \( f^{-1}(F) \) is realcompact.

(b) If \( Y \) is locally realcompact, then \( X \) is locally realcompact.

Proof. (a) \( \nu X \times F \) is realcompact. Since the graph \( G(f) \) is closed in \( \nu X \times Y \), then \( G(f) \cap (X \times F) = G(f_v) \cap (\nu X \times F) \) is realcompact. But \( f^{-1}[F] \) is homeomorphic to \( G(f) \cap (X \times F) \).

(b) \( \nu X - X = f_v^{-1}(\nu Y - Y) \). But \( Y \) is locally realcompact if and only if \( \nu Y - Y \) is closed in \( \nu Y \). Whence \( X \) is open in \( \nu X \).

Notice that it follows from Lemma 13 and 29(c) that every \( \delta \)-perfect map is R-perfect. The converse is false.

EXAMPLE 31 of an R-perfect map which is not \( \delta \)-perfect. Let \( W \) be the ordinals less than the first uncountable ordinal \( \omega_1 \), and let \( T \) be the free union of countably infinitely many copies of \( W \). Then \( \nu T \) is the free union of the one point compactifications of the \( W \)'s, so \( K_T \) is homeomorphic to \( \beta N \) (where \( N \) is the discrete space of positive integers). Let \( p \in K_T - \nu T \) and define \( X = T \cup \{ p \} \) as a subspace of \( \beta T \). Then \( \nu X = \nu T \cup \{ p \} \) and \( X \cup K_X = T \cup K_T \). Now let \( Y \) be the quotient space of \( T \cup (K_T - \nu T) \) obtained by factoring the compact set \( K_T - \nu T \) to a point \( k \). It is not difficult to see that \( Y \) is Tychonov, and \( \nu Y = \nu T \cup \{ k \} = Y \cup K_T \). Note \( K_Y \cap Y = \{ k \} \).

Let \( f: X \rightarrow Y \) be the restriction of the quotient map, so \( f(p) = k \) and \( f(x) = x \) otherwise. Moreover \( f_v \) extends \( f \) by being the identity map on \( \nu T - T \), so \( f_v^{-1}(\nu Y - Y) = \nu X - X \) and \( f \) is an R-perfect map. But \( k \in Y \) and \( f_v^{-1}(K_T - \nu T) \supseteq K_T - \nu X \neq \emptyset \). So \( f \) is not \( \delta \)-perfect.

THEOREM 32. Let \( f: X \rightarrow Y \) be an R-perfect map. If \( Y \) is locally realcompact, then \( f \) is \( \delta \)-perfect.

Proof. Since \( f \) is R-perfect, \( \nu X - X = f_v^{-1}(\nu Y - Y) \subseteq f_v^{-1}(\nu X - X) \subseteq f_v^{-1}(K_T) \), which is compact. Hence \( K_X \subseteq f_v^{-1}(K_T) \). Since \( Y \) is locally realcompact, \( \delta Y = \beta Y - K_T \). So \( f_v^{-1}(\delta Y) = f_v^{-1}(\beta Y - K_T) = \beta X - f_v^{-1}(K_T) \subseteq \beta X - K_X = \delta X \), by Theorem 30(b).

III. \( h \)-normal spaces.

DEFINITION 33. Let \( X \subseteq T \subseteq \beta X \). A set \( H \subseteq X \) is \( T \)-hard if \( H \) is closed in \( X \cup cl_{\beta X}(T - X) \). We shall call \( X \) a \( T \)-normal space if disjoint \( T \)-hard sets of \( X \) are completely separated in \( X \). Notice that for any \( T \), every normal space is always \( T \)-normal. If \( T = \nu X \), the \( T \)-hard sets of \( X \) are simply the hard sets, and we shall use the term \( h \)-normal space in this case.
It follows from Proposition 4 that a realcompact space is $h$-normal if and only if it is normal. Similarly by Proposition 3, for any $X$ we have that $\delta X$ is an $h$-normal space. In particular, every pseudocompact space is $h$-normal. Thus the Tychonov plank is an $h$-normal space which is not normal.

**Theorem 34.** Let $X \subseteq T \subseteq \beta X$. The following are equivalent:

(a) $X$ is $T$-normal.

(b) There is a $Y$, $X \subseteq Y \subseteq T$ and $Y$ is $T$-normal.

(c) $X \cup cl_{\beta x}(T - X)$ is normal.

(d) Each closed subset of $X$ is completely separated from every disjoint $T$-hard set.

**Proof.** That (c) implies (d) and (d) implies (a) are easy exercises. It suffices to show (b) implies (c). Let $A_1$ and $A_2$ be disjoint and closed in $X \cup cl_{\beta x}(T - X)$. Let $B_i = A_i \cap cl_{\beta x}(T - X)$, $i = 1, 2$. Then $B_1$ and $B_2$ are compact. By ([2], 3.11a), there are zero sets $Z_j$, $j = 1, 2, 3, 4$, of $X \cup cl_{\beta x}(T - X)$ such that

(i) $A_1 \subseteq \text{int}(Z_1)$, $B_1 \subseteq \text{int}(Z_2)$ and $Z_1 \cap Z_2 = \emptyset$, and

(ii) $A_2 \subseteq \text{int}(Z_3)$, $B_2 \subseteq \text{int}(Z_4)$ and $Z_3 \cap Z_4 = \emptyset$.

Let $H_1 = A_1 - \text{int}(Z_1)$ and $H_2 = A_2 - \text{int}(Z_2)$. If either $H_1$ or $H_2$ is empty, we have disjoint open neighborhoods of $A_1$ and $A_2$, so we are done. Otherwise $H_1$ and $H_2$ are nonempty, disjoint $T$-hard sets of $X$, hence of $Y$. Thus there are functions $f$ and $g$ in $C^*(Y)$ such that $H_1 \subseteq \text{int}_Y Z(f)$, $H_2 \subseteq \text{int}_Y Z(g)$ and $Z(f) \cap Z(g) = \emptyset$. Since $\beta Y = \beta X$ and disjoint zero sets of $Y$ have disjoint closures in $\beta Y$, we have that the $X \cup cl_{\beta x}(T - X)$-closures $Z'(f)$ and $Z'(g)$ are disjoint. Let $G_1$ and $G_2$ be the $X \cup cl_{\beta x}(T - X)$-interiors of $Z'(f)$ and $Z'(g)$ respectively. Note that $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$. Let $F_1 = [\text{int}(Z_1) \cup G_1] \cap \text{int}(Z_2)$ and $F_2 = [\text{int}(Z_3) \cup G_2] \cap \text{int}(Z_4)$. Then $A_1 \subseteq F_1$, $A_2 \subseteq F_2$ and $F_1$, $F_2$ are disjoint sets open in $X \cup cl_{\beta x}(T - X)$.

**Corollary 35.** $X$ is $h$-normal if and only if $X \cup K$ is normal.

Let $X$ be a locally realcompact and not realcompact space. Then $K$ is a nonempty compact set disjoint from $X$. In [5], it was shown that factoring $K$ to a single point gave a one-point realcompactification *$X$ of $X$. Moreover *$X$ is maximal among the one-point realcompactifications of $X$ in the sense that if $X \cup \{p\}$ is any other, then there is a map from *$X$ onto $X \cup \{p\}$ which is the identity on $X$.

**Corollary 36.** If $X$ is locally realcompact and not realcompact, then $X$ is $h$-normal if and only if *$X$ is normal.
COROLLARY 37. If $\nu X$ is normal, then every $C$-embedded subset of $X$ is $h$-normal.

Proof. Let $A$ be $C$-embedded in $X$. Then $\nu A = \text{cl}_{\nu X}(A)$ ([2], 8.10a), and a closed subset of a normal space is normal. Hence $A$ is $h$-normal by Theorem 34(b).

COROLLARY 38. If $\nu X$ is normal, then $X$ is $h$-normal. Moreover for any space $X$ for which $\nu X - X$ is closed in $\beta X - X$, $\nu X$ is normal if and only if $X$ is $h$-normal.

Proof. If $\nu X - X$ is closed in $\beta X - X$, then $\nu X = X \cup K$.

EXAMPLE 39. Corson's space $X$ ([2], p 272) is normal, hence $h$-normal, but $\nu X$ is not normal.

EXAMPLE 40. Realcompact spaces and pseudocompact spaces trivially satisfy the condition $\nu X - X$ is closed in $\beta X - X$. In general, let $Y$ by any Tychonov space and define $X = (Y \cup K_Y) - (\nu Y - Y)$. Then $Y \subseteq X \subseteq \beta Y$, so $\beta X = \beta Y$, $\nu X = Y \cup K_Y$, $\nu X - X = \nu Y - Y$ and $K_x \cap X = K_Y - \nu Y$. Thus $\nu X - X$ is closed in $\beta X - X$. By construction, $X = Y$ if and only if $\nu Y - Y$ is closed in $\beta Y - Y$, so this technique generates all the spaces with the desired property. Notice the generated space $X$ is realcompact if and only if $Y$ is realcompact, and $X$ is pseudocompact if and only if $\nu X = \beta X$ which (since $\beta X = \beta Y$) is equivalent to $Y \cup K_Y = \beta Y$, which is true if and only if $Y = \delta Y$. Hence if $Y$ is a nonrealcompact space for which $Y \neq \delta Y$, then $X$ is neither realcompact nor pseudocompact, yet $\nu X - X$ is closed in $\beta X - X$. E.g., let $Y = W \times N$, where $W$ is the usual space of ordinals with countable predecessors and $N$ is the discrete space of positive integers. The author does not have any internal characterizations for the spaces $X$ for which $\nu X - X$ is closed in $\beta X - X$.

DEFINITION 41. A subset of a space $X$ will be called an $H_\sigma$-set if it is the union of a countable family of hard sets. Every $\sigma$-compact set is an $H_\sigma$-set and every $H_\sigma$-set is an $F_\sigma$-set.

COROLLARY 42. Every $H_\sigma$-subspace of an $h$-normal space is normal.

Proof. The $H_\sigma$-sets of $X$ are $F_\sigma$-sets of $X \cup K$, and $F_\sigma$-sets of a normal space are normal.

COROLLARY 43. Every hard subset of an $h$-normal space is $C$-embedded.
Proof. A hard set $H$ is closed in normal $X \cup K$, and every closed subset of a normal space is $C$-embedded ([2], 3D1).

Example 44. The Sorgenfrey plane $S$ is a realcompact space which is not $h$-normal. Let $W$ be the space of ordinals with countable predecessors and $W^* = W \cup \{\omega_1\}$ be its compactification. Put $X = [W^* \times \beta S] - [\{\omega_1\} \times (\beta S - S)]$. Then $X$ is pseudocompact ([2], 9K) and $\{\omega_1\} \times S$ is a closed, $C^*$-embedded subset of $h$-normal $X$ which fails to be $h$-normal.

Example 45. Let $X$ be a normal, realcompact but not paracompact space. (By Moran's result [7], barring measurable cardinals, normal and metacompact imply realcompact. Hence Michael's example in [6] is such a space.) Then by Tamano's theorem ([10], Th. 2) $X \times \beta X$ is realcompact and not normal, hence not $h$-normal. Thus the product of a normal space and a compact space can fail to be $h$-normal.

Theorem 46. Let $X$ and $Y$ have nonmeasurable cardinals. If $\nu X$ is paracompact and $Y$ is a locally compact, paracompact space, then $X \times Y$ is $h$-normal.

Proof. For Tychonov spaces with nonmeasurable cardinals, paracompact implies normal and realcompact. From [1], if $Y$ is locally compact and real-compact, then for any $X$, $\nu(X \times Y) = (\nu X) \times Y$.

In [11] the following remarks are made about Zenor's maps $\varphi_A$ (see Proposition 18 above):

1. $X$ is normal if and only if $\varphi_A$ is a quotient map for each closed set $A$ in $X$.
2. Each closed set is completely separated from every disjoint zero set in $X$ if and only if $\varphi_A$ is a quotient map for each zero set $A$ in $X$.

In like vein, we observe:

Lemma 47. $X$ is $h$-normal if and only if $\varphi_A$ is a quotient map for each hard set $A$ in $X$.

From [11], we also have

Proposition 48 (Zenor). (a) $X$ is normal if and only if every $Z$-map is closed. (b) Each closed set is completely separated from every disjoint zero set in $X$ if and only if every $WZ$-map is a $Z$-map.
**Theorem 49.** For any space $X$, the following are equivalent.

(a) $X$ is $h$-normal.
(b) Every $WZ$-map on $X$ is an $H$-map.
(c) Every $\delta$-perfect $WZ$-map on $X$ is closed.

**Proof.** (a) implies (b). Let $f: X \to Y$ be a $WZ$-map and let $H$ be a hard set in $X$. Suppose $y \in Y - f(H)$. Then $f^-(y)$ is closed in $X$ and disjoint from $H$, whence $f^-(y)$ and $H$ are completely separated. So $cl_{\beta X}f^-(y) \cap cl_{\beta X}(H) = \emptyset$ and $y$ is not in $f_\beta[cl_{\beta X}H]$. But $f_\beta[cl_{\beta X}H] \cap Y$ is closed in $Y$ and contains $f(H)$. Thus $f(H)$ is closed.

(b) implies (a). Let $H$ be a hard set of $X$ and $F$ a closed set disjoint from $H$. Consider the Zenor map $\varphi_F$. It is a $WZ$-map, so $\varphi_F[H]$ is closed and $\varphi_F(F) \subseteq \varphi_F(H)$. Since $Y$ is completely regular, $\varphi_F(F)$ and $\varphi_F(H)$ are completely separated, whence $F$ and $H$ are completely separated.

(a) implies (c). Let $f: X \to Y$ be a $\delta$-perfect $WZ$-map and let $B$ be a closed subset of $X$. Let $p \in Y - f(B)$. Then $f^-(p)$ is hard in $X$ and disjoint from $B$, hence $B$ and $f^-(p)$ are completely separated. Therefore $cl_{\beta X}B$ and $cl_{\beta X}f^-(p) = f_\beta^-(p)$ are disjoint, so $p$ is not in $f_\beta[cl_{\beta X}B]$. Since $f_\beta$ is a closed map, $f_\beta[cl_{\beta X}B]$ is a closed set containing $f(B)$. Therefore $p \notin cl_Yf(B)$ and $f(B)$ is closed.

(c) implies (a). Suppose $X$ is not $h$-normal. There is a closed set $F$ and a hard set $H$ which is disjoint to it, but not completely separated from it. Consider the Zenor map $\varphi_H$. By Lemma 19, $\varphi_H$ is a $\delta$-perfect $WZ$-map. If $\varphi_H(F)$ is closed in $Y$, then there is some $Z_i = Z(f_i) \subseteq Z(Y)$ such that $\varphi_H(F) \subseteq Z_i \subseteq Y - \varphi_H(H)$. Thus $\varphi_H(H) \subseteq Y - Z_i$, which is open. Hence there is a $Z_2 = Z(f_2) \subseteq Z(Y)$ such that $\varphi_H(H) \subseteq int_Y Z_2 \subsetneq Z_2 \subseteq Y - Z_i$. Now $f_i \circ \varphi_H: X \to R$, $i = 1, 2$, and $Z(f_i \circ \varphi_H)$ and $Z(f_2 \circ \varphi_H)$ are disjoint zero sets in $X$ completely separating $F$ and $H$, contradiction. Whence $\varphi_H$ is not closed.

We observe that the closed image of a normal space is normal. If $X = \delta X$, then every map on $X$ is an $H$-map. Hence by Lemma 24, every $\delta$-perfect map $f: X \to Y$ is a $\delta$-perfect $WZ$-map. (Notice that since $\delta X$ is an $h$-normal space, such an $f$ must be closed by Theorem 49(c).) By Corollary 17, $Y = \delta Y$ is also $h$-normal. More generally,

**Theorem 50.** Let $X$ be an $h$-normal space and $f: X \to Y$ be a $\delta$-perfect $WZ$-map. Then $Y$ is $h$-normal if and only if for every $\delta$-perfect $WZ$-map $g$ on $Y$, $g \circ f$ is a $WZ$-map.
Proof. (If), \( g \circ f \) is \( \delta \)-perfect by Corollary 14. Whence by Theorem 49(c), \( g \circ f \) is closed. Thus \( g \) is closed and since \( g \) is an arbitrary \( \delta \)-perfect WZ-map on \( Y \), \( Y \) is \( h \)-normal.

(Only if). \( f \) and \( g \) are closed maps, whence \( g \circ f \) is a closed map. But closed maps are WZ.

It is not true in general that the composition of WZ-maps is a WZ-map; in fact an example due to M. Henriksen shows more.

**Example 51 (Henriksen).** A closed map and a Z-map whose composition is not a WZ-map. Consider the subspace of the product of ordinal spaces given by

\[
X = W(\omega_1 + 1) \times W(\omega_2 + 1) - \{\omega_1\} \times [W(\omega_2 + 1) - W(\omega_1)]
\]

We observe that \( X \) is pseudocompact and \( \beta X = W(\omega_1 + 1) \times W(\omega_2 + 1) \). Let \( Y = W(\omega_1 + 1) \times W(\omega_1 + 1) - \{(\omega_1, \omega_1)\} \) and define \( t: X \to Y \) by \( t(a, b) = (a, \omega_1) \) if \( b \geq \omega_1 \), \( t(a, b) = (a, b) \) otherwise. Since \( [W(\omega_2 + 1) - W(\omega_1)] \) is compact, it follows that \( t \) is a closed map.

Let \( \varphi: Y \to W(\omega_1 + 1) \) be given by \( \varphi(a, b) = a \). Isiwata has shown ([4], 3.5) that \( \varphi \) is an open Z-map which is not closed. Consider \( \varphi \circ t: X \to W(\omega_1 + 1) \). We have \( cl_{\beta X}(\varphi \circ t)^{-1}(\omega_1) = cl_{\beta X}([\omega_1] \times W(\omega_1)) = \{\omega_1\} \times W(\omega_1 + 1) \). But \( (\varphi \circ t)^{-1}(\omega_1) = \{\omega_1\} \times W(\omega_2 + 1) \), so \( \varphi \circ t \) is not a WZ-map.

**References**

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WINNIPEG, MANITOBA R3T2N2
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