CANCELLING 1-HANDLES AND SOME TOPOLOGICAL IMBEDDINGS

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In this note we use the existence of a certain type of handle decomposition (see corollary) for compact simply connected P. L. 4-manifolds and R. Edwards results on the double suspension conjecture to prove:

**Theorem 2.** Let \( \alpha \in H_2(M; \mathbb{Z}) \) where \( M \) is a compact simply connected P. L. 4-manifold. Then there is a proper topological imbedding (possible nonlocally flat) \( \theta : S^2 \times R \to M \times R \) (mapping ends to ends) with \( \theta_*[S^2 \times R] = \bar{\alpha} \in H_2(M \times R; \mathbb{Z}) \). \( \bar{\alpha} \) is the image of \( \alpha \) under \( \times R \). Proper, here, means inverse images of compact sets are compact.

In [2], we considered the problem of constructing smooth proper imbeddings, \( \theta \), and showed that if \( \alpha \) is characteristic (dual to \( w_2(\tau(M)) \)), the only obstruction to the existence of \( \theta \) is an Arf invariant which is equal to the Milnor-Kervaire number \( \left( \text{signature}(M) - \alpha \cdot \alpha/8 \right) \mod 2 \) when \( M \) is closed and that if \( \alpha \) is ordinary (not dual to \( W_2(\tau(M)) \)) there is no obstruction. This suggests two problems: (1) Can \( \theta \) always be arranged to be topologically locally flat, and (2) can \( \theta \) always be arranged to be P. L.?

Here is our "handle cancellation" theorem:

**Theorem 1.** Let \( M \) be any compact connected P. L. manifold of dimension \( m \) (assume \( M \) orientable if \( m = 3 \)). Let \( N \) be a compact connected codimension 0 submanifold of \( \partial M \). If \( \pi_1(M, N) = \emptyset \), then there is a codimension \( 0 \) submanifold, \( \bar{N} \), of \( M \) with: (1) \( N \subseteq \bar{N} \), (2) the inclusion \( N \hookrightarrow \bar{N} \) is a homotopy equivalence, (3) \( M = \bar{N} \cup 2\text{-handles} \cup 3\text{-handles} \cup \cdots \cup m\text{-handles} \).

**Note.** The P. L. category is convenient here since handle decompositions always exist.

**Proof.** If \( n \geq 5 \), the usual arguments for cancelling handles produce the desired \( \bar{N} \xrightarrow{P. L.} N \times I \) (see Appendix [3]). We need only consider the cases \( m = 3 \) or 4.

Let \( m = 4 \) and let \( \mathcal{H}(M, N) \) be a handle decomposition of \( M \) relative to \( N \). We may assume \( \mathcal{H}(M, N) \) has no zero-handles.

Let \( \{h_i\} = \{D_i \times D_i^{m-1}\} \) be the 1-handles. Let \( \{c_i\} \) be closed curves
on \( L_i \), the level after the 1-handles are attached, each consisting of \((D^i \times \text{pt.})\) for some \( \text{pt.} \in \partial D^i \) and an arc in \( N - \{D^i \times \partial D^i \} \). We claim that the latter arcs may be chosen so that each curve, \( c_i \), is null homotopic in \( X \overset{\text{def}}{=} M - (1\text{-handles of } \mathscr{H}(M, N)) \). Since \( m = 4 \), \( \pi_1(X) \to \pi_1(M) \) is an isomorphism. The arcs may be chosen (since \( \pi_1(N) \to \pi_1(M) \) is epic) so that each \( c_i \) represents \( 0 \in \pi_1(M) \) and, therefore, \( 0 \in \pi_1(X) \).

Let \( \{\gamma_j\} \) be the disjoint simple closed curves in \( L_i \) along which the 2-handles \( \{h_j\} \) are attached. Picking paths to the base \( * \), \( \{\gamma_j\} \) determines relations \( \{r_j\} \) and \( \pi_1(X) = \pi_1(L_i) / \langle r_j \rangle \). Choosing a path from \( c_i \) to \( * \), we have \( [c_i] \in \langle r_j \rangle \). So \( [c_i] = \prod_{k=1}^{n} u_kx_ku_k^{-1} \) where \( u_k \in \pi_1(L) \) and \( x_k \in \{r_j, r_j^{-1}\} \). For each curve \( c_i \), introduce a trivial oriented (2-handle, 3-handle) pair. Let \( h^j \) be the new 2-handle. Choose a path from \( \partial h^j \) to \( * \). Now perform a sequence of \( n \)-handle passings. \( h^j \) should be passed over the oriented (+ or - as \( x_k = r_j \) or \( r_j^{-1} \)) 2-handles corresponding to \( x_1, \ldots, x_n \) along arcs corresponding to the elements \( u_1 \cdots u_n \). The framing along each arc is immaterial so long as it restricts at the end points to a framing induced by the orientation of each 2-handle. Let \( \{\gamma_i\} \) be the curves along which \( \{h_j\} \) are attached after the above handle passings. \( \gamma_i \) is homotopic to \( c_i \). By the handle cancellation lemma [3], attaching 2-handles to \( \{c_i\} \) would result in a product \( N \times I \). Since homotopy type depends only on the homotopy class of attaching maps, \( N \overset{\text{def}}{=} N \cup \{h^j\} \cup \{h_i\} \overset{s.e.}{=} N \times I \). \( N \) has the desired properties.

Let \( m = 3 \). If \( \pi_1(N) = 0 \) then \( \pi_1(M) = 0 \) and \( M \) must be a homotopy \((S^3\text{-interior of closed disks})\). Let \( N = M - \{\text{closed disk } \cup \text{thickened arcs to } \partial \text{ components } \not\approx N\} \), so \( M = N \cup 2\text{-handles } \cup 3\text{-handles. We now assume } \pi_1(N) \neq 0 \).

If \( \pi_1(N) \to \pi_1(M) \) is an isomorphism, every imbedded 2-sphere in \( M \) separates \( M \), one component of the complement being a homotopy \( B^3 \) with finitely many punctures. Let \( \tilde{M} = M \cup_{\text{spherical } \partial \text{ components }} (3\text{-cells}) \). By the sphere theorem, \( \tilde{M} \) is a \( K(\pi, 1) \) so \((\tilde{M}, N)\) is an \( h \)-cobordism. But \( M \overset{\text{diff}}{=} \tilde{M} \cup 2\text{-handles, so } \tilde{M} \) satisfies the conditions for \( \tilde{N} \).

Assume \( \pi_1(N) \to \pi_1(M) \) is epi. By Dehn's lemma, if \( \pi_1(N) \to \pi_1(M) \) is not injective, there is an essential simple closed curve, \( \alpha \subset N \), bounding an imbedded 2-disk \( \beta \subset M \). Let \((M', N')\) be the result of ambient surgery (handle subtraction) along \( \beta \). \( \pi_0(N') \to \pi_0(M') \) is an isomorphism. (Proof: \( \beta(\alpha) \) disconnects \( M(N) \) if and only if there is no curve in \( M(N) \) meeting \( \beta(\alpha) \) algebraically once. Since \( H_1(N) \to H_1(M) \) is epi, there is a dual curve for \( \beta \) if and only if there is a dual curve for \( \alpha \).)
PROPOSITION. On each component, $\pi_1(N') \to \pi_1(M')$ is epi.

Proof. $M$ is obtained from $M'$ by attaching a 1-handle, and $N'$ is obtained from $N$ by the corresponding 0-surgery. The proposition can be deduced from the following group theoretic fact: Let $\theta: A \to X$, $\phi: B \to Y$ be group homomorphisms. If $\theta \ast \phi$ is epi, then $\theta$ and $\phi$ are epi.

Proceeding inductively (on the genus of the components of $N'$), we obtain $(M'', N'')$ with $\pi_1(N'') \to \pi_1(M'')$ an isomorphism on each component. This decomposes $M$ as:

Diagram 1:

\[
\begin{array}{c}
\text{2-handles} \\
\text{\textit{h}-cobordism} \\
\text{2-handles}
\end{array}
\]

By a theorem of J. Stallings [5], every $\textit{h}$-cobordism between orientable surfaces is the connected sum of a product and a homotopy 3-sphere, $\Sigma^3$. So we have:

Diagram 2:

\[
\begin{array}{c}
\text{2-handles} \\
\text{Product} - D^3 \\
\Sigma^3 - D^3 \\
\text{2 han dles}
\end{array}
\]

$\tilde{N} = N \times I \# (\Sigma^3 - D^3)$.

This completes the proof of Theorem 1.
Note. The orientation restriction in dimension 3 results from ignorance about $h$-cobordisms on $RP^2$.

Corollary. Let $M$ be a compact simply connected P. L. (= smooth) 4-manifold. $M \overset{\text{P. L.}}{\longrightarrow} K \cap 2$-handles $\cup 3$-handles $\cup 4$-handles, where $K$ is a compact contractable 4-manifold.

Proof. Apply Theorem 1 to $(M - D^4, \partial D^4)$. Let $K = \overline{N} \cup D^4$.

Remark. A. Casson has recently exhibited (unpublished work) a simply connected P. L. 4-manifold with boundary, $M$, with the property that every handle decomposition of $M$, $\mathcal{H}(M)$, must contain a 1-handle. This answers negatively a question raised in [4] on the existence of (relative) 2-spines. So the preceding corollary is all one can hope for.

Proof of Theorem 2. Let $M \overset{\text{P. L.}}{\longrightarrow} K \cup 2$-handles $\cup 3$-handles $\cup 4$-handles. Let $\tilde{M} = \text{cone}(\partial K) \cup 2$-handles $\cup 3$-handles $\cup 4$-handles. $H_2(M; Z) \cong H_2(\tilde{M}, \text{cone}(\partial K); Z) \cong H_2(\tilde{M}; Z)$. Any element of $H_2(\tilde{M}, \text{cone}(\partial K); Z)$ is represented by a relatively imbedded 2-disk constructed as a linear combination of 2-handles in the above handle decomposition by taking ambient boundary-connected-sums. So every element, $\alpha$, of $H_2(\tilde{M}; Z)$ is represented by a simplicial imbedding, $\omega$, of $S^2$ in $\tilde{M}$. By a theorem of R. Edwards, [1], $(\text{cone}(\partial K)) \times R$ is (topologically) homeomorphic to $K \times R$, $\tilde{M} \times R$ is (topologically) homeomorphic to $M \times R$. The composition:

$$S^2 \times R \xrightarrow{\omega \times \text{id}_R} \tilde{M} \times R \overset{\text{top. homeomorphism}}{\longrightarrow} M \times R$$

is the topological imbedding with the claimed properties.

References

1. R. Edwards, Unpublished results on the double suspension conjecture.
2. M. Fredman, A converse to (Milnor-Kervaire theorem) $\times R$, etc. ..., to appear.

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