THE IWASAWA IN Variant μ FOR QUADRATIC FIELDS

Frank E., III Gerth
THE IWASAWA INVARIANT $\mu$ FOR QUADRATIC FIELDS

FRANK GERTH III

We let $k_0$ be a quadratic extension field of the rational numbers, and we let $\mathfrak{l}$ be a rational prime number. In this paper we show that there exists a constant $c$ (depending on $k_0$ and $\mathfrak{l}$) such that the Iwasawa invariant $\mu(K/k_0) \leq c$ for all $\mathbb{Z}_\mathfrak{l}$-extensions $K$ of $k_0$. In certain cases we give explicit values for $c$.

1. Introduction. We let $\mathbb{Q}$ denote the field of rational numbers, and we let $\mathfrak{l}$ denote a rational prime number. We let $k_0$ be a finite extension field of $\mathbb{Q}$, and we let $K$ be a $\mathbb{Z}_\mathfrak{l}$-extension of $k_0$ (that is, $K/k_0$ is a Galois extension whose Galois group is isomorphic to the additive group of the $\mathfrak{l}$-adic integers $\mathbb{Z}_\mathfrak{l}$). We denote the intermediate fields by $k_0 \subseteq k_1 \subseteq k_2 \subseteq \cdots \subseteq k_n \subseteq \cdots \subseteq K$, where $\text{Gal}(k_n/k_0)$ is a cyclic group of order $\mathfrak{l}^n$. We let $A_n$ denote the $\mathfrak{l}$-class group of $k_n$ (that is, the Sylow $\mathfrak{l}$-subgroup of the ideal class group of $k_n$). In [5, §4.2], Iwasawa proves that $A_n = \mathfrak{l}e_n + \lambda n + \nu$

(1)

for $n$ sufficiently large, and $\mu, \lambda, \nu$ are rational integers (called the Iwasawa invariants of $K/k_0$) which are independent of $n$. Also $\mu \geq 0$ and $\lambda \geq 0$.

Next we let $W$ be the set of all $\mathbb{Z}_\mathfrak{l}$-extensions of $k_0$. If $K \in W$, we define

$$W(K, n) = \{ K' \in W | [K \cap K': k_0] \geq \mathfrak{l}^n \}.$$

Thus $W(K, n)$ consists of all $\mathbb{Z}_\mathfrak{l}$-extensions of $k_0$ that contain $k_n$, where $k_n$ is the unique subfield of $K$ such that $[k_n : k_0] = \mathfrak{l}^n$. We topologize $W$ by letting $\{ W(K, n) \}$ for $n = 1, 2, \ldots$ be a neighborhood basis for each $K \in W$. It can be proved that $W$ is compact with this topology (see [4, §3]). Next we let $W'$ be the set of $\mathbb{Z}_\mathfrak{l}$-extensions of $k_0$ with only finitely many primes lying over $\mathfrak{l}$. In [4, Proposition 3 and Theorem 4], Greenberg proves that $W'$ is an open dense subset of $W$ and that the Iwasawa invariant $\mu$ is locally bounded on $W'$. So if $K \in W'$, there exists an integer $n_0$ and a constant $c$ depending only on $K$ such that $\mu(K'/k_0) < c$ for all $\mathbb{Z}_\mathfrak{l}$-extensions $K'$ of $k_0$ with $[K \cap K': k_0] \geq \mathfrak{l}^{n_0}$. Greenberg suggests that perhaps $\mu$ is bounded on $W$; that is, perhaps there exists a constant $c$ such that $\mu(K'/k_0) < c$ for every $K' \in W$. If there is only one prime of $k_0$ above $\mathfrak{l}$, then Greenberg does prove in [4, Theorem 6] that $\mu$ is bounded on $W$

In this paper we shall prove that $\mu$ is bounded on $W$ if $k_0$ is a
quadratic extension of $Q$. We state this result as follows.

**Theorem 1.** Let $k_0$ be a quadratic extension of $Q$, and let $I$ be a rational prime number. Then there exists a constant $c$ (depending on $k_0$ and $I$) such that $\mu(K/k_0) \leq c$ for all $Z_r$-extensions $K$ of $k_0$.

2. **Proof of Theorem 1.** We let the notation be the same as in the previous section. We let $M$ be the composite of all $Z_r$-extensions of $k_0$, where $k_0$ is a finite extension field of $Q$. It is known (see [5, Theorem 3]) that $\text{Gal}(M/k_0) \approx Z^d$, where $r_1 + 1 \leq d \leq [k_0:Q]$ and $r_2$ is the number of complex archimedean primes of $k_0$. We note that when $k_0 = Q$, there is exactly one $Z_r$-extension $F$ of $Q$, and it is contained in the field obtained by adjoining to $Q$ all $n^{th}$ roots of unity for all $n$. Then for arbitrary $k_0$, the composite field $Fk_0$ is one of the $Z_r$-extensions of $k_0$. (It is called the cyclotomic $Z_r$-extension of $k_0$.)

We now specialize to the case where $k_0$ is a quadratic extension of $Q$. Then $1 \leq d \leq 2$. If $k_0$ is a real quadratic extension of $Q$, it is known that $d = 1$ (see [5, §2.3]). So there is a unique $Z_r$-extension $K$ of $k_0$, and hence the Iwasawa invariant $\mu$ is bounded on $W = \{K\}$. Next we suppose $k_0$ is an imaginary quadratic extension of $Q$. Then $d = 2$, and hence there are infinitely many $Z_r$-extensions of $k_0$, since there are infinitely many quotient groups of $Z_r$ isomorphic to $Z$. We note that when $k_0 = Q$, there is exactly one $Z_r$-extension $F$ of $Q$, and it is contained in the field obtained by adjoining to $Q$ all $n^{th}$ roots of unity for all $n$. Then for arbitrary $k_0$, the composite field $Fk_0$ is one of the $Z_r$-extensions of $k_0$. (It is called the cyclotomic $Z_r$-extension of $k_0$.)

We let $(I) = p_1p_2$, where $p_1$ and $p_2$ are primes of $k_0$. We recall from the theory of $Z_r$-extensions (see [5, Theorem 1]) that no primes other than $p_1$ and $p_2$ can ramify in a $Z_r$-extension of $k_0$. We let $L = Fk_0$, the cyclotomic $Z_r$-extension of $k_0$. Since $I$ ramifies totally in $F/Q$ and decomposes in $k_0/Q$, then $p_1$ and $p_2$ ramify totally in $L/k_0$. We let $I_1$ (resp., $I_2$) be the inertia group for $p_1$ (resp., $p_2$) for the extension $M/k_0$. (We note that we get the same inertia group for $p_i$ no matter what prime above $p_i$ in $M$ that we use because $M/k_0$ has abelian Galois group. A similar result holds for $p_2$.) Next we claim that $I_1 \approx Z_i$ and $I_2 \approx Z_i$. Since $p_1$ and $p_2$ are totally ramified in $L/k_0$, then $I_1$ and $I_2$ have quotient groups which are isomorphic to $\text{Gal}(L/k_0) \approx Z$. Also the completions of $k_0$ at $p_1$ and at $p_2$ are isomorphic to $Q$, and by local class field theory, the inertia group for the maximal abelian $l$-extension of $Q$ is isomorphic to the subgroup $U = \{1 + \alpha l | \alpha \in Z\}$ of the group of units of $Q$. Since $U \approx Z_l$ when $l \neq 2$, then $I_1$ and $I_2$ are isomorphic to quotient groups of $Z_l$ when $l \neq 2$. Combining the above results, we conclude that $I_1$ and $I_2$ are isomorphic to $Z_l$.
when \( I \neq 2 \). When \( I = 2 \), \( U \cong \mathbb{Z}_2 \times (\mathbb{Z}_2/2\mathbb{Z}_2) \), and we still get \( I_1 \cong \mathbb{Z}_2 \) and \( I_2 \cong \mathbb{Z}_2 \) since \( I_1 \) and \( I_2 \) are subgroups of \( \text{Gal}(M/k_0) \cong \mathbb{Z}_2 \).

Now since \( \text{Gal}(M/k_0) \cong \mathbb{Z}_2^2 \), \( I_1 \cong \mathbb{Z}_2 \), and \( p_1 \) and \( p_2 \) are totally ramified in \( L/k_0 \), then \( \text{Gal}(M/k_0) / I_1 \cong \mathbb{Z}_2 \) and \( \text{Gal}(M/k_0) / I_2 \cong \mathbb{Z}_2 \). Thus there exists exactly one \( \mathbb{Z}_2 \)-extension \( K_j/k_0 \) (resp., \( K_2/k_0 \)) in which \( p_1 \) (resp., \( p_2 \)) is unramified. So if \( K \) is any \( \mathbb{Z}_2 \)-extension of \( k_0 \) other than \( K_1 \) and \( K_2 \), then both \( p_1 \) and \( p_2 \) are ramified in \( K/k_0 \) (although not necessarily totally ramified). Then there are only finitely many primes of \( K \) above \( I \), and hence by the results of Greenberg in [3], there is a neighborhood of \( K \) in \( W \) on which \( \mu \) is bounded. Suppose we could show that \( K_1 \) and \( K_2 \) have neighborhoods on which \( \mu \) is bounded. Then all \( K \in W \) would have neighborhoods on which \( \mu \) is bounded. Since \( W \) is compact, \( W \) is covered by a finite number of these neighborhoods, and hence \( \mu \) would be bounded on \( W \). So to complete the proof of Theorem 1, it suffices to show that \( \mu \) is bounded on some neighborhood of \( K_1 \) and on some neighborhood of \( K_2 \).

We consider \( K_i/k_0 \) with intermediate fields \( k_0 \subset k_1 \subset k_2 \subset \cdots \subset k_n \subset \cdots \subset K_1 \). Since \( p_1 \) is unramified in \( K_1/k_0 \), then \( p_2 \) must ramify in \( K_1 \) since by class field theory the maximal unramified abelian extension of \( k_0 \) is of finite degree over \( k_0 \). So there are only finitely many primes of \( K_1 \) above \( p_2 \). Let \( t \) denote that finite number. Next we recall that \( W(K_1, n) = \{ K' \in W \mid [K_1 \cap K': k_0] \geq n \} \), and these sets \( W(K_1, n) \) for \( n = 1, 2, \ldots \), form a neighborhood basis for \( K_1 \) in \( W \). Since \( \text{Gal}(M/k_0) \cong \mathbb{Z}_2^2 \) and \( F \) and \( K_1 \) are disjoint \( \mathbb{Z}_2 \)-extensions of \( k_0 \), then it is clear that \( M = FK_1 \). If \( f_1 \) is the subfield of \( F \) such that \( [f_1:k_0] = 1 \), then every \( K' \in W(K_1, n) \) has a subfield \( k_{n+1}' \) such that \( [k_{n+1}': k_n] = 1 \) and \( k_{n+1}' \subset f_1 k_{n+1} \). We take \( n \) large enough so that \( \lambda^n > t \). Unless \( k_{n+1}' = k_{n+1} \), there are at most \( \lambda^n \) (resp., \( t \)) primes of \( k_{n+1}' \) above \( p_1 \) (resp., \( p_2 \)). Then if \( k_{n+1}' \neq k_{n+1} \), there are at most \( \lambda^n \) (resp., \( t \)) primes of \( K' \) above \( p_1 \) (resp., \( p_2 \)). If we let \( s \) denote the number of primes of \( K' \) that are ramified over \( k_0 \), then \( s \leq \lambda^n + t \). From [3, Theorem 1], we see that

\[
\mu(K'/k_0) \leq e'_{n+1}/(\lambda^{n+1} - s + 1) \leq e'_{n+1}/(\lambda^{n+1} - \lambda^n - t + 1),
\]

where \( \lambda^{n+1} \) is the order of the \( \lambda \)-class group of \( k_{n+1}' \). Since \( [f_1:k_{n+1}; k_{n+1}'] = 1 \), then by class field theory \( e'_{n+1} \leq e_{n+1} + 1 \), where \( 1^{n+1} \) is the order of the \( \lambda \)-class group of \( f_1 k_{n+1} \). So if \( K' \in W(K_1, n) \) and \( k_{n+1}' \neq k_{n+1} \), then

\[
\mu(K'/k_0) \leq (e_{n+1} + 1)/(\lambda^{n+1} - \lambda^n - t + 1).
\]

Now \( f_1 K_1 \) is a \( \mathbb{Z}_2 \)-extension of \( f_1 \). From Equation 1, \( \varepsilon_n = \mu_1 \lambda^n + \lambda_1 n + \nu_1 \) for \( n \) sufficiently large, where \( \mu_1 = \mu(f_1 K_1/f_1) \), \( \lambda_1 = \lambda(f_1 K_1/f_1) \), \( \nu_1 = \nu(f_1 K_1/f_1) \). So for \( n \) sufficiently large,

\[
\varepsilon_{n+1} + 1 = \mu_1 \lambda^{n+1} + \lambda_1 (n + 1) + \nu_1 + 1
\]
and

\[
\mu(K'/k_0) \leq (s_{n+1} + 1)/(l^{n+1} - l^n - t + 1) = \frac{\mu_l^{n+1} + \lambda_l(n + 1) + \nu_l + 1}{l^{n+1} - l^n - t + 1}.
\]

Since

\[
\lim_{n \to \infty} \frac{\mu_l^{n+1} + \lambda_l(n + 1) + \nu_l + 1}{l^{n+1} - l^n - t + 1} = \frac{\mu_l}{1 - l^{-1}} < 3\mu_l,
\]

we see that for \( n \) sufficiently large, \( \mu(K'/k_0) < 3\mu_l \) for all \( K' \in W(K, n) \). So \( \mu \) is bounded on some neighborhood of \( K \). Similarly \( \mu \) is bounded on some neighborhood of \( K_0 \). Hence our proof of Theorem 1 is complete.

3. Explicit upper bounds for \( \mu \) in certain cases. We first consider a real quadratic extension \( k_0/Q \). Then there is only one \( \mathbb{Z}_l \)-extension \( K \) of \( k_0 \), namely the cyclotomic \( \mathbb{Z}_l \)-extension of \( k_0 \). It is known that \( \mu(K/k_0) = 0 \) in this case (see [2]).

Now we consider an imaginary quadratic extension \( k_0/Q \). We first suppose that \( l \) ramifies or remains prime in \( k_0 \). We let \( H \) denote the maximal unramified abelian \( l \)-extension of \( k_0 \), and we let \( l^\alpha \) be the exponent of \( \text{Gal}(H/k_0) \). If \( K \) is any \( \mathbb{Z}_l \)-extension of \( k_0 \) with intermediate fields \( k_0 \subset k_1 \subset k_2 \subset \cdots \subset k_\alpha \subset \cdots \subset K \), then the primes above \( l \) in \( k_\alpha \) ramify totally in \( K/k_\alpha \), and there are at most \( l^\alpha \) such primes. Then from [3, Theorem 1], we see that \( \mu(K/k_0) \leq e_\alpha \), where \( l^\alpha = |A_\alpha| \). So in Theorem 1, we may take \( c \) to be the maximum of the \( e_\alpha \) obtained from the extensions \( k_\alpha \) of \( k_0 \) such that \( k_\alpha \) is contained in a \( \mathbb{Z}_l \)-extension of \( k_0 \) and \( [k_\alpha:k_0] = l^\alpha \). Frequently we can obtain a better upper bound for \( \mu \). For example, if \( M \) is the composite of all \( \mathbb{Z}_l \)-extensions of \( k_0 \) and if \( M \cap H = k_0 \), then the prime of \( k_0 \) above \( l \) is totally ramified in each \( \mathbb{Z}_l \)-extension of \( k_0 \), and hence from [3, Corollary 1], \( \mu(K/k_0) \leq e_0 \) for each \( \mathbb{Z}_l \)-extension \( K \) of \( k_0 \).

Finally we suppose that \( k_0 \) is an imaginary quadratic extension of \( Q \) and that \( l \) decomposes in \( k_0 \). In this case we shall give an explicit upper bound for \( \mu \) only under certain conditions. We let \( M \) be the composite of all \( \mathbb{Z}_l \)-extensions of \( k_0 \), and we let \( M_1 \) be the maximal extension of \( k_0 \) contained in \( M \) such that \( \text{Gal}(M_1/k_0) \) has exponent \( l \). We note that \( \text{Gal}(M_1/k_0) \approx (\mathbb{Z}_l/\mathbb{Z}_l)^3 \) since \( \text{Gal}(M/k_0) \approx \mathbb{Z}_l^3 \), and hence \( M_1 \) contains \( l + 1 \) subfields of degree \( l \) over \( k_0 \). We let \( (l) = \pi_1 \) and \( \pi_2 \) are primes in \( k_0 \). We shall assume that there is exactly one prime of \( M_1 \) above \( \pi_1 \) and exactly one prime of \( M_1 \) above \( \pi_2 \). (Note: From our discussion in §2 and our definition of \( M_1 \), we see that there is exactly one prime of \( M_1 \) above \( \pi_1 \) precisely when \( \pi_1 \) remains prime in one of the extensions of \( k_0 \) of degree \( l \) and
ramifies in the other $I$ extensions of degree $l$ over $k_0$. A similar result applies to $p_2$.) Then there is exactly one prime of $M$ above $p_1$ and exactly one prime of $M$ above $p_2$. It then follows from [3, Corollary 2] that we may take $c$ in Theorem 1 to be the maximum of the numbers $e_i/(l-1)$ obtained from the fields $k_i$ contained in $M$, with $[k_i:k_0]=l$. As usual, $l^*\iota$ is the order of the $I$-class group of $k_i$.

In some of these situations where $I$ decomposes in $k_0$, we can actually find $\mu$, $\lambda$, $\nu$ exactly for every $Z_l$-extension of $k_0$. We assume that $I$ does not divide the class number of $k_0$. We let $M_i$ be the maximal extension of $k_0$ contained in $M$ such that $\text{Gal}(M_i/k_0)$ has exponent $l^\iota$. (We note that $\text{Gal}(M_i/k_0) \cong (Z_l/l^\iota Z_l)^*$. We also assume that there is exactly one prime of $M_i$ above $p_1$ and exactly one prime of $M_i$ above $p_2$. Then there is only one prime of $M_i$ above $p_1$ for each $i$, and only one prime of $M_i$ above $p_2$ for each $i$. We recall from §2 that there is a unique $Z_l$-extension $K_i$ (resp., $K_2$) of $k_0$ in which $p_1$ (resp., $p_2$) is unramified. Since $I$ does not divide the class number of $k_0$, then $p_2$ (resp., $p_1$) is totally ramified in $K_i$ (resp., $K_2$). So $K_i$ (resp., $K_2$) is a $Z_l$-extension of $k_0$ in which exactly one prime is ramified, and that prime is totally ramified. Since $I$ does not divide the class number of $k_0$, then $I$ does not divide the class number of $k_0$, then $I$ does not divide the class number of every subfield of $K_i$ (resp., $K_2$). (See [6].) So $\mu(K_i/k_0) = \lambda(K_i/k_0) = \nu(K_i/k_0) = 0$ and $\mu(K_2/k_0) = \lambda(K_2/k_0) = \nu(K_2/k_0) = 0$. If $K_i$ has subfields $k_0 \subset k_1 \subset k_2 \subset \cdots \subset k_n \subset \cdots \subset K_i$, we note that $\text{Gal}(M_i/k_0)$ is a cyclic group of order $l^\iota$ for each $i$. Since $I$ does not divide the class number of $k_0$, then $I$ does not divide the class number of $k'_i$, and since there is only one prime of $M_i$ (namely the prime of $M_i$ above $p_1$) that is ramified over $k'_i$, we see that $I$ does not divide the class number of $M_i$, for each $i$. Now we let $K$ be any $Z_l$-extension of $k_0$ with intermediate fields $k_0 \subset k_1 \subset k_2 \subset \cdots \subset k_n \subset \cdots \subset K$, and we suppose $K_i$ has intermediate fields $k'_0 \subset k'_1 \subset k'_2 \subset \cdots \subset k'_n \subset \cdots \subset K_i$. If $K \cap K_i = k_0$ and $K \cap K_2 = k_0$, then $p_1$ and $p_2$ are totally ramified in $k'_n/k_0$, and then $M_n/k_n$ is an unramified cyclic extension of degree $l^\iota$. Since $I$ does not divide the class number of $M_n$, then $M_n$ must be the Hilbert $I$-class field of $k_n$, and hence by class field theory the $I$-class group of $k_n$ is a cyclic group of order $l^\iota$ for all $n$. So $\mu(K/k_0) = 0$, $\lambda(K/k_0) = 1$, $\nu(K/k_0) = 0$. Now suppose $K \cap K_1 = k'_j$. By arguments similar to those above, it can be proved that the $I$-class group of $k_n$ is trivial if $n \leq j$ and a cyclic group of order $l^{\iota-j}$ if $n > j$. So $\mu(K/k_0) = 0$, $\lambda(K/k_0) = 1$, $\nu(K/k_0) = -j$. Similarly if $K \cap K_2 = k'_j$, then $\mu(K/k_0) = 0$, $\lambda(K/k_0) = 1$, $\nu(K/k_0) = -j$.

We conclude with an example to which the results of the previous paragraph apply. We let $k_0 = Q(\sqrt{-11})$ and $l = 3$. We note that $3$ does not divide the class number of $k_0$, and $3$ decomposes in $k_0$ (in face, $3 = \alpha_1 \alpha_2$ with $\alpha_1 = (1 + \sqrt{-11})/2$ and $\alpha_2 = (1 - \sqrt{-11})/2$). If $M_i$ is the maximal extension of $k_0$ of exponent $I$ contained in the
composite of all $\mathbb{Z}_p$-extensions of $k_0$, we must show that there is only one prime ideal of $M_i$ above $(\alpha_1)$ and only one prime ideal of $M_i$ above $(\alpha_2)$. Then the results of the previous paragraph will apply to $k_0$. Now we let $E = \mathbb{Q}(\sqrt{-11}, \zeta)$, where $\zeta = (-1 + \sqrt{-3})/2$ (a primitive cube root of unity). Then $[E : \mathbb{Q}] = 4$, and the three quadratic subfields are $k_0, \mathbb{Q}(\sqrt{33}), \mathbb{Q}(\sqrt{-3})$. We note that there is exactly one prime of $E$ above $(\alpha_1)$ and exactly one prime of $E$ above $(\alpha_2)$. Since 3 does not divide the class numbers of the quadratic subfields of $E$, then it is easy to see that 3 does not divide the class number of $E$. It then follows from Kummer theory that the maximal abelian extension of $E$ of exponent 3 in which only primes above 3 are ramified is $E(\alpha_1^{1/3}, \alpha_2^{1/3}, \zeta^{1/3}, \epsilon^{1/3})$, where $\epsilon = 23 + 4\sqrt{33}$ is the fundamental unit of $\mathbb{Q}(\sqrt{33})$. It is not difficult to see that $M_iE = E(\zeta^{1/3}, \epsilon^{1/3})$ (cf. [1, Example 3]). Again using Kummer theory, a calculation shows that the prime of $E$ above $(\alpha_1)$ remains prime in one of the cubic extensions of $E$ contained in $M_iE$ and ramifies in the other three cubic extensions of $E$ contained in $M_iE$. A similar result is valid for the prime of $E$ above $(\alpha_2)$. It follows that there can be only one prime of $M_i$ above $(\alpha_1)$ and only one prime of $M_i$ above $(\alpha_2)$. Hence the results of the previous paragraph apply to $k_0 = \mathbb{Q}(\sqrt{-11})$.

Note. We have learned that the Russian mathematician V. A. Babaicev has obtained by other methods a proof of Theorem 1 (see Math. USSR Izvestija, 10 (1976), 675-685).

References


Received April 4, 1978. Partly supported by NSF Grant MCS 78-01459.
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Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan
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