THREE-DIMENSIONAL OPEN BOOKS CONSTRUCTED FROM THE IDENTITY MAP

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Three-dimensional manifolds are constructed as open books, using the identity diffeomorphism. The open book constructed in this way with (non)orientable page of Euler characteristic $\chi$ is the connected sum of $(1-\chi)$ copies of the (non)orientable $S^2$ bundle over $S^1$.

Introduction. We investigate orientable and nonorientable three-dimensional manifolds which are open books according to the following definition of Winkelnkemper [2].

DEFINITION. A manifold of dimension $n$ is said to have an open book description if it can be constructed using a co-dimension 2 submanifold $\partial V$ and a diffeomorphism $h:V\to V$ of an $(n-1)$-dimensional manifold with boundary $\partial V$. $h$ is required to be the identity map in a neighborhood of $\partial V$. The construction is to form the mapping torus $(V \times I)/(v,0) = (h(v),1)$ and then to identify $(v,t) = (v,t')$ for all $v$ in $\partial V$ and $t, t'$ in $I$. The image of the copies of $\partial V$ in the resulting manifold is called the binding of the open book and the circle's worth of copies of $V$ are called the pages.

Related results appear in the recent book of Rolfsen [1].

Statement of results.

THEOREM 1. If $V = S_g - n\hat{B}^2$, the surface of genus $g$ with $n$ disjoint, open discs removed from it, then the open book produced by setting $h$ equal to the identity map is the connected sum of $(2g+(n-1))$ copies of $(S^1 \times S^2)$. (Adopt the convention that zero copies of $(S^1 \times S^2)$ will refer to $S^1$.)

THEOREM 2. If $V = P_k - n\hat{B}^2$, the 2-sphere with $k$ cross-caps attached and $n$ disjoint, open discs removed from it, then the open book produced by setting $h$ equal to the identity map is the connected sum of $(k+(n-1))$ copies of the Klein bottle of dimension three. ($k \geq 1, n \geq 1$)

By the three-dimensional Klein bottle we mean the nonorientable $S^2$ bundle over $S^1$, $(S^2 \times I)/(x,y,z,0) = (-x,y,z,1)$.

Proofs of results.

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**Lemma 1.** Let $M$ be a closed, smooth manifold of dimension $(n + 1)$. If an unkotted copy of $(S^1 \times \mathbb{B}^n)$ is removed from a coordinate patch on $M$ and the identification $(\theta, x) = (\theta', x)$ is performed for all $(\theta, x)$ in $(S^1 \times S^{n-1})$ then the resulting manifold is the connected sum $M \# (S^2 \times S^{n-1})$.

**Proof.** Remove a copy of $\mathbb{B}^{n+1}$ which contains the bounding $(S^1 \times S^{n-1})$ and temporarily add a copy of $\mathbb{B}^{n+1}$ to it, giving $S^{n+1} - (S^1 \times S^*)$. The identifications glue all the meridian $(n - 1)$-spheres to one copy of $S^{n-1}$ on the boundary of the removed torus. On the bounding $(S^1 \times S^{n-1})$ in $S^{n+1} - (S^1 \times S^*) = (B^2 \times S^{n-1})$, the $(n - 1)$-spheres are parallels. When these are all identified to one $S^{n-1}$ we obtain $(S^2 \times S^{n-1})$. Now remove the superfluous copy of $\mathbb{B}^{n+1}$ and form the connected sum of $M - \mathbb{B}^{n+1}$ with $(S^2 \times S^{n-1}) - \mathbb{B}^{n+1}$ to finish the proof.

**Proof of Theorem 1.** Consider the polygonal normal form of $S_\gamma a_1 b a_1^{-1} b_1^{-1} \cdots a_n b a_n^{-1} b_n^{-1}$. Punch $n$ holes in it and form the Cartesian product with the unit interval.

![Diagram](image1)

We diffeomorph one of the inner cylinders to the outside and form the mapping torus. If we perform the required identifications on the outer copy of $(S^i \times S^i)$ we obtain $S^\gamma - \{n \text{ solid tori}\}$. The $(n - 1)$ copies of $(S^i \times S^i)$ which do not come from the $a_i b_i \cdots a_n^{-1} b_n^{-1}$ each contribute a connected sum of $S^3$ with $(S^i \times S^2)$ when the required identifications are performed. This follows from the absence of linking and Lemma 1.

The remaining $(S^1 \times S^i)$ can be surgered out in a $\mathbb{B}^3$ as in Lemma

![Diagram](image2)
1 and an extra $B^3$ added. Since the $a_t$ and $b_t$ were meridians on the removed $(S^1 \times B^3)$ they are parallels on the remaining $(S^1 \times B^3) = S^3 - (S^1 \times B^2)$. An identification such as this, pictured in Figure 2, gives the connected sum of $2g$ copies of $(S^1 \times S^2)$. The four vertical discs give the union of two $S^2$'s joined along a common equator. This configuration is $S^3 - 4B^3$ and we now attach two copies of $S^2 \times I$. A separating $S^2$ between the two handles can be constructed using four of the discs with the flanges shown in Figure 3. One quarter of the $S^2$ consists of the two curved half-flanges, and the sub-disc in a vertical disc from Figure 2.

We now complete the proof by removing the extra $B^3$ which we added above and forming the required connected sum.

Proof of Theorem 2. The proof is analogous. The two extra ingredients are to notice that the connected sum of $(S^1 \times S^n)$ with the $(n+1)$-dimensional Klein bottle is diffeomorphic to the connected sum of two $(n+1)$-dimensional Klein bottles and that an identification such as that shown in Figure 4 gives a connected sum of two Klein bottles of dimension 3.

![Figure 4](image4.png)

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