COMPACT OPERATORS OF THE FORM $uC_\phi$

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If $A$ is the disc algebra, the uniform algebra of functions analytic on the open unit disc $D$ and continuous on its closure, and if $u, \varphi \in A$ with $||\varphi|| \leq 1$, then the operator $uC\varphi$ is defined on $A$ by $uC\varphi: f(z) \to u(z)f(\varphi(z))$. In this note we characterize compact operators of this form and determine their spectra.

We recall that a bounded linear operator $T$ from a Banach space $B_1$ to a Banach space $B_2$ is compact if given a bounded sequence $\{x_n\}$ in $B_1$, there exists a subsequence $\{x_{n_k}\}$ such that $\{Tx_{n_k}\}$ converges in $B_2$.

If $\varphi: \overline{D} \to \overline{D}$, we let $\varphi_n$ denote $n^{th}$ the iterate of $\varphi$, i.e., $\varphi_0(z) = z$ and $\varphi_n(z) = \varphi(\varphi_{n-1}(z))$ for $z \in \overline{D}$ and $n \geq 1$. Our main result is the following.

**Theorem.** Let $u \in A$, $\varphi \in A$, $||\varphi|| \leq 1$ and suppose $\varphi$ is not a constant function.

I. The operator $uC\varphi$ is compact if, and only if, $|\varphi(z)| < 1$ whenever $u(z) \neq 0$.

II. Suppose $uC\varphi$ is compact and let $z_0 \in \overline{D}$ be the unique fixed point of $\varphi$ for which $\varphi_n(z) \to z_0$ for all $z \in D$. If $|z_0| = 1$, then $uC\varphi$ is quasinilpotent, while if $|z_0| < 1$, the spectrum $\sigma(uC\varphi) = \{u(z_0)\varphi'(z_0)^n | n$ is a positive integer $\} \cup \{0, u(z_0)\}$.

1. Characterization of compact $uC\varphi$. We first consider the easy case in which $\varphi$ is a constant function.

**Theorem 1.1.** Suppose $u \in A$ and $\varphi(z) = a \in \overline{D}$ for all $z \in \overline{D}$. Then $uC\varphi$ is compact.

**Proof.** Since $\varphi(z) = a$ for all $z \in \overline{D}$, $(uC\varphi)f(z) = u(z)f(\varphi(z)) = f(a)u(z)$. Therefore the range of $uC\varphi$ is one-dimensional and so $uC\varphi$ is compact.

We next give a necessary and sufficient condition that $uC\varphi$ be a compact operator for those $\varphi$ which are not constant functions.

**Theorem 1.2.** Suppose $u \in A$, $\varphi \in A$, $||\varphi|| \leq 1$ and $\varphi$ is not a constant function. Then $uC\varphi$ is a compact operator on $A$ if, and only if, $|\varphi(z)| < 1$ whenever $u(z) \neq 0$.  

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Proof. Since everything holds if \( u = 0 \), we will assume that \( u \) is not identically zero.

1. Suppose \( uC_\varphi \) is a compact operator on \( A \). We must prove that if \( z \in \bar{D} \) and \( u(z) \neq 0 \), then \( |\varphi(z)| < 1 \). Since \( \varphi \) is not a constant function, the maximum modulus principle implies that \( |\varphi(z)| < 1 \) whenever \( |z| < 1 \) and thus it suffices to show that \( |\varphi(z)| < 1 \) when \( u(z) \neq 0 \) and \( z \) lies on the unit circle. Assume the contrary and let \( \theta \) satisfy \( u(e^{i\theta}) \neq 0 \) and \( |\varphi(e^{i\theta})| = 1 \). Set \( \mu = \varphi(e^{i\theta}) \) and for each positive integer \( n \), define \( f_n \) by \( f_n(z) = (\frac{1}{2}(z + \mu))^{n} \). Then \( \|f_n\| = 1 \).

Since \( uC_\varphi \) is assumed to be compact, there exists a subsequence \( \{f_{n_k}\} \) and a function \( F \) in \( A \) with \( (uC_\varphi)f_{n_k} \to F \) in \( A \). That is, \( u(z)(\frac{1}{2}(\varphi(z) + \mu))^{n_k} \to F(z) \) uniformly for \( z \in \bar{D} \). But \( (\frac{1}{2}(\varphi(z) + \mu))^{n_k} \to 0 \) for \( |z| < 1 \) and so \( F(z) = 0 \) on \( D \). However, \( F \) is continuous on \( \bar{D} \) and therefore \( F(z) = 0 \) on \( \bar{D} \). Hence \( (uC_\varphi)f_{n_k} \to 0 \) uniformly on \( \bar{D} \). In particular, \( u(e^{i\theta})(\frac{1}{2}(\varphi(e^{i\theta}) + \mu))^{n_k} \to 0 \). But for all \( k \), we have \( |u(e^{i\theta})(\frac{1}{2}(\varphi(e^{i\theta}) + \mu))^{n_k}| = |u(e^{i\theta})| \neq 0 \). This is a contradiction. Hence if \( uC_\varphi \) is compact and \( u(z) \neq 0 \), then \( |\varphi(z)| < 1 \).

2. Conversely, assume \( |\varphi(z)| < 1 \) whenever \( u(z) \neq 0 \). To show that \( uC_\varphi \) is compact, assume \( f_n \in A \) and \( ||f_n|| \leq 1 \). Since \( \{f_n\} \) is a uniformly bounded sequence of functions on \( D \), it is a normal family in the sense of Montel and so there exists a subsequence \( \{f_{n_k}\} \) and a function \( g \) analytic on \( D \) with \( f_{n_k} \to g \) uniformly on compact subsets of the open disc \( D \). We observe that this convergence implies \( \sup_{|w| < 1} |g(w)| \leq 1 \). Now defined a function \( G \) on the closed disc \( \bar{D} \) by setting \( G(z) = 0 \) whenever \( |z| = 1 \) and \( u(z) = 0 \), and letting \( G(z) = u(z)g(\varphi(z)) \) otherwise. We claim that \( G \in A \) and \( (uC_\varphi)f_{n_k} \to G \) uniformly on \( \bar{D} \).

We first show that \( G \) is continuous on \( \bar{D} \). Indeed, \( G \) is continuous on \( \{z|u(z) \neq 0\} \) since \( |\varphi(z)| < 1 \) on this set and \( g \) is continuous on \( D \). Further, if \( |z^*| = 1 \) and \( u(z^*) = 0 \), let \( \{z_m\} \) be a sequence in \( \bar{D} \) converging to \( z^* \). For each \( m \), \( G(z_m) = 0 \) or \( G(z_m) = u(z_m)g(\varphi(z_m)) \). Since \( |g(\varphi(z_m))| \leq 1 \) it follows that \( \lim_{m \to \infty} G(z_m) = 0 = G(z^*) \) and so \( G \) is continuous at each \( z \in \bar{D} \). Also \( G \) is analytic on \( D \) since \( u \) and \( g \circ \varphi \) are analytic on \( D \). Hence \( G \in A \).

To show that \( (uC_\varphi)f_{n_k} \to G \) uniformly on \( \bar{D} \), let \( V = \{e^{i\theta}|u(e^{i\theta}) = 0\} \) and suppose \( \varepsilon > 0 \). Since \( u \) is continuous, there exists an open set \( U \supset V \) for which \( |u(t)| < \varepsilon \) for \( t \in U \). Also since \( |u(z)| < 1 \) for \( z \in U \) and \( \bar{D} \setminus U \) is a compact set, there exists \( r \), \( 0 < r < 1 \), such that \( |\varphi(z)| \leq r \) for \( z \in U \). Moreover, since \( f_{n_k} \to g \) uniformly on compact subsets of \( D \), \( u(z)f_{n_k}(\varphi(z)) \to u(z)g(\varphi(z)) \) uniformly for \( z \in U \). That is, there exists an integer \( N \) such that \( |u(z)f_{n_k}(\varphi(z)) - G(z)| = \)
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$|u(z)f_{n_k}(\phi(z)) - u(z)g(\phi(z))| < \varepsilon$ for $k \geq N$ and all $z \in U$. On the other hand, for $z \in U \setminus V$ and for all $k$,

$$|(uC\phi)f_{n_k}(z) - G(z)| = |u(z)f_{n_k}(\phi(z)) - u(z)g(\phi(z))| \leq \sup_{z \in U \setminus V} \left[ |u(z)||f_{n_k}(\phi(z)) - g(\phi(z))| \right] \leq \varepsilon[||f_{n_k}|| + ||g||_\infty] = 2\varepsilon.$$

Finally, if $z \in V$, then $(uC\phi)f_{n_k}(z) = u(z)f_{n_k}(\phi(z)) = 0 = G(z)$. Hence given $\varepsilon > 0$, there exists an integer $N$ such that $|(uC\phi)f_{n_k}(z) - G(z)| < 2\varepsilon$ for $k \geq N$ and all $z \in \bar{D}$. That is, $(uC\phi)f_{n_k} \rightarrow G$ uniformly. Thus if $|\phi(z)| < 1$ whenever $u(z) = 0$, then the operator $uC\phi$ is compact.

2. Spectra of compact $uC\phi$. If $T$ is a bounded linear operator from $A$ to $A$ we let $\sigma(T)$ denote the spectrum of $T$. As before, we first consider the case where $\phi$ is a constant function.

**Theorem 2.1.** Suppose $u \in A$ and $\phi(z) = a \in \bar{D}$ for all $z \in \bar{D}$.

Then $\sigma(uC\phi) = \{0, u(a)\}$.

**Proof.** $0$ and $u(a)$ are both eigenvalues $uC\phi$. For, if $F(z) = z - a$, then $(uC\phi)F(z) = u(z)F(\phi(z)) = u(z)F(a) = 0$, while if $G(z) = u(z)$, then $(uC\phi)G(z) = u(z)G(\phi(z)) = u(a)G(z)$. Thus $\{0, u(a)\} \subset \sigma(uC\phi)$.

On the other hand, since the range of $uC\phi$ is one-dimensional, $\sigma(uC\phi)$ contains at most two elements and therefore $\sigma(uC\phi) = \{0, u(a)\}$.

In determining the spectra of the remaining compact operators of the form $uC\phi$ we will make use of the following theorem of Denjoy and Wolf.

**Theorem A** (Denjoy [2], Wolf [6]). Suppose $\phi$ is an analytic function mapping $D$ to $D$. If $\phi$ is not conformally equivalent to a rotation about a fixed point, then there exists a unique $z' \in \bar{D}$ for which $\phi_n(z) \rightarrow z'$ for all $z \in D$. If $\phi$ is continuous at $z'$, then $\phi(z') = z'$.

Suppose $\phi \in A$ and $\phi: D \rightarrow D$. It is easy to show that if $\phi \neq z$, then there is at most one fixed point of $\phi$ in the open disc $D$. There may, however, be infinitely many fixed points on the boundary of $D$. However, if the function $\phi$ is not equivalent to a rotation, then Theorem A asserts that there exists a unique fixed point $z_0 \in \bar{D}$, which we call the Denjoy-Wolf fixed point of $\phi$, for which $\phi_n(z) \rightarrow z_0$ for all $z \in D$. The spectrum of a compact operator of the form $uC\phi$ will depend on the location of the Denjoy-Wolf fixed point of $\phi$.

**Theorem 2.2.** Suppose $u \in A$, $\phi \in A$, $||\phi|| = 1$, $\phi$ is not a constant
function and \( \varphi \) has all its fixed points on the unit circle. If \( uC_\varphi \) is a compact operator, then \( uC_\varphi \) is quasinilpotent.

**Proof.** Let \( z_0 \) be the Denjoy-Wolf fixed point of \( \varphi \), which by hypothesis has modulus 1. Since \( uC_\varphi \) is compact, Theorem 1.2 implies \( u(z_0) = 0 \). Let \( V = \{ e^{i\theta} | u(e^{i\theta}) = 0 \} \).

Choose \( \varepsilon > 0 \). As in Theorem 1.2 there exists an open set \( U \) such that \( U \supset V \) and \( |u(t)| < \varepsilon \) for all \( t \in U \). Also, since \( \overline{D} \setminus U \) is compact there exists \( r, 0 < r < \varepsilon \), such that \( |\varphi(w)| < r \) for all \( w \in \overline{D} \setminus U \). Since \( \{ \varphi_n \} \) is a bounded sequence and hence a normal family, there exists a subsequence \( \{ \varphi_{n_k} \} \) such that \( \{ \varphi_{n_k}(w) \} \) converges uniformly on compact subsets of \( D \). In particular, \( \{ \varphi_{n_k}(w) \} \) converges uniformly for \( |w| \leq r \). But \( \varphi_{n_k}(z) \to z_0 \) for all \( z \in D \). It follows that \( \{ \varphi_n(z) \} \) converges uniformly to \( z_0 \) for \( |z| \leq r \).

Now choose \( \delta > 0 \) such that \( \{ s \in \overline{D} | s - z_0 | < 2 \delta \} \subset U \). Since \( \varphi_n(w) \to z_0 \) uniformly for \( |w| \leq r \), there exists a positive integer \( N \) such that \( |\varphi_n(w) - z_0 | < \delta \) if \( n \geq N \) and \( |w| \leq r \). Thus \( \varphi_n(\{w ||w| \leq r \} ) \subset U \) for \( n \geq N \). Therefore, for each \( z \in \overline{D} \) and each positive integer \( n \), at most \( N \) elements from \( z, \varphi(z), \ldots, \varphi_n(z) \) lie in \( \overline{D} \setminus U \). From the definition of \( U \), if \( t \in U \), then \( |u(t)| < \varepsilon \). Hence for all \( z \in \overline{D} \) and \( n \geq N \),

\[
\left| \left[ (uC_\varphi)^n f \right](z) \right| = |u(z) \cdots u(\varphi_{n-1}(z))f(\varphi_n(z))| \leq ||u||^N \varepsilon^N |f|.
\]

Therefore \( ||(uC_\varphi)^n|| \leq ||u||^N \varepsilon^N \) and so \( ||uC_\varphi||_{sp} = \lim_{n \to \infty} ||(uC_\varphi)^n||^{1/n} \leq \varepsilon \).

This holds for all \( \varepsilon > 0 \); consequently \( ||uC_\varphi||_{sp} = 0 \) as required.

We next show that if \( uC_\varphi \) is a compact operator on \( A \) and if \( \varphi \) has a fixed point \( z_0 \) in \( D \), then \( \sigma(uC_\varphi) = \{ u(z_0)\varphi'(z_0)^n | n \text{ is a positive integer} \} \cup \{ 0, u(z_0) \} \). This will be proved first for \( z_0 = 0 \) and then, by a standard argument, extended to arbitrary fixed points \( z_0 \) in \( D \).

**Lemma 2.3.** Suppose \( u \in A, \varphi \in A, ||\varphi|| \leq 1 \) and \( \varphi(0) = 0 \). Then \( u(0) \in \sigma(uC_\varphi) \) and \( u(0)\varphi'(0)^n \in \sigma(uC_\varphi) \) for every positive integer \( n \).

**Proof.** (i) \( u(0) \in \sigma(uC_\varphi) \) since no \( f \in A \) satisfies \( u(0)f(z) - u(z)f(\varphi(z)) = 1 \). For, evaluating at \( z = 0 \) gives \( u(0)f(0) - u(0)f(0) = 0 \neq 1 \).

(ii) If \( \varphi'(0) = 0 \), then \( \varphi \) is not a conformal map of \( D \) onto \( D \). Therefore if \( \varphi'(0) = 0 \), the composition operator \( C_\varphi \) is not invertible and so \( uC_\varphi \) is not invertible. Thus if \( \varphi'(0) = 0 \), then \( u(0)\varphi'(0)^n = 0 \in \sigma(uC_\varphi) \) for every positive integer \( n \).

(iii) If \( u(0) = 0 \), then again \( uC_\varphi \) is not invertible and therefore if \( u(0) = 0 \), then \( u(0)\varphi'(0)^n = 0 \in \sigma(uC_\varphi) \) for every positive integer \( n \).

(iv) Finally if \( u(0)\varphi'(0) \neq 0 \), we will prove that \( u(0)\varphi'(0)^n \in \sigma(uC_\varphi) \) for every positive integer \( n \) by showing that for each such
integer $n$, the function $z^n$ is not in the range of $(u(0)\varphi'(0)^n - uC_v)$.

Suppose the contrary, that for some positive integer $n$ there exists $f \in A$ with $u(0)\varphi'(0)^nf(z) - u(z)f(\varphi(z)) = z^n$. Write $f(z) = z^nf_0(z)$ where $f_0 \in A$ and $f_0(0) \neq 0$. Then $f_0(z) = f_0(0) + O(|z|)$. Also let $u(z) = u(0) + O(|z|)$ and $\varphi(z) = \varphi'(0)z + O(|z|^2)$. Then

$$u(0)\varphi'(0)^nf(z) - u(z)f(\varphi(z)) = z^n$$

is equivalent to

$$u(0)\varphi'(0)^n[z^n[O(n)|z|)] - (u(0) + O(|z|))(\varphi'(0)^nz^m + O(|z|^{m+1}))$$

or

$$[u(0)\varphi'(0)^nf_0(0) - u(0)\varphi'(0)^nf_0(0)]z^m + O(|z|^{m+1}) = z^n. \quad (1)$$

If $m \neq n$, then the left side of (1) has order $m$ and the right side has order $n$, a contradiction. On the other hand, if $m = n$, then the left side of (1) has order at least $n + 1$ since the coefficient of $z^n$ vanishes, while the right side of (1) has order $n$, which again is a contradiction.

Hence for each positive integer $n$, $u(0)\varphi'(0)^n \in \sigma(uC_v)$.

**Lemma 2.4.** Suppose $0 \neq u \in A$, $\|\varphi\| \leq 1$, $\varphi(0) = 0$ and $\varphi$ is not a constant function. If $\lambda$ is an eigenvalue of $uC_v$, then $\lambda \in \{u(0)\varphi'(0)^n \mid n$ is a positive integer $\} \cup \{u(0)\}$.

**Proof.** Suppose $\lambda$ is an eigenvalue of $uC_v$ with $f$ as corresponding eigenvector. Then $\lambda \neq 0$ since $\varphi$ is not a constant function and the algebra $A$ has no zero divisors. Write $f(z) = az^m + O(|z|^{m+1})$, $m \geq 0$, $u(z) = bz^r + O(|z|^{r+1})$, $r \geq 0$ and $\varphi(z) = cz^s + O(|z|^{s+1})$, $s \geq 1$, where $abc \neq 0$. Then $\lambda f = (uC_v)f$ becomes

$$\lambda [az^m + O(|z|^{m+1})] = [bz^r + O(|z|^{r+1})][acz^s + O(|z|^{s+1})^m + O(|z|^{ms+1})]$$

or

$$a\lambda z^m + O(|z|^{m+1}) = abc^m z^{r+ms} + O(|z|^{r+ms+1}).$$

Equating powers, we get $m = r + ms$ and $a\lambda = abc^m$.

Since $r$ and $m$ are nonnegative integers and $s$ is a positive integer, $m = r + ms$ implies (i) $r = m = 0$ or (ii) $r = 0$ and $s = 1$. In the first case, $b = u(0)$ and so $a\lambda = abc^m$ implies $\lambda = u(0)$, while if $r = 0$ and $s = 1$, then $b = u(0)$, $c = \varphi'(0)$ and $a\lambda = abc^m$ implies $\lambda = u(0)\varphi'(0)^m$ for some positive integer $m$, concluding the proof.

**Theorem 2.5.** Suppose $0 \neq u \in A$, $\varphi \in A$, $\|\varphi\| \leq 1$, $\varphi(0) = 0$, $\varphi$ is
not a constant function and \( uC_\psi \) is a compact operator. Then \( \sigma(uC_\psi) = \{u(0)\varphi'(0)^n \mid n \text{ is a positive integer} \} \cup \{0, u(0)\} \).

**Proof.** By the Fredholm alternative for compact operators, every nonzero element in \( \sigma(uC_\psi) \) is an eigenvalue. It follows from Lemma 2.4 that the only possible eigenvalues of \( uC_\psi \) are \( u(0) \) and \( u(0)\varphi'(0)^n \) for positive integers \( n \); on the other hand Lemma 2.3 shows that each of these numbers is in \( \sigma(uC_\psi) \). Hence \( \sigma(uC_\psi) = \{u(0)\varphi'(0)^n \mid n \text{ is a positive integer} \} \cup \{0, u(0)\} \).

I should like to thank the referee for greatly simplifying my original proof of Theorem 2.5.

For arbitrary \( z_0 \in \overline{D} \) we have

**Theorem 2.6.** Let \( u \in A, \varphi \in A, \|\varphi\| \leq 1 \) and \( uC_\psi \) be a compact operator on \( A \). Suppose \( z_0 \) is the Denjoy-Wolf fixed point of \( \varphi \).

(i) If \( \varphi \) is a constant function, then \( \sigma(uC_\psi) = \{0, u(z_0)\} \).

(ii) If \( \varphi \) is not a constant function and \( |z_0| = 1 \), then \( \sigma(uC_\psi) = \{0\} \).

(iii) If \( \varphi \) is not a constant function and \( |z_0| < 1 \), then \( \sigma(uC_\psi) = \{u(z_0)\varphi'(z_0)^n \mid n \text{ is a positive integer} \} \cup \{0, u(z_0)\} \).

**Proof.** The only statement that has not been proved is (iii). Also if \( u \equiv 0 \), then certainly \( \sigma(uC_\psi) = \{0\} \).

Thus assume \( u \equiv 0, \varphi \) is not a constant function and \( \varphi(z_0) = z_0 \in \mathbb{D} \). Let \( p \) be the linear fractional transformation \( p(z) = (z_0 - z)/(1 - \overline{z}_0z) \). Then \( p \) maps \( D \) onto \( D \) and \( p \circ p = z \). If we define \( S \) by \( Sf(z) = f(p(z)) \) for \( z \in \mathbb{D} \), then \( S \) is an isometry on \( A \) and \( S = S^{-1} \). Let \( \psi = p \circ \varphi \circ p \) and \( u^*(z) = u(p(z)) \). Then \( u^* \in A \) and \( S(u^*C_\varphi)S^{-1} = uC_\psi \).

Indeed,

\[
[S(u^*C_\psi)S^{-1}]f = [S(u^*C_\psi)](f \circ p) = S[u^* \cdot f \circ p \circ \psi]
= (u^* \circ p) \cdot (f \circ p \circ \psi \circ p) = u \cdot (f \circ \varphi) = (uC_\psi)f.
\]

Consequently \( \sigma(u^*C_\psi) = \sigma(uC_\psi) \). Since \( \psi(0) = 0 \), it follows from Theorem 2.5 that \( \sigma(u^*C_\psi) = \{u^*(0)\varphi'(0)^n \mid n \text{ is a positive integer} \} \cup \{0, u^*(0)\} \). But \( u^*(0) = u(p(0)) = u(z_0) \) and \( \psi'(0) = \varphi'(z_0) \). Thus \( \sigma(uC_\psi) = \{u(z_0)\varphi'(z_0)^n \mid n \text{ is a positive integer} \} \cup \{0, u(z_0)\} \).

**Remarks.** 1. By considering the adjoint \( (uC_\psi)^* \) of \( uC_\psi \) it can be shown that each nonzero eigenvalue of \( uC_\psi \) has multiplicity one.

2. Operators of the form \( uC_\psi \) on \( A \) for those \( \varphi \) which are conformal maps of \( D \) onto \( D \) were considered in [3]. Except for the case where \( \varphi \) has finite orbit, their spectra consist of circles, discs or annuli centered at the origin.
3. Caughran and Schwartz [1], Schwartz [4], and Shapiro and Taylor [5] have considered compact composition operators on $H^p$. Included in their papers are geometric conditions on $\varphi$ insuring that $C_\varphi$ be compact. They also determine $\sigma(C_\varphi)$ when $C_\varphi$ is compact. It is shown that if $C_\varphi$ is a compact composition operator, then $\varphi$ has a fixed point $z_0$ in $D$ and $\sigma(C_\varphi) = \{\varphi'(z_0)^n | n$ is a positive integer$\} \cup \{0, 1\}$. 

4. The arguments leading to Theorem 2.5 are valid if $u \in H^\infty \varphi \in H^\infty, |\varphi(z)| < 1$ for $|z| < 1$ and $uC_\varphi$ acts on $H^p, 1 \leq p \leq \infty$. Thus for such $u$ and $\varphi$, if $\varphi(z_0) = z_0 \in D$ and $uC_\varphi$ is a compact operator on $H^p$, then again $\sigma(uC_\varphi) = \{u(z_0)\varphi'(z_0)^n | n$ is a positive integer$\} \cup \{0, u(z_0)\}$.

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Received December 1, 1977 and in revised May 5, 1978.

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