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**COMPACT OPERATORS OF THE FORM  $uC_\varphi$**

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## COMPACT OPERATORS OF THE FORM $uC_\varphi$

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If  $A$  is the disc algebra, the uniform algebra of functions analytic on the open unit disc  $D$  and continuous on its closure, and if  $u, \varphi \in A$  with  $\|\varphi\| \leq 1$ , then the operator  $uC_\varphi$  is defined on  $A$  by  $uC_\varphi: f(z) \rightarrow u(z)f(\varphi(z))$ . In this note we characterize compact operators of this form and determine their spectra.

We recall that a bounded linear operator  $T$  from a Banach space  $B_1$  to a Banach space  $B_2$  is *compact* if given a bounded sequence  $\{x_n\}$  in  $B_1$ , there exists a subsequence  $\{x_{n_k}\}$  such that  $\{Tx_{n_k}\}$  converges in  $B_2$ .

If  $\varphi: \bar{D} \rightarrow \bar{D}$ , we let  $\varphi_n$  denote  $n^{\text{th}}$  the iterate of  $\varphi$ , i.e.,  $\varphi_0(z) = z$  and  $\varphi_n(z) = \varphi(\varphi_{n-1}(z))$  for  $z \in \bar{D}$  and  $n \geq 1$ . Our main result is the following.

**THEOREM.** *Let  $u \in A$ ,  $\varphi \in A$ ,  $\|\varphi\| \leq 1$  and suppose  $\varphi$  is not a constant function.*

I. *The operator  $uC_\varphi$  is compact if, and only if,  $|\varphi(z)| < 1$  whenever  $u(z) \neq 0$ .*

II. *Suppose  $uC_\varphi$  is compact and let  $z_0 \in \bar{D}$  be the unique fixed point of  $\varphi$  for which  $\varphi_n(z) \rightarrow z_0$  for all  $z \in D$ . If  $|z_0| = 1$ , then  $uC_\varphi$  is quasinilpotent, while if  $|z_0| < 1$ , the spectrum  $\sigma(uC_\varphi) = \{u(z_0)\varphi'(z_0)^n | n \text{ is a positive integer}\} \cup \{0, u(z_0)\}$ .*

1. **Characterization of compact  $uC_\varphi$ .** We first consider the easy case in which  $\varphi$  is a constant function.

**THEOREM 1.1.** *Suppose  $u \in A$  and  $\varphi(z) = a \in \bar{D}$  for all  $z \in \bar{D}$ . Then  $uC_\varphi$  is compact.*

*Proof.* Since  $\varphi(z) = a$  for all  $z \in \bar{D}$ ,  $(uC_\varphi)f(z) = u(z)f(\varphi(z)) = f(a)u(z)$ . Therefore the range of  $uC_\varphi$  is one-dimensional and so  $uC_\varphi$  is compact.

We next give a necessary and sufficient condition that  $uC_\varphi$  be a compact operator for those  $\varphi$  which are not constant functions.

**THEOREM 1.2.** *Suppose  $u \in A$ ,  $\varphi \in A$ ,  $\|\varphi\| \leq 1$  and  $\varphi$  is not a constant function. Then  $uC_\varphi$  is a compact operator on  $A$  if, and only if,  $|\varphi(z)| < 1$  whenever  $u(z) \neq 0$ .*

*Proof.* Since everything holds if  $u \equiv 0$ , we will assume that  $u$  is not identically zero.

1. Suppose  $uC_\varphi$  is a compact operator on  $A$ . We must prove that if  $z \in \bar{D}$  and  $u(z) \neq 0$ , then  $|\varphi(z)| < 1$ . Since  $\varphi$  is not a constant function, the maximum modulus principle implies that  $|\varphi(z)| < 1$  whenever  $|z| < 1$  and thus it suffices to show that  $|\varphi(z)| < 1$  when  $u(z) \neq 0$  and  $z$  lies on the unit circle. Assume the contrary and let  $\theta$  satisfy  $u(e^{i\theta}) \neq 0$  and  $|\varphi(e^{i\theta})| = 1$ . Set  $\mu = \varphi(e^{i\theta})$  and for each positive integer  $n$ , define  $f_n$  by  $f_n(z) = (\frac{1}{2}(z + \mu))^n$ . Then  $\|f_n\| = 1$ . Since  $uC_\varphi$  is assumed to be compact, there exists a subsequence  $\{f_{n_k}\}$  and a function  $F$  in  $A$  with  $(uC_\varphi)f_{n_k} \rightarrow F$  in  $A$ . That is,  $u(z)(\frac{1}{2}(\varphi(z) + \mu))^{n_k} \rightarrow F(z)$  uniformly for  $z \in \bar{D}$ . But  $(\frac{1}{2}(\varphi(z) + \mu))^{n_k} \rightarrow 0$  for  $|z| < 1$  and so  $F(z) = 0$  on  $D$ . However,  $F$  is continuous on  $\bar{D}$  and therefore  $F(z) = 0$  on  $\bar{D}$ . Hence  $(uC_\varphi)f_{n_k} \rightarrow 0$  uniformly on  $\bar{D}$ . In particular,  $u(e^{i\theta})(\frac{1}{2}(\varphi(e^{i\theta}) + \mu))^{n_k} \rightarrow 0$ . But for all  $k$ , we have  $|u(e^{i\theta})(\frac{1}{2}(\varphi(e^{i\theta}) + \mu))^{n_k}| = |u(e^{i\theta})| \neq 0$ . This is a contradiction. Hence if  $uC_\varphi$  is compact and  $u(z) \neq 0$ , then  $|\varphi(z)| < 1$ .

2. Conversely, assume  $|\varphi(z)| < 1$  whenever  $u(z) \neq 0$ . To show that  $uC_\varphi$  is compact, assume  $f_n \in A$  and  $\|f_n\| \leq 1$ . Since  $\{f_n\}$  is a uniformly bounded sequence of functions on  $D$ , it is a normal family in the sense of Montel and so there exists a subsequence  $\{f_{n_k}\}$  and a function  $g$  analytic on  $D$  with  $f_{n_k} \rightarrow g$  uniformly on compact subsets of the open disc  $D$ . We observe that this convergence implies  $\sup_{|w| < 1} |g(w)| \leq 1$ . Now defined a function  $G$  on the closed disc  $\bar{D}$  by setting  $G(z) = 0$  whenever  $|z| = 1$  and  $u(z) = 0$ , and letting  $G(z) = u(z)g(\varphi(z))$  otherwise. We claim that  $G \in A$  and  $(uC_\varphi)f_{n_k} \rightarrow G$  uniformly on  $\bar{D}$ .

We first show that  $G$  is continuous on  $\bar{D}$ . Indeed,  $G$  is continuous on  $\{z | u(z) \neq 0\}$  since  $|\varphi(z)| < 1$  on this set and  $g$  is continuous on  $D$ . Further, if  $|z^*| = 1$  and  $u(z^*) = 0$ , let  $\{z_m\}$  be a sequence in  $\bar{D}$  converging to  $z^*$ . For each  $m$ ,  $G(z_m) = 0$  or  $G(z_m) = u(z_m)g(\varphi(z_m))$ . Since  $|g(\varphi(z_m))| \leq 1$  it follows that  $\lim_{m \rightarrow \infty} G(z_m) = 0 = G(z^*)$  and so  $G$  is continuous at each  $z \in \bar{D}$ . Also  $G$  is analytic on  $D$  since  $u$  and  $g \circ \varphi$  are analytic on  $D$ . Hence  $G \in A$ .

To show that  $(uC_\varphi)f_{n_k} \rightarrow G$  uniformly on  $\bar{D}$ , let  $V = \{e^{i\theta} | u(e^{i\theta}) = 0\}$  and suppose  $\varepsilon > 0$ . Since  $u$  is continuous, there exists an open set  $U \supset V$  for which  $|u(t)| < \varepsilon$  for  $t \in U$ . Also since  $|u(z)| < 1$  for  $z \in U$  and  $\bar{D} \setminus U$  is a compact set, there exists  $r$ ,  $0 < r < 1$ , such that  $|\varphi(z)| \leq r$  for  $z \in U$ . Moreover, since  $f_{n_k} \rightarrow g$  uniformly on compact subsets of  $D$ ,  $u(z)f_{n_k}(\varphi(z)) \rightarrow u(z)g(\varphi(z))$  uniformly for  $z \in U$ . That is, there exists an integer  $N$  such that  $|u(z)f_{n_k}(\varphi(z)) - G(z)| =$

$|u(z)f_{n_k}(\varphi(z)) - u(z)g(\varphi(z))| < \varepsilon$  for  $k \geq N$  and all  $z \in U$ . On the other hand, for  $z \in U \setminus V$  and for all  $k$ ,

$$\begin{aligned} |(uC_\varphi)f_{n_k}(z) - G(z)| &= |u(z)f_{n_k}(\varphi(z)) - u(z)g(\varphi(z))| \\ &\leq \sup_{z \in U \setminus V} [|u(z)||f_{n_k}(\varphi(z)) - g(\varphi(z))|] \leq \varepsilon[||f_{n_k}|| + ||g||_\infty] = 2\varepsilon. \end{aligned}$$

Finally, if  $z \in V$ , then  $(uC_\varphi)f_{n_k}(z) = u(z)f_{n_k}(\varphi(z)) = 0 = G(z)$ . Hence given  $\varepsilon > 0$ , there exists an integer  $N$  such that  $|(uC_\varphi)f_{n_k}(z) - G(z)| < 2\varepsilon$  for  $k \geq N$  and all  $z \in \bar{D}$ . That is,  $(uC_\varphi)f_{n_k} \rightarrow G$  uniformly. Thus if  $|\varphi(z)| < 1$  whenever  $u(z) = 0$ , then the operator  $uC_\varphi$  is compact.

**2. Spectra of compact  $uC_\varphi$ .** If  $T$  is a bounded linear operator from  $A$  to  $A$  we let  $\sigma(T)$  denote the spectrum of  $T$ . As before, we first consider the case where  $\varphi$  is a constant function.

**THEOREM 2.1.** *Suppose  $u \in A$  and  $\varphi(z) = a \in \bar{D}$  for all  $z \in \bar{D}$ . Then  $\sigma(uC_\varphi) = \{0, u(a)\}$ .*

*Proof.* 0 and  $u(a)$  are both eigenvalues  $uC_\varphi$ . For, if  $F(z) = z - a$ , then  $(uC_\varphi)F(z) = u(z)F(\varphi(z)) = u(z)F(a) = 0$ , while if  $G(z) = u(z)$ , then  $(uC_\varphi)G(z) = u(z)G(\varphi(z)) = u(a)G(z)$ . Thus  $\{0, u(a)\} \subset \sigma(uC_\varphi)$ .

On the other hand, since the range of  $uC_\varphi$  is one-dimensional,  $\sigma(uC_\varphi)$  contains at most two elements and therefore  $\sigma(uC_\varphi) = \{0, u(a)\}$ .

In determining the spectra of the remaining compact operators of the form  $uC_\varphi$  we will make use of the following theorem of Denjoy and Wolf.

**THEOREM A** (Denjoy [2], Wolf [6]). *Suppose  $\varphi$  is an analytic function mapping  $D$  to  $D$ . If  $\varphi$  is not conformally equivalent to a rotation about a fixed point, then there exists a unique  $z' \in \bar{D}$  for which  $\varphi_n(z) \rightarrow z'$  for all  $z \in D$ . If  $\varphi$  is continuous at  $z'$ , then  $\varphi(z') = z'$ .*

Suppose  $\varphi \in A$  and  $\varphi: D \rightarrow D$ . It is easy to show that if  $\varphi \neq z$ , then there is at most one fixed point of  $\varphi$  in the open disc  $D$ . There may, however, be infinitely many fixed points on the boundary of  $D$ . However, if the function  $\varphi$  is not equivalent to a rotation, then Theorem A asserts that there exists a unique fixed point  $z_0 \in \bar{D}$ , which we call the *Denjoy-Wolf fixed point* of  $\varphi$ , for which  $\varphi_n(z) \rightarrow z_0$  for all  $z \in D$ . The spectrum of a compact operator of the form  $uC_\varphi$  will depend on the location of the Denjoy-Wolf fixed point of  $\varphi$ .

**THEOREM 2.2.** *Suppose  $u \in A$ ,  $\varphi \in A$ ,  $||\varphi|| = 1$ ,  $\varphi$  is not a constant*

function and  $\varphi$  has all its fixed points on the unit circle. If  $uC_\varphi$  is a compact operator, then  $uC_\varphi$  is quasinilpotent.

*Proof.* Let  $z_0$  be the Denjoy-Wolf fixed point of  $\varphi$ , which by hypothesis has modulus 1. Since  $uC_\varphi$  is compact, Theorem 1.2 implies  $u(z_0) = 0$ . Let  $V = \{e^{i\theta} \mid u(e^{i\theta}) = 0\}$ .

Choose  $\varepsilon > 0$ . As in Theorem 1.2 there exists an open set  $U$  such that  $U \supset V$  and  $|u(t)| < \varepsilon$  for all  $t \in U$ . Also, since  $\bar{D} \setminus U$  is compact there exists  $r$ ,  $0 < r < 1$ , such that  $|\varphi(w)| < r$  for all  $w \in \bar{D} \setminus U$ . Since  $\{\varphi_n\}$  is a bounded sequence and hence a normal family, there exists a subsequence  $\{\varphi_{n_k}\}$  such that  $\{\varphi_{n_k}\}$  converges uniformly on compact subsets of  $D$ . In particular,  $\{\varphi_{n_k}\}$  converges uniformly for  $|z| \leq r$ . But  $\varphi_{n_k}(z) \rightarrow z_0$  for all  $z \in D$ . It follows that  $\{\varphi_n(z)\}$  converges uniformly to  $z_0$  for  $|z| \leq r$ .

Now choose  $\delta > 0$  such that  $\{s \in \bar{D} \mid |s - z_0| < \delta\} \subset U$ . Since  $\varphi_n(w) \rightarrow z_0$  uniformly for  $|w| \leq r$ , there exists a positive integer  $N$  such that  $|\varphi_n(w) - z_0| < \delta$  if  $n \geq N$  and  $|w| \leq r$ . Thus  $\varphi_n(\{w \mid |w| \leq r\}) \subset U$  for  $n \geq N$ . Therefore, for each  $z \in \bar{D}$  and each positive integer  $n$ , at most  $N$  elements from  $z, \varphi(z), \dots, \varphi_n(z)$  lie in  $\bar{D} \setminus U$ . From the definition of  $U$ , if  $t \in U$ , then  $|u(t)| < \varepsilon$ . Hence for all  $z \in \bar{D}$  and  $n \geq N$ ,

$$|[(uC_\varphi)^n f](z)| = |u(z) \cdots u(\varphi_{n-1}(z))f(\varphi_n(z))| \leq \|u\|^N \varepsilon^{n-N} \|f\|.$$

Therefore  $\|(uC_\varphi)^n\| \leq \|u\|^N \varepsilon^{n-N}$  and so  $\|uC_\varphi\|_{s,p} = \lim_{n \rightarrow \infty} \|(uC_\varphi)^n\|^{1/n} \leq \varepsilon$ . This holds for all  $\varepsilon > 0$ ; consequently  $\|uC_\varphi\|_{s,p} = 0$  as required.

We next show that if  $uC_\varphi$  is a compact operator on  $A$  and if  $\varphi$  has a fixed point  $z_0$  in  $D$ , then  $\sigma(uC_\varphi) = \{u(z_0)\varphi'(z_0)^n \mid n \text{ is a positive integer}\} \cup \{0, u(z_0)\}$ . This will be proved first for  $z_0 = 0$  and then, by a standard argument, extended to arbitrary fixed points  $z_0$  in  $D$ .

**LEMMA 2.3.** *Suppose  $u \in A$ ,  $\varphi \in A$ ,  $\|\varphi\| \leq 1$  and  $\varphi(0) = 0$ . Then  $u(0) \in \sigma(uC_\varphi)$  and  $u(0)\varphi'(0)^n \in \sigma(uC_\varphi)$  for every positive integer  $n$ .*

*Proof.* (i)  $u(0) \in \sigma(uC_\varphi)$  since no  $f \in A$  satisfies  $u(0)f(z) - u(z)f(\varphi(z)) = 1$ . For, evaluating at  $z = 0$  gives  $u(0)f(0) - u(0)f(0) = 0 \neq 1$ .

(ii) If  $\varphi'(0) = 0$ , then  $\varphi$  is not a conformal map of  $D$  onto  $D$ . Therefore if  $\varphi'(0) = 0$ , the composition operator  $C_\varphi$  is not invertible and so  $uC_\varphi$  is not invertible. Thus if  $\varphi'(0) = 0$ , then  $u(0)\varphi'(0)^n = 0 \in \sigma(uC_\varphi)$  for every positive integer  $n$ .

(iii) If  $u(0) = 0$ , then again  $uC_\varphi$  is not invertible and therefore if  $u(0) = 0$ , then  $u(0)\varphi'(0)^n = 0 \in \sigma(uC_\varphi)$  for every positive integer  $n$ .

(iv) Finally if  $u(0)\varphi'(0) \neq 0$ , we will prove that  $u(0)\varphi'(0)^n \in \sigma(uC_\varphi)$  for every positive integer  $n$  by showing that for each such

integer  $n$ , the function  $z^n$  is not in the range of  $(u(0)\varphi'(0)^n - uC_\varphi)$ .

Suppose the contrary, that for some positive integer  $n$  there exists  $f \in A$  with  $u(0)\varphi'(0)^n f(z) - u(z)f(\varphi(z)) = z^n$ . Write  $f(z) = z^m f_0(z)$  where  $f_0 \in A$  and  $f_0(0) \neq 0$ . Then  $f_0(z) = f_0(0) + \mathcal{O}(|z|)$ . Also let  $u(z) = u(0) + \mathcal{O}(|z|)$  and  $\varphi(z) = \varphi'(0)z + \mathcal{O}(|z|^2)$ . Then

$$u(0)\varphi'(0)^n f(z) - u(z)f(\varphi(z)) = z^n$$

is equivalent to

$$u(0)\varphi'(0)^n z^m [f_0(0) + \mathcal{O}(|z|)] - (u(0) + \mathcal{O}(|z|))(\varphi'(0)^m z^m + \mathcal{O}(|z|^{m+1})) \times (f_0(0) + \mathcal{O}(|z|)) = z^n$$

or

$$(1) \quad [u(0)\varphi'(0)^n f_0(0) - u(0)\varphi'(0)^m f_0(0)]z^m + \mathcal{O}(|z|^{m+1}) = z^n.$$

If  $m \neq n$ , then the left side of (1) has order  $m$  and the right side has order  $n$ , a contradiction. On the other hand, if  $m = n$ , then the left side of (1) has order at least  $n + 1$  since the coefficient of  $z^n$  vanishes, while the right side of (1) has order  $n$ , which again is a contradiction.

Hence for each positive integer  $n$ ,  $u(0)\varphi'(0)^n \in \sigma(uC_\varphi)$ .

LEMMA 2.4. *Suppose  $0 \neq u \in A$ ,  $\|\varphi\| \leq 1$ ,  $\varphi(0) = 0$  and  $\varphi$  is not a constant function. If  $\lambda$  is an eigenvalue of  $uC_\varphi$ , then  $\lambda \in \{u(0)\varphi'(0)^n | n \text{ is a positive integer}\} \cup \{u(0)\}$ .*

*Proof.* Suppose  $\lambda$  is an eigenvalue of  $uC_\varphi$  with  $f$  as corresponding eigenvector. Then  $\lambda \neq 0$  since  $\varphi$  is not a constant function and the algebra  $A$  has no zero divisors. Write  $f(z) = az^m + \mathcal{O}(|z|^{m+1})$ ,  $m \geq 0$ ,  $u(z) = bz^r + \mathcal{O}(|z|^{r+1})$ ,  $r \geq 0$  and  $\varphi(z) = cz^s + \mathcal{O}(|z|^{s+1})$ ,  $s \geq 1$ , where  $abc \neq 0$ . Then  $\lambda f = (uC_\varphi)f$  becomes

$$\lambda[az^m + \mathcal{O}(|z|^{m+1})] = [bz^r + \mathcal{O}(|z|^{r+1})][a(cz^s + \mathcal{O}(|z|^{s+1}))^m + \mathcal{O}(|z|^{ms+1})]$$

or

$$a\lambda z^m + \mathcal{O}(|z|^{m+1}) = abc^m z^{r+ms} + \mathcal{O}(|z|^{r+ms+1}).$$

Equating powers, we get  $m = r + ms$  and  $a\lambda = abc^m$ .

Since  $r$  and  $m$  are nonnegative integers and  $s$  is a positive integer,  $m = r + ms$  implies (i)  $r = m = 0$  or (ii)  $r = 0$  and  $s = 1$ . In the first case,  $b = u(0)$  and so  $a\lambda = abc^m$  implies  $\lambda = u(0)$ , while if  $r = 0$  and  $s = 1$ , then  $b = u(0)$ ,  $c = \varphi'(0)$  and  $a\lambda = abc^m$  implies  $\lambda = u(0)\varphi'(0)^m$  for some positive integer  $m$ , concluding the proof.

THEOREM 2.5. *Suppose  $0 \neq u \in A$ ,  $\varphi \in A$ ,  $\|\varphi\| \leq 1$ ,  $\varphi(0) = 0$ ,  $\varphi$  is*

not a constant function and  $uC_\varphi$  is a compact operator. Then  $\sigma(uC_\varphi) = \{u(0)\varphi'(0)^n \mid n \text{ is a positive integer}\} \cup \{0, u(0)\}$ .

*Proof.* By the Fredholm alternative for compact operators, every nonzero element in  $\sigma(uC_\varphi)$  is an eigenvalue. It follows from Lemma 2.4 that the only possible eigenvalues of  $uC_\varphi$  are  $u(0)$  and  $u(0)\varphi'(0)^n$  for positive integers  $n$ ; on the other hand Lemma 2.3 shows that each of these numbers is in  $\sigma(uC_\varphi)$ . Hence  $\sigma(uC_\varphi) = \{u(0)\varphi'(0)^n \mid n \text{ is a positive integer}\} \cup \{0, u(0)\}$ .

I should like to thank the referee for greatly simplifying my original proof of Theorem 2.5.

For arbitrary  $z_0 \in \bar{D}$  we have

**THEOREM 2.6.** *Let  $u \in A$ ,  $\varphi \in A$ ,  $\|\varphi\| \leq 1$  and  $uC_\varphi$  be a compact operator on  $A$ . Suppose  $z_0$  is the Denjoy-Wolff fixed point of  $\varphi$ .*

(i) *If  $\varphi$  is a constant function, then  $\sigma(uC_\varphi) = \{0, u(z_0)\}$ .*

(ii) *If  $\varphi$  is not a constant function and  $|z_0| = 1$ , then  $\sigma(uC_\varphi) = \{0\}$ .*

(iii) *If  $\varphi$  is not a constant function and  $|z_0| < 1$ , then  $\sigma(uC_\varphi) = \{u(z_0)\varphi'(z_0)^n \mid n \text{ is a positive integer}\} \cup \{0, u(z_0)\}$ .*

*Proof.* The only statement that has not been proved is (iii). Also if  $u \equiv 0$ , then certainly  $\sigma(uC_\varphi) = \{0\}$ .

Thus assume  $u \not\equiv 0$ ,  $\varphi$  is not a constant function and  $\varphi(z_0) = z_0 \in D$ . Let  $p$  be the linear fractional transformation  $p(z) = (z_0 - z)/(1 - \bar{z}_0 z)$ . Then  $p$  maps  $D$  onto  $D$  and  $p \circ p = z$ . If we define  $S$  by  $Sf(z) = f(p(z))$  for  $z \in \bar{D}$ , then  $S$  is an isometry on  $A$  and  $S = S^{-1}$ . Let  $\psi = p \circ \varphi \circ p$  and  $u^*(z) = u(p(z))$ . Then  $u^* \in A$  and  $S(u^*C_\psi)S^{-1} = uC_\varphi$ . Indeed,

$$\begin{aligned} [S(u^*C_\psi)S^{-1}]f &= [S(u^*C_\psi)](f \circ p) = S[u^* \cdot f \circ p \circ \psi] \\ &= (u^* \circ p) \cdot (f \circ p \circ \psi \circ p) = u \cdot (f \circ \varphi) = (uC_\varphi)f. \end{aligned}$$

Consequently  $\sigma(u^*C_\psi) = \sigma(uC_\varphi)$ . Since  $\psi(0) = 0$ , it follows from Theorem 2.5 that  $\sigma(u^*C_\psi) = \{u^*(0)\psi'(0)^n \mid n \text{ is a positive integer}\} \cup \{0, u^*(0)\}$ . But  $u^*(0) = u(p(0)) = u(z_0)$  and  $\psi'(0) = \varphi'(z_0)$ . Thus  $\sigma(uC_\varphi) = \{u(z_0)\varphi'(z_0)^n \mid n \text{ is a positive integer}\} \cup \{0, u(z_0)\}$ .

**REMARKS.** 1. By considering the adjoint  $(uC_\varphi)^*$  of  $uC_\varphi$  it can be shown that each nonzero eigenvalue of  $uC_\varphi$  has multiplicity one.

2. Operators of the form  $uC_\varphi$  on  $A$  for those  $\varphi$  which are conformal maps of  $D$  onto  $D$  were considered in [3]. Except for the case where  $\varphi$  has finite orbit, their spectra consist of circles, discs or annuli centered at the origin.

3. Caughran and Schwartz [1], Schwartz [4], and Shapiro and Taylor [5] have considered compact composition operators on  $H^p$ . Included in their papers are geometric conditions on  $\varphi$  insuring that  $C_\varphi$  be compact. They also determine  $\sigma(C_\varphi)$  when  $C_\varphi$  is compact. It is shown that if  $C_\varphi$  is a compact composition operator, then  $\varphi$  has a fixed point  $z_0$  in  $D$  and  $\sigma(C_\varphi) = \{\varphi'(z_0)^n \mid n \text{ is a positive integer}\} \cup \{0, 1\}$ .

4. The arguments leading to Theorem 2.5 are valid if  $u \in H^\infty$ ,  $\varphi \in H^\infty$ ,  $|\varphi(z)| < 1$  for  $|z| < 1$  and  $uC_\varphi$  acts on  $H^p$ ,  $1 \leq p \leq \infty$ . Thus for such  $u$  and  $\varphi$ , if  $\varphi(z_0) = z_0 \in D$  and  $uC_\varphi$  is a compact operator on  $H^p$ , then again  $\sigma(uC_\varphi) = \{u(z_0)\varphi'(z_0)^n \mid n \text{ is a positive integer}\} \cup \{0, u(z_0)\}$ .

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