SOME PROPERTIES OF THE CHEBYSHEV METHOD

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Several properties of the Chebyshev method of summability, defined by G. G. Bilodeau, are investigated. Specifically, it is shown that the Chebyshev method is translatable and is a Gronwall method. It is shown that the de Vallee Poussin method is stronger than the Chebyshev method, and that the Chebyshev method is not stronger than the $(C, 1)$ method. The final result shows that the Chebyshev method exhibits the Gibbs phenomenon.

Let \( \Sigma(-1)^n u_i \) be an alternating series with partial sums \( s_n = \sum_{i=0}^{\infty} (-1)^i u_i \). Define a sequence of polynomials \( \{P_n(t)\} \) by \( P_n(t) = \sum_{k=0}^{n} a_{nk} t^k \), \( P_n(1) = 1, \) \( n = 0, 1, 2, \ldots \). The series \( \Sigma(-1)^n u_i \) will be called summable \( (P_n) \) to the value \( s \) if \( \lim \sigma(P_n) = s \), where \( \sigma(P_n) = \sum_{k=0}^{\infty} a_{nk} \delta_k \). Bilodeau [1] considered the following question. What are sufficient conditions on \( P_n \) for \( \sigma(P_n) \) to speed up the rate of convergence of a convergent sequence \( \{s_n\} \)? For sequences \( \{u_n\} \) which are moment sequences, i.e., \( u_n = \int_{0}^{1} t^n d\alpha(t) \), he obtains the estimate \( |\sigma(P_n) - s|/|r_n| \leq (\mu_n/|r_n|) \int_{0}^{1} t(1 + t)^{-1} |d\alpha(t)| \), where \( s = \sum_{i=0}^{\infty} (-1)^i u_i \), \( r_n = s_n - s \), and \( \mu_n = \max_{0 \leq t \leq 1} |P_n(-t)| \). Adopting \( \mu_n \) as a measure of the value of the method \( \sigma(P_n) \), the most desirable sequence of polynomials will be those for which \( \mu_n \) is a minimum, subject to the constraint \( P_n(1) = 1 \) for each \( n \). The Chebyshev polynomials, defined by \( T_n(x) = \cos nx \), \( n = 0, 1, 2, \ldots \), \( x = \cos \theta \), form the best approximation to the zero function over the interval \([-1, 1] \). When translated to \([0, 1] \) they give \( P_n(t) = T_n(1 + 2t)/T_n(3) \) as the best polynomials to minimize \( \mu_n \), where

\[
(1) \quad T_n(x) = [(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n]/2 ,
\]

and

\[
T_n(3) = (\alpha^n + \alpha^{-n})/2, \quad \alpha = 3 + \sqrt{8} \approx 5.828 .
\]

The infinite matrix \( A = (a_{nk}) \), associated with these polynomials, has entries

\[
(2) \quad a_{nk} = \begin{cases} 
1/T_n(3), & k = 0 \\
\frac{2^{2k-1}}{T_n(3)} \left[ 2 \binom{n+k}{n} - \binom{n+k-1}{n-k} \right], & 0 < k \leq n \\
0, & k > n .
\end{cases}
\]

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Bilodeau calls the associated summability method the Chebyshev or σ-method.

We begin by establishing some properties of the maximal entry in each row of σ.

**Lemma 1.** For each positive integer \( n > 2 \), there exists an integer \( p \) such that

\[
a_{nk} < a_{n,k+1} \quad \text{for} \quad 0 \leq k < p
\]

\[
a_{nk} \geq a_{n,k+1} \quad \text{for} \quad p \leq k < n.
\]

**Proof.** For \( 0 < k \leq n \) we may write

\[
a_{nk} = \frac{2^{2k-1}n}{kT_n(3)} \left( \frac{n+k-1}{n-k} \right),
\]

so that \( a_{nk}/a_{n,k+1} = (k+1)(2k+1)/2(n^2-k^2) \). Treating \( k \) as a continuous variable and differentiating with respect to \( k \), it follows that \( a_{nk}/a_{n,k+1} \) is increasing in \( k \). The proof is completed by noting that \( a_{n0} < a_{n1} < a_{n2} \) and \( a_{n,n-1} > a_{nn} \) for each \( n > 2 \).

**Lemma 2.** For each \( n \), \( p = [x_0] \) where \( x_0 = (-3 + (32n^2 - 7)^{1/2})/8 \).

**Proof.** Since \( a_{n1} < a_{n2} \) and \( a_{n,n-1} > a_{nn} \), there exists a real positive number \( x_0 \) such that \( a_{nx0} = a_{n,x0+1} \) which implies \( 2x_0^2 + 3x_0 + 1 = 2n^2 - 2x_0^2 \). Since \( x_0 \) is positive, \( x_0 = (-3 + (32n^2 - 7)^{1/2})/8 \).

**Lemma 3.** For each \( n > 6 \), \( p = [x_0] > n/2 \).

It is sufficient to show that \( x_0 - 1 \geq n/2 \); i.e., \( 8(2n^2 - 11n - 16) \geq 0 \), for \( n > 6 \). With \( g(n) = 2n^2 - 11n - 16 \) we have \( g'(n) > 0 \) for \( n > 11/4 \), hence \( g \) is increasing for \( n > 11/4 \), and \( g \) is positive for \( n > 6 \) and \( n \) an integer.

**Lemma 4.** With \( p \) and \( a_{np} \) as defined in Lemmas 2 and 3, \( \lim_{n} a_{np} = 0 \).

From (3), and Stirling’s formula,

\[
a_{np} = \frac{n2^{p-1}r(n+p)}{PT_n(3)\Gamma(n-p+1)\Gamma(2p)}
\]

\[
\sim \frac{n2^{p-1}(n+p-1)^{n+p-1}e^{-(n+p-1)}(2\pi(n+p-1))^{1/2}}{p\alpha^n(n-p)^{n-p}e^{-(n-p)}(2\pi(n-p))^{1/2}(2p-1)^{p-1}e^{-(2p-1)}(2\pi(2p-1))^{1/2}}
\]

\[
= \frac{1}{2\sqrt{\pi}} \frac{n}{p} \left( \frac{n-p-1}{\alpha(n-1)} \right)^{n-p} \left( \frac{n+p-1}{\sqrt{\alpha(p-1/2)}} \right)^{2p}.
\]
Both \((n + p - 1)/\alpha(n - p)^{n-p}\) and \((n + p - 1)/\sqrt{\alpha (p - 1/2)^{2p}}\) are bounded above. Therefore \(\lim a_{np} = 0\).

Cooke [3, p. 119] shows that a necessary and sufficient condition for a regular matrix to be absolutely translative for all bounded sequences \(\{z_n\}\) is that the matrix \(A\) satisfies \(\lim \sum_{k=0}^n |a_{nk} - a_{n,k+1}| = 0\).

**Theorem 1.** The \(\sigma\)-method is absolutely translative for all bounded sequences.

**Proof.** Bilodeau [1, p. 296] has shown that the \(\sigma\)-method is regular. From Lemma 1,

\[
\sum_{k=0}^{\infty} |a_{nk} - a_{n,k+1}| = \sum_{k=1}^{p-1} (a_{n,k+1} - a_{nk}) + \sum_{k=p}^{n} (a_{nk} - a_{n,k+1}) = 2a_{np} - a_{n0}.
\]

The regularity of \(A\) implies that \(\lim a_{n0} = 0\), and the result follows from Lemma 4.

For unbounded sequences, we consider the class of sequences \(\{z_n\}\) satisfying \(|z_k| \leq \theta_k\) (\(\theta_k\) real, positive, and increasing), where \(\sum_{k=0}^{\infty} a_{nk}\theta_k^{p+1}\), \(\sum_{k=0}^{\infty} a_{n,k+1}\theta_k^{k+1}\), and \(\rho_n = \sum_{k=0}^{\infty} |(a_{nk} - a_{n,k+1})\theta_k^{k+1}|\) exist for each \(n\). Cooke [3, p. 119] shows that a necessary and sufficient condition for a regular matrix to be absolutely translative for all (unbounded) \(\{z_n\}\) satisfying \(|z_k| \leq \theta_k\) together with conditions stated above, is that \(\lim \rho_n = 0\).

**Theorem 2.** The \(\sigma\)-method is absolutely translative for all (unbounded) sequences \(\{z_n\}\) such that \(z_k = o(\sqrt{k})\). This result is best possible.

With \(|z_n| = \theta_n\), and using Lemma 1,

\[
\rho_n = \sum_{k=0}^{p-1} (a_{n,k+1} - a_{nk})\theta_k^{k+1} + \sum_{k=p}^{n} (a_{nk} - a_{n,k+1})\theta_k^{k+1} \leq \theta_{n-1} \sum_{k=0}^{p-1} (a_{n,k+1} - a_{nk}) + \theta_n \sum_{k=p}^{n} (a_{nk} - a_{n,k+1}) \leq \theta_n (a_{np} - a_{n0} + a_{np} - 0) = 0(\sqrt{n}) (2a_{np} - a_{n0}).
\]

It will be sufficient to show that \(\lim 2\sqrt{n}a_{np}\) is finite. But this follows immediately from (4), since \(\lim \sqrt{n(p - 1/2)^{1/2}}/p = 2^{1/4}\), and the remaining limits have already been shown to be finite.

To show that the result is best possible we shall replace \(o(\sqrt{k})\)
by $\sqrt{k}$ and verify that $\rho_n$ does not tend to zero.

From (5), $\rho_n \geq \sqrt{p} \sum_{k=p}^{n} (a_{nk} - a_{n,k+1}) = \sqrt{p} a_{np}$, which does not tend to zero.

Direct calculations verify that $\sigma$ is not a weighted mean, Nörlund, Hausdorff, or generalized Hausdorff method.

Gronwall [4, p. 102] defined a general class of summability methods, each member of which involves a pair of analytic functions $f$ and $g$. Specifically, the $(f, g)$-transform of a given series $\sum_{k=0}^{\infty} u_k$ is the sequence $\{U_n\}$ defined implicitly by the formal power series identity

$$g(w) \sum_{n=0}^{\infty} u_n [f(w)]^n = \sum_{n=0}^{\infty} b_n U_n w^n,$$

where $f$ and $g$ satisfy the following properties. Let $A = \{w \mid |w| < 1\}$. The function $z = f(w)$ is analytic in $A - \{1\}$, continuous and $1 - 1$ in $A$, with $f(0) = 0$, $f(1) = 1$, and $|f(w)| < 1$ for $w \in A$. Moreover, $w = f^{-1}(z)$ has the representation $w = 1 - (1 - z)^{\lambda} [a + a_n(1 - z) + \cdots]$, where $\lambda \geq 1$, $a > 0$, and the quantity in brackets is a power series in $1 - z$ with a positive radius of convergence. The function $g$ satisfies $g(w) \neq 0$ for $w \in A$ and has the form $g(w) = (1 - w)^{-\delta} + \gamma(w)$ for some $\delta > 0$, where $\gamma(w)$ is analytic in $A$. Also $g(w) = \sum_{n=0}^{\infty} b_n w^n$, with $b_n \neq 0$ for each $n$. The series $\sum_{k=0}^{\infty} u_k$ is said to be $(f, g)$-summable to $s$ if $\lim U_n = s$.

Examples of $(f, g)$-methods are the Cesàro methods of order $k$, $(C, k)$, for $k$ real and greater than $-1$; $(E, \beta)$ (Euler-Knopp) for $0 < \beta \leq 1$; de la Vallée Poussin summability $(V)$; a generalized $(V)$-summability $(Vk)$, introduced by Gronwall; and a method of summation of Obrechkoff. We will now show that the Chebyshev method is also a Gronwall method.

Writing $s_n = \sum_{k=0}^{n} u_k$, the $(f, g)$-method can be expressed as a sequence to sequence method by rewriting (6) in the form

$$g(w)[1 - f(w)] \sum_{n=0}^{\infty} s_n [f(w)]^n = \sum_{n=0}^{\infty} b_n U_n w^n.$$ 

Using (7), $(f, g)$ can be expressed as a triangular matrix transformation of the form $U_n = \sum_{k=0}^{n} a_{nk} s_k$, with $a_{nk} = \gamma_{nk}/b_n$, where $\gamma_{nk}$ is defined by

$$[1 - f(w)] g(w) [f(w)]^k = \sum_{n=k}^{\infty} \gamma_{nk} w^k.$$ 

(See, for example, the discussion on page 40 of [2], where the roles of $\gamma_{nk}$ and $a_{nk}$ have been interchanged.) From (8) it follows that
THEOREM 3. The Chebyshev method is a Gronwall method with 
\( f(w) = w(\alpha - 1)^2/(\alpha - w)^2 \), 
\( g(w) = (1 - w)^{-1} + \gamma(w) \), and 
\( \gamma(w) = w/(\alpha^2 - w) \), where \( \alpha = 3 + \sqrt{8} \).

Proof. If (6) is a Gronwall method, then, from (8) with \( k = 0 \) and (2),

\[
[1 - f(w)]g(w) = \sum_{n=0}^{\infty} b_n a_n w^n = \sum_{n=0}^{\infty} b_n w^n/T_n(3) .
\]

Thus

\[
f'(w) = 1 - [g(w)]^{-1} \sum_{n=0}^{\infty} b_n w^n/T_n(3) ,
\]

\[
f''(w) = [g'(w)/g^2(w)] \sum_{n=0}^{\infty} b_n w^n/T_n(3) - [g(w)]^{-1} \sum_{n=1}^{\infty} nb_n w^{n-1}/T_n(3)
\]

and \( f'(0) = [g'(0)/g^2(0)](b_0/T_0(3)) - b_1/g(0)T_1(3) = 2b_1/3b_0 \), since \( T_0(3) = 1 \) and \( T_1(3) = 3 \).

From (9) and (3),

\[
b_n = (2b_1/3b_0)^n T_n(3)/2^{2n-1} = (b_1/6b_0)^n(\alpha^n + \alpha^{-n}) .
\]

In particular, \( b_1 = b_1/b_0 \), which implies \( b_0 = 1 \), since each \( b_n \neq 0 \). One can also deduce that \( b_0 = 1 \) from (9), since \( a_{00} = 1 \).

Thus

\[
g(w) = 1 + \sum_{n=1}^{\infty} b_n w^n
\]

\[
= 1 + \sum_{n=1}^{\infty} [(b_1 \alpha w/6)^n + (b_1 w/6\alpha)^n]
\]

\[
= 1 + \frac{b_1 \alpha w}{6 - b_1 \alpha w} + \frac{b_1 w}{6\alpha - b_1 w}
\]

\[
= \frac{6}{6 - b_1 \alpha w} + \frac{b_1 w}{6\alpha - b_1 w} .
\]

For \( g \) to have the required form choose \( b_1 = 6/\alpha \).

From (10), and (11), with \( b_1 = 6/\alpha \),

\[
f(w) = 1 - [g(w)]^{-1}\left[1 + \sum_{n=1}^{\infty} 2(\alpha w)^n \right]
\]

\[
= 1 - \frac{2w}{\alpha - w}
\]

\[
= 1 - \frac{(\alpha + w)}{\alpha - w} \cdot \frac{(1 - w)(\alpha^2 - w)}{(\alpha^2 - w^2)}
\]

\[
= 1 - \frac{(1 - w)(\alpha^2 - w)}{(\alpha - w)^2} = \frac{w(\alpha - 1)^2}{(\alpha - w)^2} .
\]
We now show that \( f \) is a 1 - 1 selfmapping of \( \Delta \). If \( f(w_1) = f(w_2) \), i.e.,
\[
\frac{w_1(\alpha - 1)^2}{(\alpha - w_1)^2} = \frac{w_2(\alpha - 2)^2}{(\alpha - w_2)^2},
\]
then \((w_1 - w_2)(\alpha^2 - w_1w_2) = 0\). Since \( w_1, w_2 \in \Delta, w_1w_2 = \alpha^2 \), so \( w_1 = w_2 \).

By the Maximum Modules Theorem, it is sufficient to show that \( |f(w)| \leq 1 \) for \( w = e^{i\theta} \). \( |f(e^{i\theta})| = (\alpha - 1)^2(\alpha^2 - 2 \cos \theta + 1) \leq 1 \).

We now verify that \( w = f^{-1}(z) \) is regular on \( \Delta - \Delta \), except possibly at \( z = 1 \), and that \( 0 \in \Delta \). \( f^{-1} \) is regular except at \( z = 0 \), so now we must show
\[
\min_{0 \leq \theta < 2\pi} |f(e^{i\theta})| \geq \delta > 0.
\]

\( |f(e^{i\theta})| = (\alpha - 1)^2/T(\theta) \), where \( T(\theta) = (\alpha + 1)^2 - 4\alpha \cos \theta /2 \). A direct calculation certifies that the maximum of \( T(\theta) \) occurs at \( \theta = \pi \), and \( T(\pi) = [(\alpha - 1)/(\alpha + 1)]^2 > 0 \).

It remains to show that at \( z = 1, 1 - w = (1 - z)^4[a + a_i(1 - z) + \cdots] \), \( \lambda \geq 1, a > 0 \). \( z = f(w) = (\alpha - 1)^2w/(\alpha - w)^2 \). From the equation \( z = f(w) \) we obtain \( 1 - z = (1 - w)(\alpha^2 - w)/(\alpha - w)^2 \), which when solved for \( 1 - w \) yields
\[
1 - w = \frac{-(\alpha - 1)(1 - 2z - \alpha) \pm (\alpha - 1)(\alpha + 1)\sqrt{1 - 4\alpha(1 - z)/\alpha + 1}}{-2z}.
\]

Now divide the numerator and the denominator by \(-2\) and write \( z \) in the denominator as \( 1 - (1 - z) \).
\[
1 - w = \left\{ \frac{\alpha - 1}{2} \left[ 2(1 - z) - (\alpha + 1) \right] \pm \frac{\alpha^2 - 1}{-2} \left[ 1 - \frac{4\alpha}{2(\alpha + 1)^2}(1 - z) \right. \\
\left. + \frac{1}{8} \frac{16\alpha^2}{(\alpha + 1)^4}(1 - z)^2 + \cdots \right] \right\} \cdot \sum_{k=0}^{\infty} (1 - z)^k.
\]

Using the negative branch,
\[
1 - w = \left\{ (\alpha - 1)(1 - z) - \frac{\alpha(\alpha^2 - 1)}{(\alpha + 1)^2}(1 - z) - \frac{1}{8} \frac{16\alpha^2}{(\alpha + 1)^4}(1 - z)^2 \\
\right. \\
\left. + \cdots \right\} \cdot (1 + (1 - z) + (1 - z)^2 + \cdots).
\]
\[
= (1 - z) \left\{ (\alpha - 1) - \frac{\alpha(\alpha - 1)}{\alpha + 1} + \sum_{k=1}^{\infty} b_k(1 - z)^k \right\}
\]

Therefore \( 1 - w = (1 - z)^4[a + a_i(1 - z) + \cdots] \) where \( \lambda = 1 \) and \( a = (\alpha - 1)/(\alpha + 1) > 0 \).

Theorem 3, along with Theorems 1 and 2 of [2] show that the Chebyshev method is neither an \([F, d_n]\) nor a Sonnenschein method.
One of the important properties of \((f, g)\)-summability is the following [5, p. 267]:

Let \((f, g), (f', g')\) be two Gronwall means which map regions \(D, D_i\) and with exponents \(\lambda, \lambda_i\). If \(\lambda > \lambda_i\), and \(D\) is interior to \(D_i\), then \((f, g)\) is stronger than \((f', g')\); i.e., \((f, g) \succ (f', g')\).

The de la Vallee Poussin method \((V)\) [4, p. 103] is a Gronwall method with \(\delta = 2^{-1}\), \(f(w) = (1 - \sqrt{1 - w})/(1 - \sqrt{1 - w})\), \(g(w) = (1 - w)^{1/2}\) and \(\lambda = 2\).

**Theorem 4.** \((V) \supset (\sigma)\).

**Proof.** Since \(\lambda_{(V)} = 2\), \(\lambda_{(\sigma)} = 1\), it is enough to show that \(D(V)\) is interior to \(D(\sigma)\), that is,

\[
\frac{|1 - \sqrt{1 - w}|}{1 + \sqrt{1 - w}} \leq \frac{|(\alpha - 1)w|}{|\alpha - w|^2}.
\]

It suffices to consider \(|w| = 1\); thus we need to show

\[
\frac{1}{|(1 + \sqrt{1 - w})^2|} \leq \frac{(\alpha - 1)^2}{|\alpha - w|^2}.
\]  

Writing \(1 - w = \rho e^{i\phi}\), where \(-\pi < \phi < \pi\), (12) becomes

\[
|\alpha - 1 + \rho e^{i\phi}|^2 \leq (\alpha - 1)^2 |1 + \rho^{1/2} e^{i\phi/2}|^2,
\]

i.e.,

\[
2(\alpha - 1) \cos \phi + \rho \leq 4\alpha(2\rho^{-1/2} \cos \phi/2 + 1).
\]

Since \(\cos \phi/2 > 0\), it is sufficient to show that \(2(\alpha - 1) \cos \phi + \rho \leq 4\alpha\), which is readily verified.

**Theorem 5.** \(\sigma \not\supseteq (C, 1)\).

We shall make use of the well-known result that if \(A\) and \(B\) are regular summability methods, and \(B\) is a triangle, then \((A) \supseteq (B)\) if and only if \(AB^{-1}\) is regular.

Consider \(D = AC^{-1}\), where \(A\) is the Chebyshev method and \(C\) is \((C, 1)\). \(C^{-1}\) has entries

\[
c^{-1}_{nk} = \begin{cases} 
-n, & k = n-1 \\
n+1, & k = n \\
0, & \text{elsewhere}.
\end{cases}
\]

Then
\[ d_{nk} = \begin{cases} (k + 1)a_{nk} - (k + 1)a_{n,k+1}, & k < n \\ (n + 1)a_{nn}, & k = n \\ 0, & \text{elsewhere} \end{cases} \]

We shall show that \( D \) has infinite norm.

\[
\sum_{k=0}^{n} \left| d_{nk} \right| = \sum_{k=0}^{p-1} (k + 1)(a_{n,k+1} - a_{nk}) + \sum_{k=p}^{n-1} (k + 1)(a_{nk} - a_{n,k+1}) + a_{nn}(n + 1).
\]

Now,

\[
\sum_{k=0}^{p-1} (k + 1)(a_{n,k+1} - a_{nk}) = \sum_{k=0}^{p-1} (k + 1)a_{n,k+1} - \sum_{k=0}^{p-1} ka_{nk} - \sum_{k=0}^{p-1} a_{nk} = pa_{np} - \sum_{k=0}^{p-1} a_{nk}.
\]

\[
\sum_{k=p}^{n-1} (k + 1)(a_{nk} - a_{n,k+1}) = \sum_{k=p}^{n-1} ka_{nk} + \sum_{k=p}^{n-1} a_{nk} - \sum_{k=p}^{n-1} (k + 1)a_{n,k+1} = pa_{np} - na_{nn} + \sum_{k=p}^{n-1} a_{nk}.
\]

Therefore,

\[
\sum_{k=0}^{n} \left| d_{nk} \right| = pa_{np} - \sum_{k=0}^{p-1} a_{nk} + pa_{np} - na_{nn} + \sum_{k=p}^{n-1} a_{nk} + a_{nn}(n + 1).
\]

Since the Chebyshev method has row sums equal to 1,

\[
\sum_{k=p}^{n-1} a_{nk} = 1 - \sum_{k=0}^{p-1} a_{nk} - a_{nn}.
\]

Thus

\[
\sum_{k=0}^{n} d_{nk} = 2pa_{np} - 2 \sum_{k=0}^{p-1} a_{nk} + 1.
\]

But \( \sum_{k=0}^{p-1} a_{nk} \leq 1 \), so it is sufficient to show \( pa_{np} \to \infty \). This follows immediately from (2), since \( \lim \sqrt{n} = \infty \) and the remaining limits have already been shown to be finite and nonzero.

The Fourier series

\[
\sum_{k=1}^{\infty} \sin kt/k = (\pi - t)/2, \quad 0 < t \leq \pi,
\]

converges for all \( t \), and the function has a jump at \( t = 0 \). Hence
the convergence is nonuniform at \( t = 0 \); that is, the sequence \( \{s_n(t_n)\} \), where \( \{t_n\} \) is a positive null sequence and

\[
s_n(t) = \sum_{k=1}^{n} \sin kt/k, \quad s_0 = 0,
\]

has several limit points, depending on the manner in which \( t_n \) approaches 0.

If \( \lim nt_n = \tau \geq 0 \), then \( \lim s_n(t_n) = \int_{0}^{\tau} (\sin t/t)dt \), and the maximal limit is attained when \( \tau = \pi \), in which case

\[
\lim s_n(t_n) = \int_{0}^{\pi} \frac{\sin t}{t}dt = \frac{\pi}{2} \times 1.17897 \cdots.
\]

On the other hand, \( (\pi - t)/2 \to \pi/2 \) as \( t \downarrow 0 \). Thus the limit points of \( \{s_n(t_n)\} \) cover an interval which extends beyond \( f(0^+) \) if \( f(0^+) \neq 0 \). This situation is called the Gibbs phenomenon relative to the partial sums.

We shall now show that the corresponding phenomenon occurs for the Chebyshev means.

**Theorem 6.** The Chebyshev means of (13) satisfy

\[
\text{lim sup } \sigma_n(t_n) \leq \int_{0}^{\pi} \frac{\sin t}{t}dt.
\]

The lim sup inequality is an immediate consequence of (14) and the well-known fact that, for any totally regular matrix \( A \), and any sequence \( x = \{x_n\} \), \( \lim sup A_n(x) \leq \lim sup x_n \).

The proof of the theorem is similar to that of [6]. One may write \( s_n(t) \) in the form

\[
s_n(t) = -t/2 + \int_{0}^{t} \frac{\sin (n + 1/2)x}{2 \sin (x/2)}dx.
\]

Since \( \sin (k + 1/2)x = \mathcal{J}(\exp (i(k + 1/2)x)) \),

\[
\sigma_n(t) = -t/2 + \mathcal{J} \left[ \int_{0}^{t} \frac{1}{2 \sin (x/2)} \sum_{k=0}^{n} a_{nk}e^{ikx}e^{ix/2}dx \right].
\]

From [1, p. 297], \( \sum_{k=0}^{n} a_{nk}e^{ikx} = T_n(1 + 2e^{ix})/T_n(3) \), where \( T_n(x) \) is defined by (1).

Define
\[ n^e = 1 + 2e^ix + [(1 + 2e^ix)^2 - 1]^{1/2} = 1 + 2e^ix + 2e^ix/2e^ix/2 \cos x/2 \right)^{1/2} \]

Let \( a = (2 \cos x/2)^{1/2} \). Then \( \rho \cos \beta = 1 + 2(\cos x + \alpha \cos (3x/4)) \),

(16) \[ \rho \sin \beta = 2(\sin x + \alpha \sin (3x/4)) \]

and

(17) \[ \rho^2 = 5 + 4(\cos x + \alpha \cos (3x/4)) + 8(\cos (x/2) + \alpha \cos (x/4)) \]

Therefore \( 1 + 2e^ix - [(1 + 2e^ix)^2 - 1]^{1/2} = \rho^{-1}e^{-ix} \), and assume \( 0 < x \leq t \leq \pi/2 \).

\[
\sigma_n(t) + t/2 = \frac{1}{2T_n(3)} \int_0^t \frac{1}{2 \sin (x/2)} \left[ \rho^n \sin (n\beta + x/2) - \rho^{-n} \sin (n\beta - x/2) \right] dx = \frac{1}{4T_n(3)} \left\{ \int_0^t \rho^n \cot (x/2) \sin n\beta dx + \int_0^t \rho^{-n} \cot (x/2) \sin n\beta dx + \int_0^t \rho^{-n} \cos n\beta dx \right\} .
\]

From (17), \( \rho \) is monotone decreasing in \( x \) for \( 0 < x \leq \pi/2 \). Therefore for \( 0 < x \leq \pi/2, \rho < \alpha \). Thus

\[
\left| \frac{1}{2T_n(3)} \int_0^t \rho^n \cos n\beta dx \right| < \int_0^t (\rho/\alpha)^n dx < t ,
\]

so that there exists an \( \eta \) satisfying \( |\eta| < 1 \) such that

\[
\frac{1}{2T_n(3)} \int_0^t \rho^n \cos n\beta dx = \eta t .
\]

Now assume that \( t = t_n, nt_n \to \tau, 0 \leq \tau \leq \infty \), and \( nt_n^\varepsilon \to 0 \).

Since, from (17), \( \rho \geq \sqrt{5} \),

\[
\left| \frac{1}{4T_n(3)} \int_0^t \rho^{-n} \cos n\beta dx \right| < \frac{1}{4(\alpha \sqrt{5})^n} < \frac{1}{\sqrt{2}(\alpha \sqrt{5})^n} \int_0^t n\beta \cot (x/2) dx .
\]

We wish to show that \( \beta < x \). For \( 0 < x \leq \pi/2 \), from (16), \( \rho \sin \beta < 2(1 + \alpha) \sin x \). From (17), if \( \cos (3x/4) + 2 \cos (x/4) \geq 2 \), then \( \rho > 2(\alpha + 1) \). In the interval \([0, \pi/2]\),

\[
\cos (3x/4) + 2 \cos (x/4) \geq \cos (3\pi/8) + \cos (\pi/8) = \cos (\pi/8)(4 \cos^2 (\pi/8) - 1) .
\]

Since \( \cos (\pi/8) = \sqrt{2 + \sqrt{2}/2} \), it is sufficient to show that
which is easily verified. Therefore \(0 < \sin < \beta(\rho/2(1 + a))\) \(\sin \beta < \sin x\), and \(\beta < x\).

For \(0 < x \leq \pi/2\), \(2 \leq x/\sin(x/2) \leq \pi/\sqrt{2}\). Substituting in (18) we have

\[
\left| \frac{1}{4T_n(3)} \int_0^t \rho^n \cot(x/2) \sin n\beta dx \right| < \frac{n}{2(\alpha\sqrt{5})^n} \int_0^{\pi/2} \cos(x/2) \cdot \frac{x}{\sin(x/2)} dx
\]

\[
< \frac{n\pi^2}{4\sqrt{2(\alpha\sqrt{5})^n}} = o(1) ,
\]

and

\[
\sigma_n(t) + (1 - \eta)t/2 = \frac{1}{4T_n(3)} \int_0^t \rho^n \cot(x/2) \sin n\beta dx + o(1) .
\]

Using (17), and the values of \(a\) and \(\alpha\),

\[
1 - (\rho/\alpha)^2 = [17 + 12\sqrt{2} - 5 - 4(\cos x + a \cos(3x/4)) - 8(\cos(x/2) + a \cos(x/4))] / \alpha^2
\]

\[
= \frac{4}{\alpha^2}[1 - \cos x + 2(1 - \cos(x/2)) + \sqrt{2}(1 - \cos(3x/4)\sqrt{\cos(x/2)}) + 2\sqrt{2}(1 - \cos(x/4)\sqrt{\cos(x/2)})].
\]

Since \(0 < \cos(x/2) < 1\),

\[
1 - \cos(x/4)\sqrt{\cos(x/2)} \leq 1 - \cos(x/4) \cos(x/2)
\]

\[
= 1 - (\cos(3x/4) + \cos(x/4))/2 .
\]

Similarly, \(1 - \cos(3x/4)\sqrt{\cos(x/2)} \leq 1 - (\cos(5x/4) + \cos(x/4))/2\). Therefore,

\[
1 - (\rho/\alpha)^2 \leq \frac{4}{\alpha^2}[2 \sin^2(x/2) + 4 \sin^2(x/4) + \sqrt{2}(2 \sin^2(5x/8)
\]

\[
+ 2 \sin^2(x/8))/2 + \sqrt{2}(2 \sin^2(3x/8) + 2 \sin^2(x/8))] \leq \frac{4}{\alpha^2}[2(x/2)^2 + 4(x/4)^2 + \sqrt{2}((5x/8)^2 + (x/8)^2)
\]

\[
+ 2\sqrt{2}(3x/8)^2 + (x/8)^2)] = \frac{4}{\alpha^2}\left(3/4 + \frac{46\sqrt{2}}{64}\right) x^2 < \frac{x^2}{4} .
\]

Since \(0 < \rho/\alpha < 1\), \(1 - \rho/\alpha \leq 1 - (\rho/\alpha)^2\), so that \(1 - \rho/\alpha < x^2/4\). 0 <
\[ 1 - (\rho/\alpha)^n = (1 - \rho/\alpha) \sum_{k=0}^{n-1} (\rho/\alpha)^k < n(1 - \rho/\alpha) < nx^2/4. \]

Therefore \( 1 - (\rho/\alpha)^n = \lambda nx^2 \) for some \( \lambda \) satisfying \( 0 < \lambda < 1/4 \), so that we may write

\[
\frac{1}{2T_\alpha(3)} \int_0^t \rho^n \cot(x/2) \sin n\beta dx = \frac{\alpha^n}{2T_\alpha(3)} \left[ \int_0^t \cot(x/2) \sin n\beta dx \right] \\
- n \left[ \int_0^t \lambda x^2 \cot(x/2) \sin n\beta dx \right] \\
\times \left[ \int_0^t \lambda x^2 \cot(x/2) \sin n\beta dx \right] < n \left[ \int_0^t x^2 \cot(x/2) dx \right] \\
\leq \frac{nt\pi}{\sqrt{2}} \int_0^t dx < nt^2 = o(1),
\]

since \( \lim nt^2 = 0 \). Note that \( \lim \alpha^n/2T_\alpha(3) = 1 \).

Using (17),

\[
\frac{\rho\beta}{2} - \frac{2}{\alpha} \left( 1 + \frac{3\sqrt{2}}{4} \right)x = \frac{\rho}{\alpha} (\beta - \sin \beta - \frac{2}{\alpha} (x - \sin x) \\
- \frac{2\sqrt{2}}{\alpha} \left( \frac{3x}{4} - \sin (3x/4)\sqrt{\cos(x/2)} \right),
\]

so that

\[
|\rho\beta/\alpha - rx| \leq \frac{\rho}{\alpha} |\beta - \sin \beta| + \frac{2}{\alpha} |x - \sin x| \\
+ \frac{2\sqrt{2}}{\alpha} \left| \frac{3x}{4} - \sin (3x/4)\sqrt{\cos(x/2)} \right|,
\]

where \( r = 2(1 + 3\sqrt{2}/4)/\alpha = (4 + 3\sqrt{2})/2\alpha = (4 + 3\sqrt{2})(3 - 2\sqrt{2})/2 = 1/\sqrt{2} \).

But \( 0 \leq 3x/4 - \sin (3x/4)\sqrt{\cos(x/2)} \leq 3x/4 - \sin (3x/4) \cos(x/2) \), \( \sin (3x/4) \geq 3x/4 - (3x/4)^3/3! \), and \( \cos(x/2) \geq 1 - x^2/4 \), so that

\[
|3x/4 - \sin (3x/4)\sqrt{\cos(x/2)}| \leq 3x/4 - (3x/4 - (3x/4)^3/6)(1 - x^2/4) \\
= 33x^5/128.
\]

Since \( 0 < x - \sin x < x^3 \) and \( \beta < x \),

\[
|\rho\beta/\alpha - x/\sqrt{2}| \leq (\rho\beta^3 + 2x^3 + 33\sqrt{2}x^5/64)/\alpha < 2x^3.
\]

Also, \( |\beta - x/\sqrt{2}| \leq |\rho\beta/\alpha - x/\sqrt{2}| + (1 - \rho/\alpha)\beta < 2x^3 + x^3 = x^3 \), so that \( \beta = x/\sqrt{2} + \mu x^3 \), where \( |\mu| < 3 \).

The remainder of the proof of (15) is the same as that of [6], beginning with formula (2.7), and will therefore be omitted.
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