

# Pacific Journal of Mathematics

## **A LOWER BOUND FOR THE NUMBER OF CONJUGACY CLASSES IN A FINITE NILPOTENT GROUP**

GARY JOSEPH SHERMAN

# A LOWER BOUND FOR THE NUMBER OF CONJUGACY CLASSES IN A FINITE NILPOTENT GROUP

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**A lower bound is given for the number of conjugacy classes in a finite nilpotent group which reflects the nilpotency class of the group.**

The problem of estimating the number of conjugacy classes,  $k$ , in a finite group  $G$ , has been around since the turn of the century. Probably the earliest version of the problem is the question: Do there exist groups of arbitrarily large finite order with a fixed number of conjugacy classes? In 1903 Landau [4] answered this question in the negative by showing  $k(G)$  goes to infinity with  $|G|$ . By refining Landau's technique, Erdos and Turan [2] proved  $k(G) > \log_2 \log_2 |G|$ . The known lower bound for  $k(G)$  when  $G$  is nilpotent is somewhat better,  $k(G) > \log_2 |G|$ . This follows from a parametric equation for  $k(G)$  when  $G$  is a  $p$ -group given by Poland [5].

In [3] Gustafson posed the problem of finding improved lower bounds for  $k(G)$ . Recently, Bertram [1] provided a substantial improvement of the  $\log_2 \log_2 |G|$  bound which holds for "most" group orders. The purpose of this note is to give a lower bound for  $k(G)$  when  $G$  is nilpotent which reflects the nilpotency class of  $G$  and improves the  $\log_2 |G|$  bound.

**THEOREM.** *If  $G$  is a finite nilpotent group of nilpotency class  $n$ , then  $k(G) \geq n|G|^{1/n} - n + 1$ .*

*Proof.* We observe that

$$(1) \quad G = Z_0 \cup \left( \bigcup_{i=1}^n Z_i - Z_{i-1} \right)$$

where  $e = Z_0 \subseteq Z_1 \subseteq \dots \subseteq Z_n = G$  is the upper central series of  $G$ . Since  $Z_i$  and  $Z_{i-1}$  are normal subsets of  $G$ ,  $Z_i - Z_{i-1}$  is a union of conjugacy classes of  $G$ . Indeed, for  $x \in Z_i - Z_{i-1}$  and  $g \in G$  we have  $x^{-1}g^{-1}xg \in Z_{i-1}$  because  $Z_i/Z_{i-1}$  is the center of  $G/Z_{i-1}$ . This implies  $g^{-1}xg \in xZ_{i-1}$  and we conclude  $\bar{x}$ , the conjugacy class of  $x$  in  $G$ , is contained in  $xZ_{i-1}$ . Thus  $|\bar{x}| \leq |xZ_{i-1}| = |Z_{i-1}|$  and therefore  $Z_i - Z_{i-1}$  is a union of at least  $|Z_i|/|Z_{i-1}| - 1$  conjugacy classes. It follows from (1) that

$$k(G) \geq 1 + \sum_{i=1}^n (|Z_i|/|Z_{i-1}| - 1)$$

$$\begin{aligned}
 &= \left( \sum_{i=1}^n |Z_i|/|Z_{i-1}| \right) - n + 1 \\
 (2) \quad &= \frac{1}{n} \left( \sum_{i=1}^n n |Z_i|/|Z_{i-1}| \right) - n + 1.
 \end{aligned}$$

The arithmetic-geometric means inequality applied to the sum in (2) yields

$$\begin{aligned}
 k(G) &\geq \left( \prod_{i=1}^n n |Z_i|/|Z_{i-1}| \right)^{1/n} - n + 1 \\
 &= n |G|^{1/n} - n + 1.
 \end{aligned}$$

Let us illustrate how this result can be used to sharpen the  $\log_2 |G|$  bound for  $k(G)$ . Specifically, suppose  $G$  is a nilpotent group of order  $2^5 5^7 7^4$ . We note that  $k(G) \geq 33$  since  $\log_2(2^5 5^7 7^4) > 32$ .

Can we determine the nilpotency class of  $G$ ? Not exactly, but the class of a nilpotent group is the maximum of the classes of its  $p$ -Sylow subgroups and the class of a  $p$ -group of order  $p^m$ ,  $m \geq 3$ , is at most  $m - 1$  so the class of  $G$  is at most 6. Fortunately  $n |G|^{1/n} - n + 1$  is a decreasing function of  $n$  and therefore  $k(G) \geq 6(2^5 5^7 7^4)^{1/6} - 5 > 250$ . Thus  $k(G) \geq 251$ . To improve this bound we make use of the fact that  $k(G)$  is multiplicative; i.e., the number of conjugacy classes in a direct product is the product of the number of conjugacy classes in each factor. This implies  $k(G) \geq (4 \cdot 2^{5/4} - 3)(6 \cdot 5^{7/6} - 5)(3 \cdot 7^{4/3} - 2) > 8510$ . Thus  $k(G) \geq 8511$ .

As a corollary to the theorem and the preceding remarks:

**THEOREM.** *If  $G$  is a finite nilpotent of order  $p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$  and nilpotency class  $n$ , then*

$$k(G) \geq \prod_{i=1}^s (t_i(p_i^{r_i/t_i}) - t_i + 1) \geq n |G|^{1/n} - n + 1 > \log_2 |G|,$$

where the  $p_i$ 's are distinct primes and  $t_i = \max\{1, r_i - 1\}$ .

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