A LOWER BOUND FOR THE NUMBER OF CONJUGACY CLASSES IN A FINITE NILPOTENT GROUP

GARY JOSEPH SHERMAN
A LOWER BOUND FOR THE NUMBER OF CONJUGACY CLASSES IN A FINITE NILPOTENT GROUP

GARY SHERMAN

A lower bound is given for the number of conjugacy classes in a finite nilpotent group which reflects the nilpotency class of the group.

The problem of estimating the number of conjugacy classes, $k$, in a finite group $G$, has been around since the turn of the century. Probably the earliest version of the problem is the question: Do there exist groups of arbitrarily large finite order with a fixed number of conjugacy classes? In 1903 Landau [4] answered this question in the negative by showing $k(G)$ goes to infinity with $|G|$. By refining Landau's technique, Erdos and Turan [2] proved $k(G) > \log_2 \log_2 |G|$. The known lower bound for $k(G)$ when $G$ is nilpotent is somewhat better, $k(G) > \log_2 |G|$. This follows from a parametric equation for $k(G)$ when $G$ is a $p$-group given by Poland [5].

In [3] Gustafson posed the problem of finding improved lower bounds for $k(G)$. Recently, Bertram [1] provided a substantial improvement of the $\log_2 \log_2 |G|$ bound which holds for "most" group orders. The purpose of this note is to give a lower bound for $k(G)$ when $G$ is nilpotent which reflects the nilpotency class of $G$ and improves the $\log_2 |G|$ bound.

**Theorem.** If $G$ is a finite nilpotent group of nilpotency class $n$, then $k(G) \geq n |G|^{1/n} - n + 1$.

**Proof.** We observe that

$$G = Z_0 \cup \left( \bigcup_{i=1}^{n} Z_i - Z_{i-1} \right)$$

where $e = Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_n = G$ is the upper central series of $G$. Since $Z_i$ and $Z_{i-1}$ are normal subsets of $G$, $Z_i - Z_{i-1}$ is a union of conjugacy classes of $G$. Indeed, for $x \in Z_i - Z_{i-1}$ and $g \in G$ we have $x^{-1}g^{-1}xg \in Z_{i-1}$ because $Z_i/Z_{i-1}$ is the center of $G/Z_{i-1}$. This implies $g^{-1}xg \in xZ_{i-1}$ and we conclude $x$, the conjugacy class of $x$ in $G$, is contained in $xZ_{i-1}$. Thus $|x| \leq |xZ_{i-1}| = |Z_{i-1}|$ and therefore $Z_i - Z_{i-1}$ is a union of at least $|Z_i|/|Z_{i-1}| - 1$ conjugacy classes. It follows from (1) that

$$k(G) \geq 1 + \sum_{i=1}^{n} (|Z_i|/|Z_{i-1}| - 1)$$
The arithmetic-geometric means inequality applied to the sum in (2) yields

\[ k(G) \geq \left( \prod_{i=1}^{n} n \frac{|Z_i|}{|Z_{i-1}|} \right)^{1/n} - n + 1 = n|G|^{1/n} - n + 1. \]

Let us illustrate how this result can be used to sharpen the \( \log_2 |G| \) bound for \( k(G) \). Specifically, suppose \( G \) is a nilpotent group of order \( 2^55^77^4 \). We note that \( k(G) \geq 33 \) since \( \log_2(2^{55}5^77^4) > 32 \).

Can we determine the nilpotency class of \( G \)? Not exactly, but the class of a nilpotent group is the maximum of the classes of its \( p \)-Sylow subgroups and the class of a \( p \)-group of order \( p^m \), \( m \geq 3 \), is at most \( m - 1 \) so the class of \( G \) is at most 6. Fortunately \( n|G|^{1/n} - n + 1 \) is a decreasing function of \( n \) and therefore \( k(G) \geq 6(2^{55}5^77^4)^{1/6} - 5 > 250 \). Thus \( k(G) \geq 251 \). To improve this bound we make use of the fact that \( k(G) \) is multiplicative; i.e., the number of conjugacy classes in a direct product is the product of the number of conjugacy classes in each factor. This implies \( k(G) \geq (4 \cdot 2^{55}4 - 3)(6 \cdot 5^76 - 5)(3 \cdot 7^43 - 2) > 8510 \). Thus \( k(G) \geq 8511 \).

As a corollary to the theorem and the preceding remarks:

**Theorem.** If \( G \) is a finite nilpotent of order \( p_1^{r_1}p_2^{r_2} \cdots p_s^{r_s} \) and nilpotency class \( n \), then

\[ k(G) \geq \prod_{i=1}^{s} (t_i(p_i^{r_i/t_i}) - t_i + 1) \geq n|G|^{1/n} - n + 1 > \log_2 |G|, \]

where the \( p_i \)'s are distinct primes and \( t_i = \max \{1, r_i - 1\} \).

**References**

Jeroen Bruijning and Jun-iti Nagata, *A characterization of covering dimension by use of $\Delta_k(X)$* ................................................................. 1
Thomas Ashland Chapman, *Homotopy conditions which detect simple homotopy equivalences* .............................................................. 13
John Albert Chatfield, *Solution for an integral equation with continuous interval functions* ................................................................. 47
Ajit Kaur Chilana and Ajay Kumar, *Spectral synthesis in Segal algebras on hypergroups* ........................................................................ 59
Lung O. Chung, Jiang Luh and Anthony N. Richoux, *Derivations and commutativity of rings* ................................................................. 77
Michael George Cowling and Paul Rodway, *Restrictions of certain function spaces to closed subgroups of locally compact groups* .......... 91
David Dixon, *The fundamental divisor of normal double points of surfaces* ..... 105
Hans Georg Feichtinger, Colin C. Graham and Eric Howard Lakien, *Nonfactorization in commutative, weakly selfadjoint Banach algebras* ......................... 117
Michael Freedman, *Cancelling 1-handles and some topological imbeddings* .... 127
Frank E., III Gerth, *The Iwasawa invariant $\mu$ for quadratic fields* .............. 131
Maurice Gilmore, *Three-dimensional open books constructed from the identity map* ........................................................................ 137
Stanley P. Gudder, *A Radon-Nikodým theorem for $\ast$-algebras* .................. 141
Peter Wamer Harley, III and George Frank McNulty, *When is a point Borel?* 151
Charles Henry Heiberg, *Fourier series with bounded convolution powers* .... 159
Rebecca A. Herb, *Characters of averaged discrete series on semisimple real Lie groups* ....................................................................... 169
Hideo Imai, *On singular indices of rotation free densities* .............................. 179
Sushil Jajodia, *On 2-dimensional CW-complexes with a single 2-cell* ............ 191
Herbert Meyer Kamowitz, *Compact operators of the form $uC_\phi$* ............. 205
Matthew Liu and Billy E. Rhoades, *Some properties of the Chebyshev method* ..... 213
George Edgar Parker, *Semigroups of continuous transformations and generating inverse limit sequences* ........................................... 227
Samuel Murray Rankin, III, *Oscillation results for a nonhomogeneous equation* .................................................................................. 237
Martin Scharlemann, *Transverse Whitehead triangulations* .......................... 245
Gary Joseph Sherman, *A lower bound for the number of conjugacy classes in a finite nilpotent group* ...................................................... 253
Richard Arthur Shoop, *The Lebesgue constants for $(f, d_n)$-summability* .... 255
Stuart Jay Sidney, *Functions which operate on the real part of a uniform algebra* .................................................................................... 265
Tim Eden Traynor, *The group-valued Lebesgue decomposition* .................... 273
Tavan Thomas Trent, *$H^2(\mu)$ spaces and bounded point evaluations* ....... 279
James Li-Ming Wang, *Approximation by rational modules on nowhere dense sets* ........................................................................ 293