Pacific Journal of Mathematics

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Vol. 80, No. 1 September 1979

PACIFIC JOURNAL OF MATHEMATICS
Vol. 80, No. 1, 1979

$H^2(\mu)$ SPACES AND BOUNDED POINT EVALUATIONS

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Let $H^2(\mu)$ denote the closure of the polynomials in $L^2(\mu)$, where μ is a positive finite compactly supported Borel measure carried by the closed unit disc $\bar{D}$. For $\lambda \in \bar{D}$, define $E(\lambda)=\sup\{|p(\lambda)|/\|p\|_\mu\}$, where the suprenum is taken over all polynomials whose $L^2(\mu)$ norm is not zero. If $E(\lambda)<\infty$ we say that μ has a bounded point evaluation at λ, abbreviated b.p.e. at λ. Whenever $E(\lambda)<\infty$ we may fix the value of $f \in H^2(\mu)$ at λ. We determine the set on which all functions in $H^2(\mu)$ have (fixed) analytic values in terms of the parts of the spectrum of a certain operator.

In the case that the support of μ has a hole H bounded by an exposed arc Γ contained in ∂D and $E(z)$ is finite in H, we show how to recover the absolutely continuous part (with respect to Lebesgue measure on ∂D) of $d\mu|_\Gamma$ from a knowledge of the $E(z)$'s in H. A corollary of this is that for such measures μ the functions in $H^2(\mu)$ behave locally near Γ like those of classical Hardy space. That is, they have boundary values and their zero sets near Γ satisfy a Blaschke type growth condition. We apply this corollary to measures of the form $d\nu = GdA + wd\sigma$ to study the local behavior of functions in $H^2(\nu)$ near Γ (A denotes planar measure on $\bar{D}$, $d\sigma$ denotes linear Lebesgue measure on ∂D, and G and w are in an appropriate sense not too small on D and Γ respectively).

1. Bounded evaluations and analytic extensions of functions in $H^2(\mu)$. Let μ be a finite positive compactly supported Borel measure carried by the closed unit disc $\bar{D}$. We note that for λ a complex number, the point evaluation functional defined on polynomials by

$$p \longrightarrow p(\lambda)$$

is bounded with respect to the $L^2(\mu)$ norm if and only if $E(\lambda)<\infty$. In this latter case, by the Riesz representation theorem there is a unique element of $H^2(\mu)$, denoted by k_λ, satisfying

$$p(\lambda) = \langle p, k_\lambda \rangle$$

for all polynomials p and $\|k_\lambda\| = E(\lambda)$. We call k_λ the bounded evaluation functional for μ at λ, abbreviated b.e.f. for μ at λ.

If μ has a b.p.e. at λ with b.e.f. k_λ and $f \in H^2(\mu)$, then we fix the value of f at λ by

(1) $$\tilde{f}(\lambda) = \langle f, k_\lambda \rangle .$$

We remark that if μ has b.p.e's on a set of positive μ measure then the values $\tilde{f}$ of f fixed by (1) agree μ-a.e. with any representative of f. Also the "filling in holes" theorem due to Bram [1], interpreted in this context, says that if H is a hole of the support of μ then either

(2) $$\mu \text{ has b.p.e.'s at every } \lambda \in H$$

or else

(3) $$\mu \text{ has no b.p.e.'s in } H .$$

Whenever (2) occurs the functions in $H^2(\mu)$ can be extended into the hole H.

It is well known that if $f \in H^2(\mu)$ then $\tilde{f}$ is analytic in any holes satisfying (2). We specify the largest open set on which all extensions of functions in $H^2(\mu)$ are analytic.

Let M_μ denote the bounded linear operator multiplication by z on $H^2(\mu)$. $\Lambda(M_\mu)$, $\Gamma(M_\mu)$, and $\Pi(M_\mu)$ will designate the spectrum, the compression spectrum, and the approximate point spectrum of M_μ, respectively [see 12]. If O is an open set on which all extensions of functions in $H^2(\mu)$ are analytic, then we call O an *analytic set for* μ. If $G \subset C$ then we denote the interior of G by int G.

THEOREM 1.1. *The largest analytic set for* μ *is* $\operatorname{int}(\Gamma(M_\mu) - \Pi(M_\mu))$.

Proof. If O is any analytic set for μ and $F \subset O$ is compact, then using the Banach Steinhaus theorem [16] we see that

$$\sup\{\|k_\lambda\| : \lambda \in F\} < \infty .$$

Also if O is an open set and $\lambda \to \|k_\lambda\|$ is bounded on compact subsets of O, then using (1) and the Cauchy-Schwartz inequality it follows that O is an analytic set for μ.

Assume that O is an analytic set for μ. It is well known that $O \subset \Gamma(M_\mu)$. (This is just the statement that $M_\mu^* k_\lambda = \bar{\lambda} k_\lambda$ for $\lambda \in O$.) We show that

(4) $$O \cap \Pi(M_\mu) = \varnothing .$$

If (4) fails then there exists a λ in O and a sequence of polynomials p_n satisfying

(5) $$\|(z - \lambda)p_n(z)\|^2 < \frac{1}{n}$$

and

$$\|p_n\|^2 \geqq \frac{1}{2}\ . \tag{6}$$

Let B be the closed disc of radius r centered at λ and contained in O. Since O is an analytic set for μ,

$$\sup\{\|k_z\| \colon z \in B\} = C < \infty\ .$$

For w with $|w - \lambda| = r$,

$$\frac{1}{n} > \|(z - \lambda)p_n(z)\|^2 \geqq \frac{|w - \lambda|^2 |p_n(w)|^2}{\|k_w\|^2} \geqq \frac{r^2}{C^2}|p_n(w)|^2\ .$$

So by the maximum modulus principle,

$$|p_n(u)|^2 \leqq \frac{C^2}{nr^2} \tag{7}$$

for all $u \in B$. But using (5) and (7),

$$\begin{aligned}\|p_n\|^2 &= \int_{\bar{D}-B} |p_n|^2 d\mu + \int_B |p_n|^2 d\mu \\ &\leqq \frac{1}{r^2 n} + \frac{C^2}{nr^2}\mu(B)\ .\end{aligned}$$

Letting $n \to \infty$, we see that (6) is contradicted so (4) holds.

Conversely, assume that O is an open set satisfying $O \cap \Pi(M_\mu) = \varnothing$ and $O \subset \Gamma(M_\mu)$. By our opening remark in the proof, it will be sufficient to show that $\lambda \to \|k_\lambda\|$ is bounded in a neighborhood of λ. Fix $a \in O$. Since $a \notin \Pi(M_\mu)$ there is a $C < \infty$ so that

$$\|f\| \leqq C\|(z - a)f(z)\|$$

for all $f \in H^2(\mu)$. A computation shows that

$$\|f\| \leqq 2C\|(z - w)f(z)\| \tag{8}$$

whenever $|w - a| \leqq 1/2C$.

Let $q(z) = (p(z) - p(\lambda))/(z - \lambda)$ for p a polynomial and let $C_1 = \min\{1/2C, 1/(4C\|k_a\|)\}$. By (8), for $|\lambda - a| < C_1 \leqq 1/2C$,

$$\begin{aligned}|q(a)| &\leqq \|k_a\|\,\|q\| \leqq \|k_a\|\,2C\|(z - \lambda)q(z)\| \\ &\leqq 2C\|k_a\|[\|p\| + |p(\lambda)|]\ .\end{aligned}$$

Hence

$$|p(\lambda)| \leqq |p(a)| + |\lambda - a|\,2C\|k_a\|[\|p\| + |p(\lambda)|]\ .$$

So for $|\lambda - a| < C_1$,

$$|p(\lambda)| \leqq ||k_a|| \, ||p|| + \frac{1}{2}[||p|| + |p(\lambda)|] .$$

Thus

$$||k_\lambda|| \leqq 2\,||k_a|| + 1$$

so we are done.

COROLLARY 1.1. *If H is a hole of the support of μ and $H \subset \Lambda(M_\mu)$ then H is an analytic set for μ.*

Proof. $\Lambda(M_\mu) = \Pi(M_\mu) \cup \Gamma(M_\mu)$. If $\lambda \in H$ then $1/(z-\lambda) \in L^\infty(\mu)$ and hence $\lambda \notin \Pi(M_\mu)$.

Denote the essential spectrum of M_μ by $\Lambda_e(M_\mu)$ [9].

COROLLARY 1.2. *If M_μ has no point spectrum, then the maximal analytic set for μ is $\Lambda(M_\mu) - \Lambda_e(M_\mu)$.*

Proof. If M_μ has no point spectrum then [9] says that $\operatorname{int}(\Gamma(M_\mu) - \Pi(M_\mu)) = \Lambda(M_\mu) - \Lambda_e(M_\mu)$. Now apply Theorem 1.1.

Let M'_μ denote the pure subnormal part of M_μ [7].

COROLLARY 1.3. *The maximal analytic set for μ is $\Lambda(M'_\mu) - \Lambda_e(M'_\mu)$.*

Proof. It is easy to see that the maximal analytic sets of M_μ and M'_μ are equal. If M'_μ is a pure subnormal operator, then M'_μ has empty point spectrum so Corollary 1.2 applies.

If $\mathscr{B}$ denotes the set of b.p.e.'s for μ, the obvious question is whether int $\mathscr{B}$ is the largest analytic set for μ. While we cannot answer this, we have the following partial result.

THEOREM 1.2. *There exists a dense open subset $\mathscr{S}$ of $\mathscr{B}$ so that $\mathscr{S}$ is an analytic set for μ.*

Proof. We show that if $\mathscr{S} = \{z \in \mathscr{B}\colon$ there is some neighborhood U of z with $\bar{U} \subset \mathscr{B}$ and $\sup\{||k_\lambda||\colon \lambda \in U\} < \infty\}$ then $\mathscr{S}$ is a dense subset of $\mathscr{B}$. Let V be any open subset of $\mathscr{B}$ with $\bar{V} \subset \mathscr{B}$. We are done if we show that $\bar{V} \cap \mathscr{S} \neq \varnothing$. Define

$$E_N = \{z \in \bar{V}\colon ||k_z|| \leqq N\} .$$

Clearly,

$$\bigcup_{N=1}^{\infty} E_N = \bar{V}\,.$$

Now

$$\|k_z\| = E(z) = \sup\{|p(z)|/\|p\|\}$$

where the suprenum is taken over polynomials p with rational complex coefficients and $\|p\| \neq 0$. Thus $z \to \|k_z\|$ is a lower semi-continuous function on $\mathscr{B}$, so E_N is a closed set. An application of the Baire category theorem completes the proof.

It may be useful to note that by Corollary 1.3 $\mathscr{S} = \{z \in D$: $z - M_\mu$ is a Fredholm operator and $\operatorname{ind}(z - M_\mu) = -1\}$.

2. **Recovering a part of the measure μ from $E(z)$.** It is a well known result of Bram [1] that the operator M_μ, multiplication by z on $H^2(\mu)$, is a model for a general contractive cyclic subnormal operator. Some subnormal operators have been shown to have (nontrivial, closed) invariant subspaces by establishing that if $H^2(\mu) \neq L^2(\mu)$ then μ has a bounded point evaluation [2], [3], [4]. This provides a basic motivation for the study of the relationship of the measure μ to the possible existence of b.p.e.'s.

Let $d\sigma$ denote normalized Lebesgue measure on ∂D. For a measure ν carried by ∂D, it is a classical result of Szegö and Kolomogorov [see 13] that $H^2(\nu) \neq L^2(\nu)$ if and only if $\log h \in L^1(d\sigma)$, where h denotes the absolutely continuous part of ν with respect to σ. Whenever $H^2(\nu) \neq L^2(\nu)$, then ν has b.p.e.'s in D with b.e.f.'s k_λ for $\lambda \in D$. It was observed in [14] that h can be recovered from $\|k_\lambda\|$ as follows:

$$\lim_{\lambda \to e^{i\theta}} (1 - |\lambda|^2)\|k_\lambda\|^2 = \frac{1}{h(e^{i\theta})} \text{ for } \sigma - \text{a.e. } e^{i\theta} \tag{9}$$

where $\lambda \to e^{i\theta}$ nontangentially. Suppose that μ is a measure carried by $\bar{D}$. Let

$$d\mu = d\mu|_D + \left(\frac{d\mu}{d\sigma}\right)d\sigma + d\mu_s$$

where $d\mu_s$ is carried by ∂D and is singular to $d\sigma$. Just as in the classical case (ν as above) a result of Clary [6] says that μ has a b.p.e. at $\lambda \in D$ if and only if $d\mu - d\mu_s$ does. Since $d\mu_s$ is not involved in the existence of b.p.e.'s, it is clear that there is no hope of recovering $d\mu_s$ from a knowledge of the norms of b.e.f.'s for

$d\mu$ (in fact, $E^{\mu}(\lambda) \equiv E^{\mu-\mu_s}(\lambda)$ for all λ).

We are interested in the interplay between $\mu|_D$ and $\mu|_{\partial D}$ and the existence of b.p.e.'s in D. By the previous discussion μ_s has no bearing on this problem. We investigate a class of measures μ for which the absolutely continuous part of μ with respect to σ can be recovered on an arc of ∂D in an analogous fashion to (9).

DEFINITION. Let K be a compact set. Then K contains *an exposed arc* J if there exists a simply connected open set $\mathscr{D}$ such that $\mathscr{D} \cap K = J$ and J is the range of a smooth Jordan curve.

Let μ be a measure carried by $\bar{D}$ satisfying:

(A) there is a hole H of the support of μ so that $\bar{H}$ has an exposed arc Γ with $\Gamma \subset \partial D$.

(B) μ has b.p.e.'s in the hole H.

We remark that by a result of Brown, Shields, and Zellar [5], it is possible to construct a measure μ carried by D whose support has a hole H for which (B) holds, $\mu(\partial D) = 0$, and $\partial H \supset \partial D$. For such a measure, it is clear that $\mu|_{\partial D}$ is not involved in the existence of b.p.e.'s in H. Thus condition (A) is a guarantee that if (B) is to hold, then $\mu|_\Gamma$ and $\mu|_D$ must interrelate in some way. Hence if μ satisfies (A) and (B), it is plausible that a knowledge of the norms of b.e.f.'s in H would lead to a recovery of the absolutely continuous part of μ with respect to σ restricted to Γ. This is indeed the case. Before proving this result, we will need a few lemmas.

Suppose that α is any measure whose support contains a hole H. Assume, furthermore, that α has b.p.e.'s in H. For $\lambda \in H$, k_λ is the b.e.f. of α at λ. Denote the orthogonal projections of $L^2(\alpha)$ onto $H^2(\alpha)$ and $H^2(\alpha)^\perp$ by P_1 and P_2, respectively. We have the following lemma.

LEMMA 2.1. (i) *Let $a \in H$. If $g \in H^2(\alpha)^\perp$ and $\langle 1/(z-a), g\rangle \neq 0$ then*

$$k_a = P_1\left(\frac{g(z)}{\bar{z}-\bar{a}} + f\right)\Big/\left\langle g, \frac{1}{z-a}\right\rangle \tag{10}$$

where f is any element of $H^2(\alpha)^\perp$.

(ii) *If $g = P_2(1/(z-a))$ then $\langle 1/(z-\lambda), g(z)\rangle = 0$ for at most a countable number of λ's in H.*

Proof. Let $\hat{g}(a)$ denote $\langle 1/(z-a), g(z)\rangle$. If p is a polynomial then $(p(z)-p(a))/(z-a)$ is a polynomial so

$$0 = \left\langle \frac{p(z)-p(a)}{z-a}, g\right\rangle = \left\langle p, \frac{g(z)}{\bar{z}-\bar{a}}\right\rangle - p(a)\hat{g}(a).$$

Hence

$$p(a) = \left\langle p, \left(\frac{g}{\bar{z} - \bar{a}}\right)\Big/ \overline{\hat{g}(a)} \right\rangle$$

for all polynomials p. Now $1/(z - a)$ is in $L^\infty(\alpha)$ since $a \in H$, so $g/(\bar{z} - \bar{a}) \in L^2(\alpha)$. Thus (10) follows by the uniqueness of the b.e.f. at a.

Let $g = P_2(1/(z - a))$. Since α has a b.p.e. at a, $1/(z-a) \notin H^2(\alpha)$. (Else we would have $1 = \langle 1, k_a \rangle = \langle (z - a)(1/(z - a)), k_a \rangle = (a - a) \langle 1/(z - a), k_a \rangle = 0$.) Thus

$$\left\langle \frac{1}{z - a}, g \right\rangle = \left\| P_2\left(\frac{1}{z - a}\right) \right\|^2 > 0 \, .$$

Now we need only notice that $\lambda \to \langle 1/(z - \lambda), g(z) \rangle$ is analytic and not identically zero in H to complete the proof of (ii).

Suppose that μ is a measure supported on $\bar{D}$ satisfying (A) and (B) for a hole H of the support of μ with exposed arc Γ. Let $a \in H$ and denote $P_2(1/(z - a))$ by g and $\langle 1/(z - a), g \rangle$ by $\hat{g}(a)$.

LEMMA 2.2. *g vanishes on no subset of Γ with positive Lebesgue measure.*

Proof. Define

$$d\beta = \frac{\overline{g(z)}}{(z - a)\hat{g}(a)} \, d\mu \, .$$

Then $d\beta$ is a complex representing measure for evaluation at a on the space of the polynomials with respect to sup norm on the support of μ [see 10]. It follows from Theorem 2.2 of [10] that there exists a positive representing measure $d\nu$ for evaluation at a which is absolutely continuous with respect to $|d\beta|$. It is easy to see that ν has a b.p.e. at a. Applying Lemma 2 of [17] shows that

$$\int_{\Gamma_1} \log \frac{d\nu}{d\sigma} d\sigma > -\infty$$

for every closed subarc Γ_1 of Γ. This completes the proof.

We are now ready for the main result of this section. Assume that μ is a measure supported on $\bar{D}$ satisfying (A) and (B) for a hole H of the support of μ with exposed arc Γ. Let w denote the Radon-Nikodym derivative of the absolutely continuous part of $\mu|_{\partial D}$ with respect to σ. Fix a point $a \in H$ and again denote

$P_2(1/(z-a))$ by g and $\langle 1/(z-a), g\rangle$ by $\hat{g}(a)$.

THEOREM 2.1.

$$\lim_{\lambda\to e^{i\theta}} (1-|\lambda|^2)\|k_\lambda\|^2 = \frac{1}{w(e^{i\theta})} \text{ for } \sigma\text{-a.e. } e^{i\theta}\in\Gamma \tag{11}$$

as $\lambda\to e^{i\theta}$ *nontangentially.*

Proof. By a theorem of [14] it is shown that for any measure β on $\bar{D}$,

$$\underline{\lim}_{\lambda\to e^{i\theta}} (1-|\lambda|^2)(E^\beta(\lambda))^2 \geqq 1\Big/\frac{d\beta}{d\sigma}(e^{i\theta}) \text{ for } \sigma\text{-a.e. } e^{i\theta}\in\partial D \tag{12}$$

where $\lambda\to e^{i\theta}$ nontangentially. Thus we need only show that

$$\overline{\lim}_{\lambda\to e^{i\theta}} (1-|\lambda|^2)\|k_\lambda\|^2 \leqq \frac{1}{w(e^{i\theta})} \text{ for } \sigma\text{-a.e. } e^{i\theta}\in\Gamma \tag{13}$$

where $\lambda\to e^{i\theta}$ nontangentially. From Lemma 2.1 we see that

$$\|k_\lambda\|^2 \leqq \left\|\frac{g}{z-\lambda}\right\|^2 \Big/ |\hat{g}(\lambda)|^2 . \tag{14}$$

(Note that from Lemma 2.1, $\hat{g}(\lambda)$ can vanish on at most a countable set of H. If for some $\lambda\in H$, $\hat{g}(\lambda)=0$, then the right hand side of (14) is to be interpreted as ∞.) Denote $(1-|\lambda|^2)/|1-\lambda e^{-i\theta}|^2$ by $P(\lambda, e^{i\theta})$. Define Ω to be the support of μ minus Γ. Then

$$(1-|\lambda|^2)\left\|\frac{g(z)}{z-\lambda}\right\|^2 = \int_\Gamma P(\lambda, e^{it})|g(e^{it})|^2 w(e^{it})d\sigma(t) + \int_\Omega \frac{1-|\lambda|^2}{|\lambda-z|^2}|g(z)|^2 d\mu(z) . \tag{15}$$

Now

$$\hat{g}(\lambda) = \left\langle\frac{1}{z-\lambda}, g\right\rangle = \left\langle\frac{1}{z-\lambda}+\frac{\bar{\lambda}}{1-\bar{\lambda}z}, g\right\rangle$$

since $z\to\bar{\lambda}/(1-\bar{\lambda}z)$ is analytic in $\bar{D}$ and $g=P_2(1/(z-a))$ is in $H^2(\mu)^\perp$. Writing this out, we see that

$$\hat{g}(\lambda) = \int_\Gamma P(\lambda, e^{it})e^{-it}\overline{g(e^{it})}w(e^{it})d\sigma(t) + \int_\Omega \frac{1-|\lambda|^2}{(z-\lambda)(1-\bar{\lambda}z)}\, g(z)d\mu(z) . \tag{16}$$

Since $e^{i\theta}\notin\bar{\Omega}$ it is easy to see that the second integrals of (15) and (16) converge to 0 as $\lambda \to e^{i\theta}$. Hence by a theorem of Fatou [see 13], we get

$$\text{(17)} \qquad \lim_{\lambda \to e^{i\theta}} (1 - |\lambda|^2) \left\| \frac{g(z)}{z - \lambda} \right\|^2 = |g(e^{i\theta})|^2 w(e^{i\theta}) ,$$

$$\text{(18)} \qquad \lim_{\lambda \to e^{i\theta}} \hat{g}(\lambda) = e^{-i\theta} \overline{g(e^{i\theta})} w(e^{i\theta}) \quad \text{for } \sigma\text{-a.e. } e^{i\theta} \in \Gamma$$

where $\lambda \to e^{i\theta}$ nontangentially. Recall that by Lemma 2.2, g cannot vanish on a subset of Γ with positive Lebesgue measure. Thus, combining (14), (17), and (18), we establish (13) to complete the proof.

Suppose that μ is a measure on $\bar{D}$ satisfying (A) and (B) for a hole H of the support of μ with exposed arc Γ. Assume that $d\mu|_\Gamma$ is absolutely continuous with respect to $d\sigma$. In [17] it was shown that if $f \in H^2(\mu)$ and f does not vanish identically on Γ then

$$\int_{\Gamma_1} \log |f| d\sigma > -\infty$$

for Γ_1 any closed subarc of Γ. Thus the functions of $H^2(\mu)$ exhibit one of the properties of Hardy space functions locally on Γ. Thus if $f \in H^2(\mu)$ the question is raised as to whether f can be recovered as the boundary values of $\tilde{f}$ on Γ. J. Thompson and R. Olin have informed us that the answer to this question is yes. Subsequently, we have established this result together with a Blaschke type growth condition based on Theorem 2.1 and a result of Kriete and Trutt [15].

Let μ satisfy the hypothesis of Theorem 2.1. Also assume that $d\mu|_\Gamma$ is absolutely continuous with respect to Lebesgue measure. We have the following regularity theorem for extensions of functions in $H^2(\mu)$.

THEOREM 2.2. *Let $f \in H^2(\mu)$.*

(i) $\lim_{\lambda \to e^{i\theta}} \tilde{f}(\lambda) = f(e^{i\theta})$ *for* σ-a.e. $e^{i\theta} \in \Gamma$ *where* $\lambda \to e^{i\theta}$ *nontangentially.*

(ii) *Assume that f is not equal to* 0 σ-a.e. *on Γ. If Γ_1 is any proper closed subarc of Γ and $\tilde{f}$ vanishes on the set $\{z_n\}_1^\infty$ which has no limit points outside of Γ_1 then*

$$\sum_1^\infty (1 - |z_n|) p_n < \infty$$

where p_n is the multiplicity of z_n as a zero of $\tilde{f}$.

Proof. The proof will be established by showing that any f in $H^2(\mu)$ may be viewed as an element of a space $H^2(\beta)$. The corresponding extensions of f as an element of $H^2(\mu)$ and $H^2(\beta)$ have the same values at the points which are bounded point evaluations of both μ and β. Once this is done it will be sufficient to show that extensions of functions in $H^2(\beta)$ satisfy (i) and (ii). This will follow from a conformal mapping argument.

Let Γ_1 be any closed subarc of Γ. Let a and b be elements of $\Gamma - \Gamma_1$, one on each side of Γ_1, for which equality holds in (11). Let M denote the arc connecting a with b and containing Γ_1. By hypothesis (B) on the support of μ, we can find a polar rectangle R with int $R \subset H$, and $\partial R \cap \partial D = M$. Let L denote $\partial R \cap D$.

Define a finite Borel measure, $d\beta$, with support ∂R by

$$d\beta = (1 - |z|^2)|dz|\Big|_L + \frac{w(z)}{2\pi}|dz|\Big|_M$$

where $|dz|$ denotes arc length measure.

Let p be a polynomial. Then

$$\begin{aligned}\|p\|_\beta^2 &= \int_L |p|^2(1 - |z|^2)|dz| + \int_M |p|^2 w\, d\sigma \\ &\leqq \|p\|_\mu^2 \int_L \|k_z^\mu\|^2(1 - |z|^2)|dz| + \|p\|_\mu^2 .\end{aligned}$$

Now the hypothesis that a and b satisfy the equality in (11) enables us to find a constant $K < \infty$ so that

$$\|p\|_\beta^2 \leqq K\|p\|_\mu^2 . \tag{19}$$

Hence by (19), the mapping defined on polynomials by $p \to p$ extends to a bounded linear map T of $H^2(\mu)$ into $H^2(\beta)$.

Notice that

$$\int_M |\log w|\,|dz| + \int_L |\log(1 - |z|^2)|\,|dz| < \infty .$$

The first integral is finite by Lemma 2 of [4] since μ has b.p.e.'s in H and the second integral is finite by a routine computation. Thus if

$$W(z) = \begin{cases} w(z) & z \in M \\ (1 - |z|^2) & \qquad\qquad z \in L \end{cases}$$

then

$$d\beta = W(z)|dz| \text{ where } \int_{\partial R} |\log W(z)| < \infty . \tag{20}$$

If ψ is a simple conformal mapping of D onto R extended to a mapping of $\bar{D}$ onto $\bar{R}$ then $\psi^{-1\prime}$ is bounded above by a modification of Theorem 9.8 of [18]. Using a theorem of Szegö [see 13] and a conformal mapping argument, it is not hard to show that β has b.p.e.'s in R if and only if $\log[(W\circ\psi)|\psi'|]\in L^1(d\sigma)$. By Theorem 3.12 of [8] (since ψ is rectifiable), $\psi'\in H^1(d\sigma)$ so $\log|\psi'|\in L^1(d\sigma)$. Combining (20) and the boundedness of $\psi^{-1\prime}$ we see that

$$\int_{\partial D}|\log W\circ\psi|d\sigma = \frac{1}{2\pi}\int_{\partial R}|\log W|\,|\psi^{-1\prime}|\,|dz| < \infty\,.$$

Fix $f\in H^2(\mu)$. By the definition of T, a sequence of polynomials converging to f in $H^2(\mu)$ will converge to Tf in $H^2(\beta)$. Also the existence of b.p.e.'s in the hole R implies by Theorem 1.1 that the convergence of polynomials is uniform on compact subsets of int R. Hence $\tilde{f} = T\tilde{f}$ in R.

To show that extensions of functions in $H^2(\beta)$ satisfy (i) and (ii), we refer to the proof of Theorem 8 in [15]. This completes the proof.

3. An application. Let dA denote planar Lebesgue measure on D and let Γ be an open subarc of ∂D. We shall apply the results of §2 to finite positive measures of the form

$$d\nu = GdA + wd\sigma$$

satisfying

$$\log G \text{ is in } L^1(dA) \text{ and } \int_\Gamma \log wd\sigma > -\infty\,. \tag{21}$$

THEOREM 3.1. *Suppose that* $d\nu = GdA + wd\sigma$ *satisfies* (21). *Then*

$$\operatorname*{Lim}_{\lambda\to e^{i\theta}}(1-|\lambda|^2)\|k_\lambda\|^2 = \frac{1}{w(e^{i\theta})}\ \textit{for}\ \sigma\text{-a.e. } e^{i\theta}\in\Gamma$$

where $\lambda\to e^{i\theta}$ *nontangentially.*

Proof. Remove the open region S from D which is bounded by a proper closed subarc Γ_1 of Γ and the chord connecting the endpoints of Γ_1. Define $\tau = \nu|_{\bar{D}-S}$. Clearly, $\|p\|_\tau \leqq \|p\|_\nu$ so by definition

$$E^\nu(z) = \|k_z^\nu\| \leqq E^\tau(z)\,.$$

Appealing to (12), it is enough to show that

$$\overline{\operatorname{Lim}}_{\lambda\to e^{i\theta}} (1-|\lambda|^2)(E^\tau(\lambda))^2 \leqq \frac{1}{w(e^{i\theta})} \text{ for } \sigma\text{-a.e. } e^{i\theta}\in\Gamma_1 \tag{22}$$

where $\lambda \to e^{i\theta}$ nontangentially.

The support of the measure τ satisfies condition (A) with respect to S and Γ_1 by definition. If we show that τ satisfies (B), then we may apply Theorem 2.1 to establish (22). The remainder of the proof is a lengthy calculation to show that (B) holds.

First we need some notation. Without loss of generality let us assume that for some α with $-1<\alpha<1$, $S=\{z\in D: \alpha<\operatorname{Re} z<1\}$. For $-1<x<\alpha$, let L_x denote the chord $\{z\in\bar{D}: \operatorname{Re} z = x\}$. Choose $-1<\beta<\alpha$ so that for every x with $\beta\leqq x\leqq\alpha$ L_x intersects $\Gamma-\Gamma_1$ in two points. (Since Γ is an open arc and Γ_1 is a proper closed subarc of Γ this can be done.) For $-1<x<\alpha$, let S_x denote the open segment of D with chord L_x and containing S. Denote $\partial S_x\cap\partial D$ by Γ_x.

Let $E_n=\{t\in[\beta,\alpha]: \int_{L_t} G(t+iy)|dy|<\infty$ and $\int_{\Gamma_t}|\log w/2\pi||dz| + \int_{L_t}|\log G(t+iy)||dy|<n\}$. It is clear from the hypotheses on G and w that for some $n<\infty$, $m(E_n)>0$, where m is linear Lebesgue measure. Let E be any set E_n with $m(E_n)>0$. If $t\in E$, define the measures ν_t with support ∂S_t by

$$d\nu_t = \frac{w}{2\pi}|dz|\,|_{\Gamma_t} + m(E)G(t+iy)|dy|\,|_{L_t}\,.$$

Let

$$h_t = \begin{cases} \dfrac{w}{2\pi} \text{ on } \Gamma_t \\ m(E)G(t+iy) \text{ on } L_t\,. \end{cases}$$

Then

$$d\nu_t = h_t|dz|\,|_{\partial S_t}$$

and

$$\int_{\partial S_t}|\log h_t||dz| \leqq n < \infty\,.$$

Notice that ν_t has b.p.e.'s in S_t (and hence in S) by an argument similar to that employed in the proof of Theorem 2.2.

Fix any $a\in S$. For any polynomial p

$$|p(a)|^2 \leqq \|k_a^{\nu_t}\|^2\,\|p\|_{\nu_t}^2 \tag{23}$$

where $t\in E$. Integrating (23) on E with respect to dm, we obtain

$$m(E)\,|p(a)|^2 \leqq \sup_{t\in E} \|k_a^{\nu_t}\|^2\left[\int_E\int_{L_t} |p|^2 G m(E)|\,dy\,|dm + m(E)\int_{\Gamma_t} |p|^2 w d\sigma\right]$$
$$\leqq \sup_{t\in E} \|k_a^{\nu_t}\|^2 m(E)\,\|p\|_\tau^2 \,.$$

We need only show that $\sup_{t\in E}\|k_a^{\nu_t}\|^2$ is finite to establish (B). Let ψ_t denote the simple conformal map of D onto S_t with $\psi_t(a) = a$ and $\psi_t'(a) > 0$. Denote $\sup\{|\psi_t^{-1\prime}(z)|\colon \beta \leqq t \leqq \alpha,\, z \in \bar{S}_t\}$ by C. Let A stand for the set of angles measured in radians of the corners of S_t with $t \in [\beta, \alpha]$. Referring to the proof of Theorem 9.8 of [18], we see that $C < \infty$, since $0 < \inf A \leqq \sup A < \pi$. (Because these conformal maps can be given explicitly, this also follows by a direct computation.) It follows from a conformal mapping and a theorem of Szegö [see 13] that

$$\|k_a^{\nu_t}\|^2 = \frac{\exp - \displaystyle\int_{\partial S_t} P(a, \psi_t^{-1}(z))\log h(z)\,|\psi_t^{-1\prime}(z)|\,\frac{|dz|}{2\pi}}{2\pi(1-|a|^2)|\psi_t'(a)|}$$

so

$$\sup_{t\in E} \|k_a^{\nu_t}\|^2 \leqq \frac{\exp\left(\dfrac{1+|a|}{1-|a|}\right)Cn}{2\pi(1-|a|)^2}\,C\,.$$

This completes the proof.

We remark that functions in $H^2(dA)$ do not in general have Hardy space properties. However, if $d\nu = GdA + wd\sigma$ satisfies (21) then we have the following theorem.

THEOREM 3.2. *Suppose that $d\nu = GdA + wd\sigma$ satisfies* (21). *Let $f \in H^2(\nu)$.*

(i) $\mathrm{Lim}_{\lambda\to e^{i\theta}} \tilde{f}(z) = f(e^{i\theta})$ *for* σ-a.e. $e^{i\theta}\in\Gamma$.

(ii) *Suppose that f is not the zero function. If Γ_1 is any proper closed subarc of Γ and $\tilde{f}$ vanishes on the set $\{z_n\}_1^\infty$ which has no limit points not in Γ_1, then*

$$\sum_1^\infty (1-|z_n|)p_n < \infty$$

where the p_n is the multiplicity of z_n as a zero of $\tilde{f}$.

(iii) *Suppose f is not the zero function. Let Γ_1 be any proper closed subarc of Γ, then*

$$\int_{\Gamma_1} \log|f|\,d\sigma > -\infty\,.$$

Proof. The proof is similar to that given for Theorem 3.1 and will be omitted.

These results extend a part of the author's dissertation submitted in partial fulfillment of the requirements for the Ph.D. degree at the University of Virginia. The author wishes to express his appreciation for the encouragement and guidance of Professor Thomas L. Kriete, III.

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Received March 13, 1978 and in revised form July 21, 1978.

UNIVERSITY OF NORTH CAROLINA
CHAPEL HILL, NC 27514

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 80, No. 1 September, 1979

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