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CONNECTIVITY PROPERTIES OF METRIC SPACES

DOUGLAS S. BRIDGES

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We discuss various connectivity properties of a metric space, and investigate how far their equivalence carries over from the classical to the constructive setting. In passing, we obtain interesting relations between connectivity and convexity for subsets of R , and a result on preservation of connectivity by continuous mappings.

1. The primary object of this note is a constructive examination of the relationship between several, classically equivalent connectivity properties of a metric space (E, d) . In order to make sense of the statements of these properties, we recall that a subset A of E is *located* (in E) if

$$\text{dist}(x, A) \equiv \inf \{d(x, a) : a \in A\}$$

is computable for each x in E ; in which case the *metric complement* of A in E is defined to be

$$E - A \equiv \{x \in E : \text{dist}(x, A) > 0\}.$$

Note that a located set A is nonvoid, in the sense that we can construct at least one of its elements. For further properties of located sets, and general background material in constructive analysis, we refer the reader to [1] and [2].

In [3], we introduced the following types of connectivity of a metric space:

C-connectivity: if A is a closed, located subset of E with nonvoid metric complement, then there exists a point ξ in $A \cap (E - A)^-$;

0-connectivity: if A is an open, located subset of E with nonvoid metric complement, then there exists ξ in \bar{A} such that $d(\xi, x) > 0$ for each x in A ;

Connectivity: if A is an open, closed and located subset of E , then $A = E$.

We then showed that

$$C\text{-connectivity} \implies 0\text{-connectivity} \implies \text{connectivity}.$$

In this section, we shall show that these implications cannot be reversed

within a constructive proof-theoretic framework. To do this, we first characterize located C - and 0 -connected subsets of the real line R , and prove connectivity of subsets of R of the form $[a, b] \cup]b, c]$, where $a < b < c$.

LEMMA 1. *Let S be a subset of R with the property: $S \cap]a, b[$ is dense in $[a, b]$ whenever a, b belong to S and $a < b$. Let A be a located subset of the metric space S and let $b \in S - A$. Then there exist a in A and ξ in $\bar{A} \cap (S - A)^-$ such that either $a \leq \xi < b$ or $b < \xi \leq a$.*

We first note that if $x \in S - A$, then

$$\min(\text{dist}(x - \text{dist}(x, A), A), \text{dist}(x + \text{dist}(x, A), A)) = 0.$$

In particular, it follows that, if $r \equiv \text{dist}(b, A)$, then either $\text{dist}(b - r, A) < (1/2)r$ or $\text{dist}(b + r, A) < (1/2)r$. Taking, for example, the former case (the latter produces the second alternative of the conclusion of the lemma), we compute a in $]b - 3r/2, b - r] \cap A$. As $a \in S, b \in S$, and $a < b$, there exists x_1 in $S \cap]b - r, b - (1/2)r]$. Let $\rho \equiv \text{dist}(x_1, A)$ and $\xi \equiv x_1 - \rho$. Then $0 < \rho \leq x_1 - a < r$; so that $x_1 + \rho$ belongs to $]x_1, b + (1/2)r[$, and therefore

$$\text{dist}(x_1 + \rho, A) \geq \min(\rho, r) > 0.$$

Hence $\text{dist}(\xi, A) = 0$, and $\xi \in \bar{A}$. On the other hand, as $\xi \geq a, S \cap]a, b[$ is dense in $[a, b]$, and $|x_1 - \xi| = \text{dist}(x_1, A)$, it follows that $] \xi, x_1[\subset S - A$, and therefore that $\xi \in (S - A)^-$.

THEOREM 1. *A necessary and sufficient condition that a located subset S of R be C -connected is that $S \supset]a, b]$ whenever a, b are points of S and $a < b$.*

If S is C -connected, and a, b are points of S with $a < b$, and $x \in [a, b]$, we have either $a < x$ or $x < b$. Without loss of generality, we suppose the latter. Then $A \equiv S \cap]-\infty, x]$ is a closed, located subset of S such that $b \in S - A$. Thus there exists ξ in $\bar{A} \cap S \cap (S - A)^-$. It is easy to see that $\xi = x$, whence $x \in S$.

Conversely, suppose the stated condition holds, and let A be a closed, located subset of S with $S - A$ nonvoid. Choosing b in $S - A$, compute a in A and ξ in $\bar{A} \cap (S - A)^-$ such that either $a \leq \xi < b$ or $b < \xi \leq a$. Then $\xi \in S$, and so $\xi \in A \cap (S - A)^-$. Thus S is C -connected.

THEOREM 2. *A necessary and sufficient condition that a located subset S of R be 0 -connected is that $S \supset]a, b[$ whenever a, b are points of S and $a < b$.*

If S is 0-connected, and a, b are points of S with $a < b$, and $x \in]a, b[$, we apply the 0-connectivity condition to $A \equiv S \cap]-\infty, x[$, to obtain ξ in S with $d(\xi, y) > 0$ for each y in A . As ξ is clearly equal to x , we have $x \in S$, as required.

Conversely, suppose the stated condition holds, and let A be an open, located subset of S with $S - A$ nonvoid. Choosing b in $S - A$, compute a in A and ξ in $\bar{A} \cap (S - A)^-$ such that either $a \leq \xi < b$ or $b < \xi \leq a$. As $\xi \in A$ entails $A \cap (S - A)$ nonvoid, and A is open, we see that $d(\xi, x) > 0$ for each x in A . In particular, either $a < \xi < b$ or $b < \xi < a$; so that $\xi \in S$, and S is 0-connected.

PROPOSITION 1. *Let a, b, c be real numbers with $a < b < c$. Then $[a, b] \cup]b, c[$ is connected.*

Let A be an open, closed, located (and therefore totally bounded) subset of $S \equiv [a, b] \cup]b, c[$. We first prove that, if $A \cap [a, b]$ is nonvoid, then $A \supset [a, b]$. Indeed, given x_0 in $A \cap [a, b]$ and x in $[a, b]$, we have either $x_0 \leq x - r$ or $x + r \leq x_0$. Without loss of generality, we may assume the former. Letting

$$B \equiv A \cap [a, x] = A \cap [a, x[,$$

we see that B is open and closed in $[a, b]$. On the other hand, if $0 < \varepsilon < r$ and $\{x_1, \dots, x_s\}$ is an ε -net of A , we may assume that x_1, \dots, x_s belong to $A \cap [a, x - r]$, and that x_{s+1}, \dots, x_v belong to $A \cap [x + r, c]$. It is now easy to show that $\{x_1, \dots, x_s\}$ is an ε -net of B ; whence B is totally bounded, and therefore located in $[a, b]$. By connectivity of $[a, b]$, we now have $B = [a, b]$; whence we obtain the contradiction $x \in A$. Thus $r = 0$, $x \in \bar{A} \cap [a, b]$, and so $A \supset [a, b]$.

In a similar manner, we can show that if $A \cap]b, c[$ is nonvoid, then $A \supset]b, c[$. Given ξ in A , we now see that either $\xi \in [a, b]$, in which case $[a, b] \subset A$ and therefore (as A is open in S) $A \cap]b, c[$ is nonvoid; or $\xi \in]b, c[$, when $]b, c[\subset A$, and therefore (as A is closed in S) $b \in A$. In either case, we have $A \supset [a, b] \cup]b, c[$, and therefore $A = S$. Thus S is connected.

THEOREM 3. *The proposition,*

a located, 0-connected subset of R is C -connected,

is essentially nonconstructive.

Consider the located subset $S \equiv \{0\} \cup]0, 1[$ of R . It follows from Theorem 2 that S is 0-connected. On the other hand, by Theorem 1, the C -connectivity of S would entail the proposition

$$\forall x \in [0, 1](x > 0 \vee x = 0),$$

which is known to be essentially nonconstructive.

THEOREM 4. *The proposition,*

a located, connected subset of R is 0-connected,

is essentially nonconstructive.

Consider the located subset $S \equiv [-1, 0] \cup]0, 1]$ of R . By Proposition 1, S is connected. However, the 0-connectivity of S would entail the proposition

$$\forall x \in [-1, 1](x > 0 \vee x \leq 0),$$

which is known to be essentially nonconstructive.

2. A subset U of the metric space E is *colocated* (in E) if it is the metric complement of a located set. U is then an open subset of E . Colocated sets, like located sets (although to a lesser degree), are easier to handle than general subsets of E . It therefore seems reasonable to investigate what happens when we formulate alternative connectivity properties in terms of colocated sets.

When we do so, we find that the natural analogue of C -connectivity is just a condition of disconnectedness. That of 0-connectivity is given by the property,

if U is a nonvoid, colocated subset of E , then exists ξ in \bar{U} such that $d(\xi, x) > 0$ for each x in U ,

a property easily shown to be equivalent to that of C -connectivity. Finally, there is no direct analogue of connectivity, although a natural property (readily seen to be equivalent to that of connectivity) is that any open, closed, colocated subset of E is empty.

Of greater interest is the following property, analogous to that of M -connectivity (defined in [4], and there shown to be equivalent to 0-connectivity):

(*) if U, V are nonvoid, disjoint subsets of E with U colocated and V open, then there exists ξ in E such that $d(\xi, x) > 0$ for each x in $U \cup V$.

(By "disjoint" here, we mean that $d(u, v) > 0$ whenever $u \in U$ and $v \in V$.) We have

$$C\text{-connectivity} \implies (*) \implies 0\text{-connectivity}.$$

To see this, suppose first that E is C -connected, and let A be a

located subset of E such that $U \equiv E - A = E - \bar{A}$ is nonvoid. Then there exists ξ in $\bar{A} \cap \bar{U}$. As U is open, $d(\xi, x) > 0$ for each x in U . If also V is a nonvoid open subset of E such that U, V are disjoint, then $\xi \in V$ entails $V \cap U$ nonvoid; whence, as V is open, $d(\xi, x) > 0$ for each x in V . Thus E satisfies (*).

On the other hand, if E satisfies (*) and A is an open, located subset of E with $U \equiv E - A$ nonvoid, then there exists ξ in E such that $d(\xi, x) > 0$ for each x in $U \cup A$. Were $\text{dist}(\xi, A) > 0$, we would have the contradiction $\xi \in U$; hence $\text{dist}(\xi, A) = 0$, $\xi \in \bar{A}$, and so E is 0-connected.

On the real line, we can say more:

THEOREM 5. *A necessary and sufficient condition that a located subset E of R satisfy (*) is that E be 0-connected.*

Let E be 0-connected. Let U be a nonvoid, collocated subset of E , V a nonvoid, open subset of E such that U, V are disjoint, and choose u in U, v in V . We may assume that $u < v$. By the lemma in [5], the set

$$B \equiv \{x \in [u, v]: [u, x] \subset U\}$$

is totally bounded. Let $\xi \equiv \sup B$. Then (as $U \cup V$ is open in E) it is clear that $|\xi - x| > 0$ for each x in $U \cup V$. Thus $u < \xi < v$, and so, by Theorem 2, $\xi \in E$. Hence E satisfies (*). Reference to the remarks preceding this theorem completes the proof.

THEOREM 6. *Let E be either an open ball in a Banach space, or a complete, convex subset of a normed space. Then E satisfies (*).*

Let A be a located subset of E , with $U \equiv E - A$ nonvoid. Using the argument of the proof of 2.1 of [3], we can construct a point ξ of $E \cap \bar{A} \cap \bar{U}$. It is easy to see that, if V is a nonvoid, open subset of E such that U, V are disjoint, then $\|\xi - x\| > 0$ for each x in $U \cup V$.

Theorems 5 and 6 support the (classically true) conjecture that 0-connectivity and (*) are equivalent properties of a metric space.

3. An immediate consequence of Theorems 1 and 3 is that the proposition,

if S is a located, 0-connected subset of R , and a, b are points of S with $a < b$, then $[a, b] \subset S$,

is essentially nonconstructive. This, and Theorem 1 itself, extends

the work of Mandelker [6] in response to the first of two questions with which we ended [3]. On the other hand, Proposition 1 enables us to progress towards an answer to the second of these questions, which we shall consider in a form slightly different to that found in [3]:

if f is a uniformly continuous mapping of $[0, 1]$ into R , what connectivity properties obtain for $f([0, 1])$?

LEMMA 2. *Let K be a compact, connected metric space, $f: K \rightarrow R$ a uniformly continuous mapping, and a, b points of $f(K)$ with $a \leq b$. Then $f(K) \cap [a, b]$ is dense in $[a, b]$.*

Let $y \in [a, b]$, and suppose that $0 < r \equiv \text{dist}(y, f(K))$. Then

$$a \leq y - r < y < y + r \leq b.$$

Compute α in $]0, r[$ so that

$$A \equiv f^{-1}(]-\infty, y - \alpha]) = f^{-1}(]-\infty, y])$$

is compact [1, Ch. 4, Thm. 8]. Then A is an open, closed and located subset of K . Hence $A = K$, and so $y \in A$ — a contradiction. Thus $\text{dist}(y, f(K)) = 0$.

THEOREM 7. *The proposition,*

a uniformly continuous mapping $f: [0, 1] \rightarrow R$ has 0-connected range,

is essentially nonconstructive.

Let $\alpha \in [-1, 1]$, and define a uniformly continuous mapping $f: [0, 1] \rightarrow R$ so that $f(0) = -1$, $f(1/3) = f(2/3) = \alpha$, $f(1) = 1$, and f is linear in each of the intervals $[0, 1/3]$, $[1/3, 2/3]$, $[2/3, 1]$. Let $S \equiv f([0, 1])$ and $A \equiv [-1, 0] \cap S$. Then A is open in S . As S is dense in $[-1, 1]$ (by Lemma 2), A is dense in $[-1, 0]$, and therefore totally bounded. Hence A is located in S . Also, $\text{dist}(1, A) > 0$, and $1 \in S$. Suppose that S is 0-connected. Then there exists ξ in $\bar{A} \cap S$ with $|\xi - x| > 0$ for each x in A . It is clear that $\xi = 0$; whence $0 \in S$, and we can compute z in $[0, 1]$ with $f(z) = 0$. Either $1/3 < z$ or $z < 2/3$. In the former case, we have $\alpha = f(1/3) \leq f(z) = 0$; in the latter, $\alpha = f(2/3) \geq f(z) = 0$. Thus we see that the proposition in question entails

$$\forall \alpha \in [-1, 1] \quad (\alpha \geq 0 \vee \alpha \leq 0),$$

a proposition known to be essentially nonconstructive.

We have yet to answer the final question of [3] in its original form:

if f is a uniformly continuous mapping of an interval I in R into a metric space, is $f(I)$ connected?

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K. Adachi, <i>On the multiplicative Cousin problems for $N^P(D)$</i>	297
Howard Banilower, <i>Isomorphisms and simultaneous extensions in $C(S)$</i>	305
B. R. Bhonsle and R. A. Prabhu, <i>An inversion formula for a distributional finite-Hankel-Laplace transformation</i>	313
Douglas S. Bridges, <i>Connectivity properties of metric spaces</i>	325
John Patton Burgess, <i>A selection theorem for group actions</i>	333
Carl Claudius Cowen, <i>Commutants and the operator equations $AX = \lambda XA$</i>	337
Thomas Curtis Craven, <i>Characterizing reduced Witt rings. II</i>	341
J. Csima, <i>Embedding partial idempotent d-ary quasigroups</i>	351
Sheldon Davis, <i>A cushioning-type weak covering property</i>	359
Micheal Neal Dyer, <i>Nonminimal roots in homotopy trees</i>	371
John Erik Fornæss, <i>Plurisubharmonic defining functions</i>	381
John Fuelberth and James J. Kuzmanovich, <i>On the structure of finitely generated splitting rings</i>	389
Irving Leonard Glicksberg, <i>Boundary continuity of some holomorphic functions</i>	425
Frank Harary and Robert William Robinson, <i>Generalized Ramsey theory. IX. Isomorphic factorizations. IV. Isomorphic Ramsey numbers</i>	435
Frank Harary and Allen John Carl Schwenk, <i>The spectral approach to determining the number of walks in a graph</i>	443
David Kent Harrison, <i>Double coset and orbit spaces</i>	451
Shiro Ishikawa, <i>Common fixed points and iteration of commuting nonexpansive mappings</i>	493
Philip G. Laird, <i>On characterizations of exponential polynomials</i>	503
Y. C. Lee, <i>A Witt's theorem for unimodular lattices</i>	509
Teck Cheong Lim, <i>On common fixed point sets of commutative mappings</i>	517
R. S. Pathak, <i>On the Meijer transform of generalized functions</i>	523
T. S. Ravisankar and U. S. Shukla, <i>Structure of Γ-rings</i>	537
Olaf von Grudzinski, <i>Examples of solvable and nonsolvable convolution equations in \mathcal{K}'_p, $p \geq 1$</i>	561
Roy Westwick, <i>Irreducible lengths of trivectors of rank seven and eight</i>	575