

# Pacific Journal of Mathematics

**A SELECTION THEOREM FOR GROUP ACTIONS**

JOHN PATTON BURGESS

## A SELECTION THEOREM FOR GROUP ACTIONS

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**Let a Polish group  $G$  act continuously on a Polish space  $X$ , inducing an equivalence relation  $E$ . Let  $E_Y$  be the restriction of  $E$  to an invariant Borel subset  $Y$  of  $X$ . Assume  $E_Y$  is countably separated. Then it has a Borel transversal.**

Throughout, let  $\Gamma$  be a continuous action of a Polish group  $G$  on a Polish space  $X$ . Thus  $X$  is a separable space admitting a complete metric, while  $G$  is a topological group whose topology is separable and admits a complete metric, and  $\Gamma$  is a continuous function  $G \times X \rightarrow X$  satisfying  $\Gamma(g^{-1}, \Gamma(g, x)) = x$  and  $\Gamma(g, \Gamma(h, x)) = \Gamma(gh, x)$  for all  $x \in X$  and  $g, h \in G$ . We write  $gx$  for  $\Gamma(g, x)$ , and for subsets of  $X$  write  $gA$  for  $\{gx: x \in A\}$ .  $\Gamma$  induces an equivalence relation  $E$  on  $X$ :  $xEy$  iff  $gx = y$  for some  $g \in G$ .  $W \subset X$  is *invariant* if  $gW = W$  for all  $g \in G$ . Let  $Y \subset X$  be an invariant Borel set,  $E_Y$  the restriction of  $E$  to  $Y$ . A *transversal* or *selector-set* for an equivalence relation is a set composed of exactly one representative from each equivalence class. Let us assume  $E_Y$  is *countably separated*, i.e., that there exist invariant Borel  $Z_0, Z_1, Z_2, \dots \subset Y$  such that for all  $x, y \in Y$ :

$$(0) \quad xEy \iff \forall m(x \in Z_m \iff y \in Z_m)$$

our goal is to prove the following selection result:

**THEOREM.** Under the above hypotheses,  $E_Y$  has a Borel transversal. It should be mentioned that a number of special cases and overlapping results have been known to and applied by  $C^*$ -algebraists for some time now. The construction of the required transversal proceeds in four stages.

*Stage A.* It will prove convenient to reserve the letters  $m, n$  plain and with subscripts to range over the set  $I$  of natural numbers, and to reserve  $s, t$  plain and with subscripts to range over the set  $Q$  of finite sequences of natural numbers. We let  $s^*m$  denote the *concatenation* of  $s$  and  $m$ , i.e.,  $s$  with  $m$  tacked on at the end. We wish to define Borel sets  $A(s)$  for every  $s \in Q$  of even length.

*Case 1.*  $s =$  the empty sequence  $\emptyset$ . Set  $A(\emptyset) = Y$ .

*Case 2.*  $s =$  a sequence  $(m, n)$  of length two. Set  $A((m, n)) = Z_m$

if  $n = 0$ , and  $Y - Z_m$  if  $n > 0$ .

*Case 3.*  $s = a$  sequence of form  $t^*m^*n$ , where  $t$  has length  $\geq 2$ , and  $A(t)$  is a closed set. For such  $t$  we wish to define  $A(t^*m^*n)$  for all  $m$  and  $n$  at once. In order to do so, we first fix a complete metric  $\rho$  compatible with the topology of  $X$ . For each  $m$  we then let  $\{A(t^*m^*n): n \in I\}$  be a family of closed sets of  $\rho$ -diameter  $< 1/m$  whose union is  $A(t)$ .

Note that in every case so far we have:

$$(1) \quad A(t) = \bigcap_m \bigcup_n A(t^*m^*n).$$

*Case 4.*  $s = a$  sequence of form  $t^*m^*n$ , where  $t$  has length  $\geq 2$ , and  $A(t)$  is not closed. Again, for such  $t$  we define all  $A(t^*m^*n)$  at once.

But first we introduce by induction on countable ordinals  $\alpha$  a slight modification of the usual hierarchies of Borel sets. Let  $\Theta_0$  be the family of all closed subsets of  $X$ . For a countable ordinal  $\alpha > 0$ , let  $\Theta_\alpha$  be the family of all sets of form  $\bigcap_m \bigcup_n W_{m_n}$  with the  $W_{m_n} \in \bigcup_{\beta < \alpha} \Theta_\beta$ . Thus  $\Theta_1 = F_{oi}$ ,  $\Theta_2 = F_{oioi}$ . For present purposes the rank of a Borel set  $W$  will mean the least  $\alpha$  with  $W \in \Theta_\alpha$ .

Now returning to our Borel set  $A(t)$  of rank  $\alpha > 0$ , we let the  $A(t^*m^*n)$  be sets of rank  $< \alpha$  satisfying (1) above. This completes the opening stage of the construction.

*Stage B.* Let us fix an enumeration  $s_0, s_1, s_2, \dots$  of the nonempty members of  $Q$ , such that if  $s_m$  is an initial segment of  $s_n$ , then  $m < n$ . Let  $F_n$  denote the set of all functions  $\{s_0, \dots, s_{n-1}\} \rightarrow I$ . (So  $F_0$  contains only the empty function  $\emptyset$ .) Let  $F = \bigcup_n F_n$ , and let  $F_\infty$  be the set of all functions  $\{s_i: i \in I\} \rightarrow I$ . We reserve the letters  $\sigma, \tau$  plain and with subscripts to range over  $F$ . We say  $\tau$  is an immediate proper extension of  $\sigma$ , and write  $\sigma \subseteq \tau_\alpha$ , if for some  $n$ ,  $\sigma \in F_n, \tau \in F_{n+1}$ , and  $\tau$  extends  $\sigma$ .

For  $\psi \in F \cup F_\infty$  and  $s = (m_0, m_1, \dots, m_{k-1}) \in \text{dom } \psi$  we define:

$$\psi^+(s) = (m_0, n_0, m_1, n_1, \dots, m_{k-1}, n_{k-1}), \text{ where}$$

$$n_0 = \psi((m_0)) \text{ and } n_1 = \psi((m_0, m_1)), \dots, n_{k-1} = \psi(s).$$

To complete stage B of the construction, we define  $B(\sigma)$  to be the intersection of all  $A(\sigma^+(s))$  for  $s \in \text{dom } \sigma$ . Unpacking all these definitions, one readily verifies that:

$$(2) \quad B(\sigma) = \bigcup_{\sigma \subseteq \tau} B(\tau).$$

Another glance at the definitions (especially stage A, case 2) discloses:

$$(3) \quad x \in B(\sigma) \ \& \ (m) \in \text{dom } \sigma \longrightarrow (x \in Z_m \iff \sigma((m)) = 0).$$

*Stage C.* Before launching into the next stage of the construction, we define for any  $W \subset X$  the *Vaught transform*  $W^\#$  of  $W$  to be  $\{x \in X: \{g \in G: gx \in W\}$  is nonmeager (2nd category) in  $G\}$ . One readily verifies that:

$W^\#$  is invariant.

$W$  is invariant  $\rightarrow W = W^\#$ .

$(\bigcup_n W_n)^\# = \bigcup_n (W_n)^\#$ .

It is shown in [1] that

$$W \text{ is Borel} \longrightarrow W^\# \text{ is Borel}$$

which will be all-important for us.

Now let us define  $C(\sigma) = B(\sigma)^\#$ . The above facts from Vaught's theory of group actions imply that each  $C(\sigma)$  is an invariant Borel set, that  $C(\emptyset) = Y$ , and that:

$$(4) \quad C(\sigma) = \bigcup_{\sigma \in \tau} C(\tau) .$$

Now if  $x \in C(\sigma)$ , then some  $gx \in B(\sigma)$ , so applying (3) above, and recalling that the  $Z_m$  are invariant, we conclude:

$$(5) \quad x \in C(\sigma) \ \& \ (m) \in \text{dom } \sigma \longrightarrow (x \in Z_m \longleftrightarrow \sigma((m)) = 0) .$$

*Stage D.* We say  $\sigma$  *lexicographically precedes*  $\tau$ , and write  $\sigma \triangleleft \tau$ , if for some  $n$  and  $i < n$  we have  $\sigma \in F_n, \tau \in F_n, \sigma(s_j) = \tau(s_j)$  for all  $j < i$ , and  $\sigma(s_i) < \tau(s_i)$ . The relation  $\triangleleft$  well orders each  $F_n$ .

Let  $D(\sigma)$  be the invariant Borel set  $C(\sigma) \cdot \bigcup \{C(\tau): \tau \triangleleft \sigma\}$ . Thus  $D(\emptyset) = Y$  and by (4) and (5) we have:

$$(6) \quad D(\sigma) = \sum_{\sigma \in \tau} D(\tau)$$

$$(7) \quad x \in D(\sigma) \text{ and } (m) \in \text{dom } \sigma \longrightarrow (x \in Z_m \longleftrightarrow \sigma((m)) = 0) .$$

In (6),  $\Sigma$  denotes *disjoint* union.

Finally we are in a position to introduce the Borel set:

$$T = \bigcap_n \bigcup_{\sigma \in F_n} (B(\sigma) \cap D(\sigma)) .$$

We aim to show that  $T$  is the required transversal for  $E_Y$ . To this end we consider an arbitrary  $E$ -equivalence class  $K \subset Y$  and verify that  $T \cap K$  is a singleton.

To begin with, from (6) it is evident that there exists a sequence  $\emptyset = \sigma_0 \subset \sigma_1 \subset \sigma_2 \subset \dots$  of elements of  $F$  such that  $K \in D(\sigma_n)$  for each  $n$ , but  $K \cap D(\sigma) = \emptyset$  for any other  $\sigma \in F$ . Let  $\psi \in F_\infty$  be the union of these  $\sigma_n$ .

Recall that:

$$B(\sigma_n) = \bigcap \{A(\sigma_n^+(s_i)): i < n\} = \bigcap \{A(\psi^+(s_i)): i < n\} .$$

Let us consider the closely related sets:

$$L_n = \bigcap \{A(\psi^+(s_i)) : i < n \text{ and } A(\psi^+(s_i)) \text{ is a closed set}\}.$$

Manifestly the  $L_n$  are closed and nested,  $L_{n+1} \subset L_n$ . They are also nonempty. (To see this, note that  $K \subset D(\sigma_n) \subset C(\sigma_n)$  implies  $K \cap B(\sigma_n) \neq \emptyset$ , and that  $L_n \supset B(\sigma_n)$ .) Finally, the  $\rho$ -diameters of the  $L_n$  converge to zero. (To see this, consider for any given  $m$  the sets  $A(\psi^+(m))$ ,  $A(\psi^+((m, m)))$ ,  $A(\psi^+((m, m, m)))$ ,  $\dots$ . By stage A, case 4 of our construction, the ranks of these sets decrease until at some step we reach a closed set; then by stage A, case 3, at the very next step we get a closed set of  $\rho$ -diameter  $< 1/m$ .) Since  $\rho$  is complete,  $\bigcap_n L_n$  is a singleton  $\{y\}$ .

*Claim.*  $y \in A(\psi^+(s))$  for all  $s$ .

This is established by induction on the rank of the set involved: we know it already for rank 0, i.e., closed, sets. Suppose then  $A(\psi^+(s))$  has rank  $\alpha > 0$ , and assume as induction hypothesis that the claim holds for sets of rank  $< \alpha$ , e.g., for the various  $A(\psi^+(s)^*m^*n)$ . Then for any  $m$ , letting  $n = \psi(s^*m)$ , we have  $\psi^+(s^*m) = \psi^+(s)^*m^*n$ , and so by induction hypothesis,  $y \in A(\psi^+(s)^*m^*n)$ . This shows  $y \in \bigcap_m \bigcup_n A(\psi^+(s)^*m^*n) = A(\psi^+(s))$  as required to prove the claim.

From the claim it is immediate that  $y \in \bigcap_n B(\sigma_n)$ , and also that for any  $m$ ,  $y \in Z_m$  iff  $\psi(m) = 0$ . On the other hand, by (7) above, for any  $m$ ,  $K \subset Z_m$  iff  $\psi(m) = 0$ . But then by (0),  $y \in K$ . And this implies  $y \in \bigcap_n D(\sigma_n)$ . Putting everything together,  $T \cap K = \{y\}$  as required.

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