Pacific Journal of Mathematics

# EMBEDDING PARTIAL IDEMPOTENT $d$-ARY QUASIGROUPS

J. CSIMA

Vol. 80, No. 2 October 1979

PACIFIC JOURNAL OF MATHEMATICS
Vol. 80, No. 2, 1979

# EMBEDDING PARTIAL IDEMPOTENT $d$-ARY QUASIGROUPS

J. CSIMA

**It is shown that every finite partial idempotent $d$-quasigroup is embedded in a finite idempotent $d$-quasigroup.**

1. Introduction. Evans [3] has proved that every partial Latin square of order $n$ can be embedded in a Latin square of order $2n$. Equivalently, every partial quasigroups of order $n$ can be embedded in a quasigroup of order $2n$. The connection between Latin squares and quasigroups is explained in [2]. Lindner [5] has proved that every idempotent partial quasigroup of order $n$ can be embedded in an idempotent quasigroup of order $2^n$, while Hilton [4], using a different technique, reduced this order to $4n$. After Cruse [1] gave a multidimensional analogue of Evans' theorem, Lindner [6] succeeded in proving an embedding theorem for idempotent ternary quasigroups. In the present paper, denoting by $N(p)$ the minimal order of $d$-quasigroups in which the partial idempotent $d$-quasigroup $(P, p)$ is embedded, we show that $(P, p)$ is embedded in an idempotent $d$-quasigroup $(Q, q)$, such that $|Q| \leqq 2N(p)$ if $d$ is odd and $|Q| \leqq 3N(p)$ if $d$ is even.

For $d = 3$ this is an improvement on Lindner's result, but when $d = 2$ our construction gives a higher upper bound than Hilton's. The reason for this is that Hilton's construction relies on the fact that a partial quasigroup can be embedded in a quasigroup with the order doubled. This is not known to be true when $d > 2$ and a direct generalization of Hilton's construction cannot be applied.

2. Notation and definitions. If $A$ is a set and $x \in A^d$, then $x_i$ denotes the $i$th component of $x = (x_1, x_2, \cdots, x_d)$. If $x \in A$, $\bar{x} \in A^d$ is defined as $\bar{x} = (x, x, \cdots, x)$. Similar notation applies to the functions $f: X \to Y^d$ and $g: X \to Y$. For every $x \in X$

$$f(x) = (f_1(x), f_2(x), \cdots, f_d(x))$$

and for every $x \in X^d$, $\bar{g}(x) = (g(x_1), g(x_2), \cdots, g(x_d))$. The function $\Delta_A$: $A \to A^d$ is defined as $\Delta_A(x) = \bar{x}$ for all $x \in A$. The restriction of $f: S \to T$ to $A \subseteq S$ is denoted by $f|A$. We may take exception when $f$ is a $d$-ary operation, in which case $f|A$ will often be abbreviated by $f$. When no danger of ambiguity exists, we do not distinguish between $h: S \to T$ and $g: S \to U$ if $h(x) = g(x)$ for every $x \in S$. The symbol $[x, y]$ denotes the $d$-tuple

$$((x_1, y_1), (x_2, y_2), \cdots, (x_d, y_d)) ,$$

${}_xU$ stands for $\{[x, y]: y \in U\}$ and $S_x$ denotes the Cartesian product $\{x\} \times S$.

If $Q$ is a nonempty finite set of cardinality $n$ and $d$ is a natural number, we say that $q: U \to Q$ is a *partial d-quasigroup of order n*, provided $U \subseteqq Q^d$ and the equation $q(x) = q(y)$ implies that either $x = y$ or else $x$ and $y$ differ in at least two of their components. The partial $d$-quasigroup $q$ may also be denoted by $(Q, q)$ or $(Q, U, q)$. If $U = Q^d$, then $q$ is a *d-quasigroup of order n*.

We observe that if $(Q, q)$ is a finite $d$-quasigroup, then given $x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_d$ and $y$ in $Q$, there exists a unique $x_i \in Q$ such that

$$q(x_1, x_2, \cdots, x_d) = y .$$

A partial $d$-quasigroup $(Q, U, q)$ is *idempotent* if $x \in Q$ implies $\bar{x} \in U$ and $q(\bar{x}) = x$.

In order to simplify our terminology we refer to ordinary finite quasigroups by calling them binary quasigroups and use the word "quasigroup" to abbreviate the expression "finite $d$-quasigroup".

$(S, T, s)$ is a *partial subquasigroup* of the partial quasigroup $(P, U, q)$, *if* $S \subseteqq Q$ and $s = q \mid T$. A partial quasigroup $(S, T, s)$ is *isomorphic* to $(Q, U, q)$, if there exists a bijection $\phi: S \to Q$ such that $\bar{\phi}(T) = U$ and $q(\bar{\phi}(x)) = \phi(s(x))$ for all $x \in T$. $(S, T, s)$ is *embedded* ("can be embedded") in $(Q, U, q)$ if there exists an injection $\phi: S \to Q$ such that $\bar{\phi}(T) \subseteqq U$ and $q(\bar{\phi}(x)) = \bar{\phi}(s(x))$ for all $x \in T$. Evidently, $(S, T, s)$ is embedded in $(Q, U, q)$ if and only if the latter has a partial subquasigroup isomorphic to the former.

A function $t: Q \to Q^d$ is a *transversal* of the quasigroup $(Q, q)$ if

(i) $q(t(x)) = x$ for all $x \in Q$

(ii) $x \neq y$ implies $t_i(x) \neq t_i(y)$ for $i = 1, 2, \cdots, d$. We observe that if $(Q, q)$ is idempotent, then $\Delta_Q$ is a transversal of $(Q, q)$. Some quasigroups do not possess transversals. A transversal $t$ of $(Q, q)$ is an *offbeat* transversal if $t(x) \neq \bar{y}$ for all $x, y \in Q$. We say that $f: Q \to Q^d$ *fixes* $P$ if $P \subseteqq Q$ and $f(x) = \bar{x}$ for all $x \in P$.

## 3. Transversals and embedding.

LEMMA 1. *Let $n \geqq 2$. Then for every odd $d \geqq 3$ there exists an idempotent $d$-quasigroup $(Q, q)$ of order $n$ possessing an offbeat transversal.*

*Proof.* Let $Q = \{0, 1, \cdots, n-1\}$, let

$$q(x) = x_1 + \sum_{i=1}^{(d-1)2} (x_{2i} - x_{2i+1}) \pmod{n}$$

and let

$$t(x) = (x, x+1, x+1, \cdots, x+1) \pmod{n}.$$

Then $(Q, q)$ is an idempotent quasigroup with $t$ as an offbeat transversal.

LEMMA 2. *Let $n \geqq 3$. Then for every $d \geqq 2$ there exists an idempotent $d$-quasigroup of order $n$ with an offbeat transversal.*

*Proof.* We may assume that $d$ is even as Lemma 1 covers the case when $d$ is odd. We first deal with the case when $d = 2$. Figure 1 shows an idempotent binary quasigroup of order 6 with an offbeat transversal $\tau$.

| | 0 | 1 | 2 | 3 | 4 | 5 | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 2 | [1] | 5 | 3 | 4 | $\tau(0) = (1, 4)$ |
| 1 | 4 | 1 | 5 | 2 | [0] | 3 | $\tau(1) = (0, 2)$ |
| 2 | [3] | 4 | 2 | 1 | 5 | 0 | $\tau(2) = (3, 5)$ |
| 3 | 5 | 0 | 4 | 3 | 1 | [2] | $\tau(3) = (2, 0)$ |
| 4 | 2 | [5] | 3 | 0 | 4 | 1 | $\tau(4) = (5, 3)$ |
| 5 | 1 | 3 | 0 | [4] | 2 | 5 | $\tau(5) = (4, 1)$ |

FIGURE 1

For all other orders $n \geqq 3$ the desired binary quasigroups can be constructed with the help of orthogonnl Latin squares. Now let $d \geqq 4$, $d$ even and $n \geqq 3$. Let $Q = \{0, 1, \cdots, n-1\}$ and let $(Q, l)$ be an idempotent binary quasigroup (of order $n$) with an offbeat transversal $\tau$. Let

$$q(x) = l(x_1, x_2) + \sum_{i=2}^{d/2} (x_{2i} - x_{2i-1}) \pmod{n}$$

and let

$$t(x) = (\tau_1(x), \tau_2(x), x, x, \cdots, x).$$

Then $(Q, q)$ is an idempotent $d$-quasigroup with $t$ as an offbeat transversal.

LEMMA 3. *Let $(Q, q)$ be a $d$-quasiguoup with a transversal $t$ and let*

$$q_t(x) = q(t_1(x_1), t_2(x_2), \cdots, t_d(x_d)) .$$

*Then $(Q, q_t)$ is an idempotent quasigroup.*

*Proof.* It is clear that $q_t$ maps $Q^d$ into $Q$. Suppose that $x \neq y$ and $q_t(x) = q_t(y)$. Let $i$ be such that $x_i \neq y_i$. Then $t_i(x_i) \neq t_i(y_i)$. Since

$q(t_1(x_1), t_2(x_2), \cdots, t_d(x_d)) = q(t_1(y_1), t_2(y_2), \cdots, t_d(y_d))$, $(t_1(x_1), t_2(x_2), \cdots, t_d(x_d))$ and $(t_1(y_1), t_2(y_2), \cdots, t_d(y_d))$ must differ in at least two components. Hence there exists a $j \neq i$ such that $t_j(x_j) \neq t_j(y_j)$ implying $x_j \neq y_j$. Thus $x$ and $y$ differ in at least two components and $(Q, q_t)$ is a quasigroup. If $z \in Q$, then

$$q_t(\bar{z}) = q(t_1(z), t_2(z), \cdots, t_d(z)) = q(t(z)) = z$$

and $(Q, q_t)$ is idempotent.

LEMMA 4. *Let $(P, p)$ be an idempotent partial subquasigroup of a (not necessarily idempotent) $d$-quasigroup $(Q, q)$ and let $t$ be a transversal of $(Q, q)$ fixing $P$. Then $(P, p)$ is a partial subquasigroup of $(Q, q_t)$.*

*Proof.* It suffices to show that $q$ and $q_t$ agree on $P^d$. Let $x \in P^d$. Then indeed

$$q_t(x) = q(t_1(x_1), t_2(x_2), \cdots, t_d(x_d)) = q(x_1, x_2, \cdots, x_d) = q(x) .$$

DEFINITION. The *product* $(Q, q)$ of the $d$-quasigroups $(R, r)$ and $(S, s)$ is defined as follows. $Q = R \times S$ and for every

$$z = [x, y] \in (R \times S)^d$$
$$q(z) = (r(x), s(y)) .$$

If $t'$ and $t''$ are transversals in $(R, r)$ and $(S, s)$ respectively, their product $t$ is defined by

$$t(x, y) = [t'(x), t''(y)] .$$

LEMMA 5. *The product $(Q, q)$ of the quasigroups $(R, r)$ and $(S, s)$ is a quasigroup. If $t'$ and $t''$ are transversal of $(R, r)$ and $(S, s)$ respectively, then their product $t$ is a transversal of $(Q, q)$. If $(V, r)$ is a subquasigroup of $(R, r)$, then $q|(V \times S)^d$ is a subquasigroup of $(Q, q)$. If $(R, r)$ is idempotent and $x \in R$, then $(S_x, q)$ is isomorphic to $(S, s)$. The product of idempotent quasigroups is idempotent.*

*Proof,* Let $(Q, q)$ be the product of $(R, r)$ and $(S, s)$. Suppose

$q([x, y]) = q([u, v])$ and $[x, y] \neq [u, v]$. Then $r(x) = r(u)$ and $s(y) = s(v)$. If $x \neq u$, then $x$ and $u$ differ in at least two components and so do $[x, y]$ and $[u, v]$. If $x = u$, then $y \neq v$ and again $[x, y]$ and $[u, v]$ differ in at least two components. Thus $(Q, q)$ is a quasigroup. Suppose $t'$ and $t''$ are transversals of $(R, r)$ and $(S, s)$ respectively and $t$ is their product. Then

$$q(t(x, y)) = q[t'(x), t''(y)] = (r(t'(x)), s(t''(y))) = (x, y) .$$

Suppose $(x, y) \neq (u, v)$. If $x \neq u$, then $t_i'(x) \neq t_i'(u)$ for $i = 1, 2, \cdots, d$; and if $y \neq v$, then $t_i''(y) \neq t_i''(v)$. In any event, if $(x, y) \neq (u, v)$, we have

$$t_i(x, y) = (t_i'(x), t_i''(y)) \neq (t_i'(u), t_i''(v)) = t_i(u, v)$$

for all $i$. Thus $t$ is a transversal of $(Q, q)$. Suppose $(V, r)$ is a subquasigroup of $(R, r)$. Then the range of $q|(V \times S)^d$ is $V \times S$, so $q|(V \times S)^d$ is a subquasigroup of $(Q, q)$. If $(R, r)$ is idempotent, then $y \mapsto (x, y)$ is an isomorphism from $(S, s)$ to $(S_x, q)$ for every $x \in R$. If $(R, r)$ and $(S, s)$ are both idempotent and $z = (x, y) \in Q$, then

$$q(\bar{z}) = (r(\bar{x}), s(\bar{y})) = (x, y) = z$$

and $(Q, q)$ is idempotent.

LEMMA 6. *Let $(R, r)$ and $(S, g)$ be idempotent quasigroups and let $(Q, f)$ be their product. Let $P \subseteq S$ and let $\tau$ be an offbeat transversal of $(R, r)$. For every $z = (x, y) \in Q$ let*

$$t_i(z) = \begin{cases} (x, y) & \text{if } y \in P \\ (\tau_i(x), y) & \text{if } y \notin P \end{cases}$$

*for $i = 1, 2, \cdots, d$. Then $t$ is a transversal of $(Q, f)$, fixing $R \times P$. Furthermore, if $(x, y) \in Q$ and $a \in R$, then $t(x, y) \in S_a^d$ if and only if $x = a$ and $y \in P$.*

*Proof.* Let $(x, y) \in Q$ and $(u, v) \in Q$ be such that $t_i(x, y) = t_i(u, v)$ for some $i$. Then necessarily $y = v$. If $y \in P$, then

$$(x, y) = t_i(x, y) = t_i(u, v) = t_i(u, y) = (u, y) = (u, v) .$$

If $y \notin P$, then $(\tau_i(x), y) = (\tau_i(u), v)$ implies $(x, y) = (u, v)$. If $y \in P$, then

$$f(t(x, y)) = f([\bar{x}, \bar{y}]) = (r(\bar{x}), g(\bar{y})) = (x, y) .$$

If $y \notin P$, then

$$f(t(x, y)) = f([\tau(x), \bar{y}]) = (r(\tau(x)), g(\bar{y})) = (x, y) .$$

Thus $t$ is a transversal of $(Q, f)$. It is evident from the definition of $t$, that $t$ fixes $R \times P$. If $a \in R$ and $y \in P$, then of course $t(a, y) \in S_a^d$. On the other hand if $(x, y) \in Q$, $a \in R$ and $y \notin P$, then $t(x) \notin S_a^d$ because $\tau(x) = \bar{a}$ is impossible as $\tau$ is an offbeat transversal.

LEMMA 7. *Let $(Q, r)$ be a quasigroup with a subquasigroup $(S, r)$ and let $(S, s)$ be an arbitrary quasigroup (on the set $S$). For each $x \in Q^d$ let*

$$q(x) = \begin{cases} s(x) & \text{if } x \in S^d \\ r(x) & \text{if } x \notin S^d \end{cases}.$$

*Then $(Q, q)$ is a quasigroup.*

*Proof.* Let $x \in Q^d$ and $y \in Q^d$ such that $x \neq y$ and $q(x) = q(y)$. If both $x$ and $y$ belong to $S^d$, then $s(x) = s(y)$ implies that $x$ and $y$ differ in at least two components. The same is true if neither $x$ nor $y$ belong to $S^d$. If, say $x \in S^d$ and $y \notin S^d$, assume that $x$ and $y$ differ in exactly one component, say their first. Then $x_1 \neq y_1$ and $x_i = y_i$ if $i \geqq 2$. It follows then, that $y_1 \notin S$. Let $x_1' \in S$ be such that

$$r(x_1', x_2, \cdots, x_d) = s(x_1, x_2, \cdots, x_d) .$$

Then $x_1' \neq y_1$. On the other hand,

$$r(x_1', x_2, \cdots, x_d) = s(x) = r(y) = r(y_1, x_2, \cdots, x_d) ,$$

implying $x_1' = y_1$, a contradiction. Thus $(Q, q)$ is a quasigroup.

DEFINITION. If $(Q, r)$, $(S, r)$, $(S, s)$ and $(Q, q)$ are as in Lemma 7, then $(Q, q)$ is called the *replacement of $(S, r)$ by $(S, s)$ in $(Q, r)$*.

THEOREM 1. *Let $(P, U, p)$ be a partial idempotent sub-d-quasigroup of a d-quasigroup $(S, s)$. Then $(P, U, p)$ can be embedded in an idempotent d-quasigroup $(Q, q)$ such that $|Q| \leqq 3|S|$ if $d$ is even and $|Q| \leqq 2|S|$ if $d$ is odd.*

*Proof.* Let $(P, U, p)$ be a partial idempotent subquasigroup of $(S, s)$. First we deal with the case when $|S| \geqq 3$. Let $g$ be such that $(S, g)$ is an idempotent quasigroup and let $(R, r)$ be an idempotent quasigroup with an offbeat transversal $\tau$. Let $(Q, f)$ be the product of $(R, r)$ and $(S, g)$. Define $t$ as in Lemma 6. Then $t$ is a transversal of $(Q, f)$. Let $a \in R$. Then $t$ fixes $P_a (\subseteqq R \times P)$. Define $s' : S_a^d \to S_a$ as follows: $s'([\bar{a}, z]) = (a, s(z))$ for all $z \in S^d$. Then $(S, s)$ is isomorphic to $(S_a, s')$ via $\phi(y) = (a, y)$ for all $y \in S$. Indeed, $\bar{\phi}(S^d) = S_a^d$ and $s'(\bar{\phi}(z)) = s'([\bar{a}, z]) = (a, s(z)) = \phi(s(z))$ for all $z \in S^d$. Let $(Q, q)$ be the replace-

ment of $(S_a, f)$ by $(S_a, s')$ in $(Q, f)$. Then $\phi | P$ establishes an isomorphism from $(P, U, p)$ to $(P_{a'a}U, q)$. Thus $(P, U, p)$ is embedded in $(Q, q)$. Next we will show, that $t$ is a transversal of $(Q, q)$. It suffices to verify that $q(t(x, y)) = (x, y)$ for every $(x, y) \in Q$. Suppose $(x, y) \in Q$. If $t(x, y) \notin S_a^d$, then $q(t(x, y)) = f(t(x, y)) = (x, y)$. If $t(x, y) \in S_a^d$, we must have $x = a$ and $y \in P$ by Lemma 6. But then

$$q(t(x, y)) = q(t(a, y)) = q([\bar{a}, \bar{y}]) = s'([\bar{a}, \bar{y}]) = (a, s(\bar{y}))$$
$$= (a, p(\bar{y})) = (a, y) = (x, y) .$$

Thus $t$ is a transversal of $(Q, q)$. By Lemma 4 $(P, U, p)$ is embedded in the idempotent $(Q, q_t)$. Clearly, $|Q| = |R|\,|S|$ and the smallest idempotent quasigroup $(R, r)$ with an offbeat transversal is of order 3 or 2, depending on the parity of $d$.

Now let us look at the case when the order of $(S, s)$ is one or two. Then, if $P = S$, $(P, U, p)$ is embedded in the idempotent $(S, s)$. If $P \neq S$, then $(P, U, p)$ is the unique (idempotent) quasigroup or order one, embedded in itself.

THEOREM 2. *Let $(P, p)$ be a finite partial idempotent $d$-quasigroup. Then $(P, p)$ can be embedded in a finite idempotent $d$-quasigroup $(Q, q)$. Furthermore, if $N(p)$ denotes the minimal order of $d$-quasigroups into which $(P, p)$ can be embedded, then $Q$ can be chosen so that $|Q| \leqq 2N(p)$ if $d$ is odd and $|Q| \leqq 3N(p)$ if $d$ is even.*

*Proof.* Using Cruse's result [1] that every finite partial $d$-quasigroup is embedded in a finite $d$-quasigroup, our theorem immediately follows from Theorem 1.

## REFERENCES

1. A. B. Cruse, *On the finite completion of partial Latin cubes*, J. Combinatorial Theory—(A), **17** (1974), 112-119.
2. J. Dénes and A. D. Keedwell, *Latin squares and their applications*, Akadémiai Kiadó, Budapest 1974.
3. T. Evans, *Embedding incomplete Latin squares*, Amer. Math. Monthly, **67** (1960), 958-961.
4. A. J. W. Hilton, *Embedding an incomplete diagonal Latin square in a complete diagonal Latin square*, J. Combinatorial Theory—(A), **15** (1973), 121-128.
5. C. C. Lindner, *Embedding partial idempotent Latin squares*, J. Combinatorial Theory—(A),**10** (1971), 240-245.
6. ———, *A finite partial idempotent Latin cube can be embedded in a finite idempotent Latin cube*, J. Combinatorial Theory—(A) **21** (1976), 104-109.

Received June 8, 1977 and in revised form October 20, 1977, Research supported by the National Research Council of Canada, Grant No. A4078.

McMASTER UNIVERSITY
HAMILTON, ONTARIO L8S 4KI, CANADA

# PACIFIC JOURNAL OF MATHEMATICS



## EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, CA 90024

CHARLES W. CURTIS
University of Oregon
Eugene, OR 97403

C. C. MOORE
University of California
Berkeley, CA 94720

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, CA 90007

R. FINN and J. MILGRAM
Stanford University
Stanford, CA 94305

## ASSOCIATE EDITORS

E. F. BECKENBACH  B. H. NEUMANN  F. WOLF  K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.


Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.


The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Older back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.



Copyright © 1978 by Pacific Journal of Mathematics
Manufactured and first issued in Japan

**Pacific Journal of Mathematics**
Vol. 80, No. 2 October, 1979

K. Adachi, *On the multiplicative Cousin problems for $N^p(D)$* ..... 297
Howard Banilower, *Isomorphisms and simultaneous extensions in $C(S)$* .... 305
B. R. Bhonsle and R. A. Prabhu, *An inversion formula for a distributional finite-Hankel-Laplace transformation* ..... 313
Douglas S. Bridges, *Connectivity properties of metric spaces* ..... 325
John Patton Burgess, *A selection theorem for group actions* ..... 333
Carl Claudius Cowen, *Commutants and the operator equations $AX = \lambda XA$* ..... 337
Thomas Curtis Craven, *Characterizing reduced Witt rings. II* ..... 341
J. Csima, *Embedding partial idempotent d-ary quasigroups* ..... 351
Sheldon Davis, *A cushioning-type weak covering property* ..... 359
Micheal Neal Dyer, *Nonminimal roots in homotopy trees* ..... 371
John Erik Fornaess, *Plurisubharmonic defining functions* ..... 381
John Fuelberth and James J. Kuzmanovich, *On the structure of finitely generated splitting rings* ..... 389
Irving Leonard Glicksberg, *Boundary continuity of some holomorphic functions* ..... 425
Frank Harary and Robert William Robinson, *Generalized Ramsey theory. IX. Isomorphic factorizations. IV. Isomorphic Ramsey numbers* ..... 435
Frank Harary and Allen John Carl Schwenk, *The spectral approach to determining the number of walks in a graph* ..... 443
David Kent Harrison, *Double coset and orbit spaces* ..... 451
Shiro Ishikawa, *Common fixed points and iteration of commuting nonexpansive mappings* ..... 493
Philip G. Laird, *On characterizations of exponential polynomials* ..... 503
Y. C. Lee, *A Witt's theorem for unimodular lattices* ..... 509
Teck Cheong Lim, *On common fixed point sets of commutative mappings* ..... 517
R. S. Pathak, *On the Meijer transform of generalized functions* ..... 523
T. S. Ravisankar and U. S. Shukla, *Structure of $\Gamma$-rings* ..... 537
Olaf von Grudzinski, *Examples of solvable and nonsolvable convolution equations in $\mathcal{H}'_p$, $p \geq 1$* ..... 561
Roy Westwick, *Irreducible lengths of trivectors of rank seven and eight* .... 575