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Let π be a finite group which does not satisfy the Eichler condition and let M be a π-module. A π-module M' is a noncancellation example of M if $M\oplus(Z\pi)^2\cong M'\oplus(Z\pi)^2$ but $M\not\cong M'$. This note classifies the set $\mathscr{N}\mathscr{C}_M(\pi)$ of isomorphism classes of noncancellation examples for $M=Z\oplus Z\pi$, where Z is the trivial π-module, $M=A(\pi)$, the augmentation ideal, and $M=Z\pi/(N)$, where (N) is the ideal generated by the norm element $N=\sum_{x\in\pi}x$. It is shown that these noncancellation examples yield nonminimal roots of the homotopy tree $HT(\pi, m)$ of (π, m)-complexes.

1. Introduction. Let π be a finite group. We say that a π-module M satisfies the *Eichler condition* if the endomorphism ring End(QM) has no simple component which is a totally definite quaternion algebra over its center (see [11, page 176] for a definition). A finitely generated, Z-torsion free (left) π-module M has the *cancellation property* (CP) iff for any π-module M' such that $M\oplus(Z\pi)^2\cong M'\oplus(Z\pi)^2$ we have $M\cong M'$. If $M\oplus(Z\pi)^2\cong M'\oplus(Z\pi)^2$, we say that M and M' are *stably isomorphic*. Note that this is completely general, for by Bass' cancellation [1, Corollary 10.2], $M\oplus(Z\pi)^2\cong M'\oplus(Z\pi)^2$ iff $M\oplus(Z\pi)^n\cong M'\oplus(Z\pi)^n (n\geqq 2)$. If M has the Eichler condition, and $M\cong N\oplus Z\pi$, then M has the cancellation property [7], [11, Theorem 19.8].

In this paper we are interested in noncancellation examples. A module M' is a *noncancellation example for* M iff M' is stably isomorphic to, but not isomorphic to M. We determine in § 2 the set $\mathscr{N}\mathscr{C}_M(\pi)$ of isomorphism classes on noncancellation examples of certain modules M. In § 3, we show that the Swan counterexample [10, Theorem 3] for the generalized quaternion group of order 32 gives rise to noncancellation examples.

We apply this to the homotopy classification of (π, m)-complexes. A (π, m)-complex is a finite, connected, m-dimensional CW-complex with $\pi_1X\cong\pi$ and $\pi_iX=0$ for $1<i<m$. A (π, m)-conplex X is called a *root* if there is *no* other (π, m)-complex Y such that $Y\vee S^m\simeq X$; *a minimal root* if the number $(-1)^m\chi(X)$ is minimal over all (π, m)-complexes; otherwise a *nonminimal root*. In § 4, we show that the Swan counterexample gives rise to nonminimal roots for (GQ(32), 4i-1)-complexes.

For π a finite group, a recent theorem of W. Browning [2] (generalizing the Jacobinski cancellation theorem to the category of pointed modules) shows that such nonminimal roots occur very rarely. In fact, for π finite, nonminimal roots for (π, m)-complexes oncur only if π is periodic and $m = k - 1$, where k is a period of π. The situation for infinite groups is much less clear. However, M. J. Dunwoody [3] has constructed an example of a nonminimal root for $(T, 2)$-complexes, where T is the trefoil knot group.

I would like to thank R. Swan for his proof of the crucial Lemma 3.4 in this paper and the referee for simplifying the hypotheses in Lemmas 2.4 and 2.8.

2. Noncancellation. Let π be a finite group of order n. For each integer p prime to n, let (p, N) denote the ideal of the integral group ring $Z\pi$ generated by p and the norm element $N = \sum_{x \in \pi} x$. Each (p, N) is projective [8, Lemma 6.1]. If Z_n^* denotes the units of the ring of integers modulo n and $\bar{p}$ is the residue class of an integer p modulo n, then the correspondence $\bar{p} \to$ class $[(p, N)]$ of (p, N) in the (reduced) projective class group $\tilde{K}_0 Z\pi$ of $Z\pi$ defines a homomorphism

$$\partial \colon Z_n^* \longrightarrow \tilde{K}_0 Z\pi$$

(see [8, Lemma 6.1]).

Note. For any $p \in Z_n^*$, we will abuse the notation and write (p, N). This is well-defined up to isomorphism because if $r \equiv s (\mathrm{mod}\, n)$, then $(r, N) \cong (s, N)$.

Let $\mathscr{P}'(\pi)$ denote the set of isomorphism classes of projective (left) ideals in the integral group ring $Z\pi$ of π. By Theorem A of [9], $\mathscr{P}'(\pi)$ is also the set of isomorphism classes of rank 1 projective π-modules. Let $\{P\}$ denote the isomorphism class of the projective ideal P. Let $SF(\mathscr{P}')$ (respectively $SW(\mathscr{P}')$) denote the subset of $\mathscr{P}'(\pi)$ consisting of those isomorphism classes $\{P\}$ such that the element $[P]$ in $\tilde{K}_0 Z\pi$ is zero (respectively, $[P] \in \operatorname{im} \partial$). Furthermore, let $F(\pi) = \{p \in Z_n^* \mid (p, N) \cong Z\pi\}$ and $SF(\pi) = \ker \partial = \{p \in Z_n^* \mid (p, N) \oplus Z\pi \cong (Z\pi)^2\}$.

We may identify the groups $SF(\pi)/F(\pi) \hookrightarrow Z_n^*/F(\pi)$ as subgroups of the set $\mathscr{P}'(\pi)$ via $p \to \{(p, N)\}$. The group action is given by $\{(p, N)\} \cdot \{(q, N)\} = \{(p, N) \otimes_\pi (q, N)\} = \{(pq, N)\}$. Thus

$$\begin{array}{ccccc} \dfrac{SF(\pi)}{F(\pi)} & \subset & \dfrac{Z_n^*}{F(\pi)} & & \\ \cap & & \cap & \subset & \mathscr{P}^1(\pi)\,. \\ SF(\mathscr{P}) & \subset & SW(\mathscr{P}^1) & & \end{array}$$

Furthermore, the group $Z_n^*/F(\pi)$ (respectively $SF(\pi)/F(\pi)$) acts on the set $SW(\mathscr{P}^1)$ (respectively $SF(\mathscr{P}^1)$) as follows: for each projective ideal P and $p \in Z_n^*$, define $P_p = (p, N) \otimes_\pi P$. Then let $p \cdot \{P\} =$

$\{P_p\}$. In order to define the above tensor product, we note that (p, N) is a 2-sided ideal, hence a right π-module ((p, N) is also an invertible bimodule). Then P_p has a left π-module structure using the left module structure of P, because $(p, N) \otimes_\pi P \cong Z\pi q \cdot P + Z\pi N \cdot P (q \in p$ is an integer) and hence is the left ideal generated by $(q \cdot P, N \cdot P)$.

DEFINITION. Let M be a π-module. Let $*_M$ denote the class of modules isomorphic to M. Let $\mathscr{N}\mathscr{C}_M(\pi)$ be the set whose elements consist of $*_M$ together with the set of isomorphism classes of non-cancellation examples of M. Thus

$$\mathscr{N}\mathscr{C}_M(\pi) = \{*_M\} \cup \{\{M'\} \mid M' \oplus (Z\pi)^2 \cong M \oplus (Z\pi)^2 \text{ but } M' \neq M\}.$$

In this section we will compute $\mathscr{N}\mathscr{C}_M(\pi)$ for $M = Z \oplus Z\pi$, where Z is the trivial π-module, $M = A(\pi)$, the augmentation ideal in $Z\pi$, and $M = Z\pi/(N)$, where (N) is the ideal generated by the norm element N. If a group G acts on a set S (on the left) as a group of permutations, we denote the set of orbits by S/G.

THEOREM 2.1. *The following sets are isomorphic*:
(a) $\mathscr{N}\mathscr{C}_{Z \oplus Z\pi}(\pi) \cong SW(\mathscr{P}^1(\pi))/(Z_n^*/F(\pi))$
(b) $\mathscr{N}\mathscr{C}_{A(\pi)}(\pi) \cong \mathscr{N}\mathscr{C}_{Z\pi/(N)}(\pi) \cong SF(\mathscr{P}^1(\pi))/(SF(\pi)/F(\pi))$.

Note 2.2. (a) It follows from [11, Theorem 9.7], [4, Propositions 5.3, 5.4, 5.5] that if M is $Z \oplus Z\pi$, $A(\pi)$, or $Z\pi/(N)$, then M' is stably isomorphic to M iff $M' \oplus Z\pi \cong M \oplus Z\pi$.

(b) Lemma 6.2 of [8] and Proposition 5.5 of [4] show that

$$M' \oplus Z\pi \cong (Z \oplus Z\pi) \oplus Z\pi$$

iff $M' \cong Z \oplus P$ where P is a projective ideal and $[P] \in \operatorname{im} \partial$ in $\tilde{K}_0 Z\pi$.

We will prove Theorem 2.1 after a series of propositions and lemmas.

LEMMA 2.3. *For any $\bar{q} \in Z_n^*$, and any projective (left) ideal $P \subset Z\pi$, $P_{\bar{q}}/P_{\bar{q}}^\pi \cong P/P\pi$, where $P_{\bar{q}} = (q, N) \otimes_\pi P$ and $P^\pi = \{p \in P \mid xp = p,\ \forall x \in \pi\}$.*

Proof. Let $N \cdot P = t_P \cdot Z \cdot N \subset (N) \cap P = s_P \cdot Z \cdot N = P^\pi$, where t_P and s_P are positive integers such that s_P divides t_P. Then

$$\begin{aligned} P_{\bar{q}}/P_{\bar{q}}^\pi &= \frac{(q, N) \otimes_\pi P}{((q, N) \otimes_\pi P)^\pi} \cong \frac{q \cdot P + P \cdot N}{(qs_pZ + t_pZ)N} \\ &\cong \frac{1 \cdot P + P \cdot N}{(s_PZ + t_PZ)N} \cong P/(s_PZ)N = P/P^\pi . \end{aligned}$$

The second isomorphism is given by carrying $q\alpha + \alpha' N \to \alpha + \alpha' N$ for any $\alpha, \alpha' \in P$.

PROPOSITION 2.4. *Let P be a projective (left) ideal in $Z\pi$ and n be the order of π. Then $\operatorname{Ext}^1_{Z\pi}(P/P^\pi, P^\pi) \cong Z_n$. Furthermore, the projective extensions of $P^\pi \cong Z$ by P/P^π are $Z_n^* = \{0 \to \underset{\underset{Z}{\|}}{P_q^\pi} \to P_q \to P_q/P_q^\pi \to 0 \mid q \in Z_n^*\}$, where $P/P^\pi \cong P_q/P_q^\pi$.*

Proof. To prove the first statement we localize: $\operatorname{Ext}^1_{Z\pi}(P/P^\pi, P^\pi) \cong \bigoplus_{p|n} \operatorname{Ext}_{Z_{(p)}\pi}(P_{(p)}/P^\pi, P^\pi_{(p)})$. [6, Corollary 3.12, page 16.] Theorem 4.4 of [6] yields that $Z_{(p)}\pi$ is isomorphic to $P_{(p)}$. Thus $\operatorname{Ext}^1_{Z\pi}(P/P^\pi, P^\pi) \cong \operatorname{Ext}^1_{Z\pi}(Z\pi/Z\pi^\pi, Z\pi^\pi) \cong Z_n$.

The projective extensions are necessarily the units Z_n^* of Z_n [5, 1.1] and hence are given by the diagram below. Choose $s \in q \in Z_n^*$.

$$\begin{array}{ccccccccc}
 & & Z & & & & & & \\
 & & \| & & & & & & \\
0 & \longrightarrow & P^\pi & \longrightarrow & P & \longrightarrow & P/P^\pi & \longrightarrow & 0 \\
 & & \downarrow{\cdot s} & & \downarrow{\cdot s} & & \| & & \\
0 & \longrightarrow & P_q^\pi & \longrightarrow & P_q & \longrightarrow & (P_q/P_q^\pi \xrightarrow{\cong}) \, P/P^\pi & \longrightarrow & 0 \\
 & & & & \| & & & & \\
 & & & & \{sP + N\cdot P\} & & & &
\end{array}$$

Thus P_q represents the element $q \in \operatorname{Ext}(P/P^\pi, P^\pi) = Z_n$.

Note. We observe that the function $Z = \operatorname{End} P^\pi \to \operatorname{Ext}^1_{Z\pi}(P/P^\pi, P^\pi)$ given by pushouts is surjective because P is projective.

LEMMA 2.5. *If $h: P \oplus Z\pi \xrightarrow{\cong} (Z\pi)^2$, then $P/P^\pi \oplus Z\pi \cong Z\pi/(N) \oplus Z\pi$.*

Proof. It is easy to see that $P^\pi = (N) \cap P$ and that $(P \oplus Z\pi)^\pi = ((N) \cap P) \oplus (N)$. Consider $\bar{h} = h|(P \oplus Z\pi)^\pi$. $\bar{h}$ is an automorphism of $Z \oplus Z$. By diagonalizing the (integer) matrix of $\bar{h}$, one may obtain a basis $\{e_1, e_2\}$ for $Z\pi^2$ with respect to which $h((N) \cap P) = N \cdot Z\pi \cdot e_1$. Thus $P/P^\pi \oplus Z\pi \cong Z\pi/(N) \oplus Z\pi$.

LEMMA 2.6. *For each $p \in Z_n^*$, $P_p \oplus Z\pi \cong P \oplus (p, N)$.*

Proof. Choose an integer $q \in p \in Z_n^*$ and consider the following commutative diagram with exact rows:

$$(*)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & (N) & \longrightarrow & Z\pi & \longrightarrow & Z\pi/(N) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \| & & \\ 0 & \longrightarrow & (N) & \longrightarrow & (q, N) & \xrightarrow{h} & (Z\pi/(N) & \longrightarrow & 0 \\ & & \| & & & & & & \\ & & Z & & & & & & \end{array}$$

where $h: (q, N) \to Z\pi/(N)$ carries $\alpha q + \alpha' N \mapsto \alpha + (N)$ and $f(\alpha) = q \cdot \alpha (\alpha, \alpha' \in Z\pi)$. Then g is multiplication by q, also. By tensoring the above diagram (*) on the right by P we obtain

$$\begin{array}{ccccccccc} & & Z & & & & & & \\ & & \| & & & & & & \\ 0 & \longrightarrow & (N) \otimes_\pi P & \longrightarrow & P & \longrightarrow & Z\pi/(N) \otimes_\pi P & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \bar{g}} & & \downarrow{\scriptstyle \bar{h}} & & \| & & \\ 0 & \longrightarrow & (N) \otimes_\pi P & \longrightarrow & (q, N) \oplus_\pi P & \longrightarrow & Z\pi/(N) \otimes_\pi P & \longrightarrow & 0\,. \end{array}$$

Thus by Schanuel's lemma [8, § 1], $Z \oplus P_p \cong Z \oplus P$ by a map of degree q (multiplication by q on the left factor). By Lemma 6.4 of [8], $[P_p] = [P] + [(p, N)]$ in $\tilde{K}_0 Z\pi$ and hence $P_p \oplus Z\pi \cong P \oplus (p, N)$ follows from Bass' cancellation theorem [11, Theorem 9.7].

LEMMA 2.7. *If* $[P]$ *is a member of* $\operatorname{im} \partial$, *then* $P/P^\pi \oplus Z\pi \cong Z\pi/(N) \oplus Z\pi$.

Proof.

$$\begin{aligned} P \oplus Z\pi \cong (p, N) \oplus Z\pi &\Longrightarrow P \oplus Z\pi \oplus (q, N) \cong (Z\pi)^3 (q = p^{-1}) \\ &\Longrightarrow P_q \oplus (Z\pi)^2 \cong (Z\pi)^3 \ (2.6) \\ &\Longrightarrow P_q \oplus Z\pi \cong (Z\pi)^2 \ (\text{Bass cancellation}) \\ &\Longrightarrow P_q/P_q^\pi \oplus Z\pi \cong Z\pi/(N) \oplus Z\pi \ (2.5) \\ &\Longrightarrow P/P^\pi \oplus Z\pi \cong Z\pi/(N) \oplus Z\pi \ (2.3). \end{aligned}$$

PROPOSITION 2.8. *If* P *and* Q *are projective ideals in* $Z\pi$, *then* $Z \oplus P \cong Z \oplus Q$ *iff* $Q \cong P_p$ *for some* $p \in Z_n^*$.

Proof. If $Q \cong P_p$, then $Z \oplus P \cong Z \oplus P_p$ follows from the proof of Lemma 2.6. $Z \oplus P \cong Z \oplus Q$ implies that $P/P^\pi \cong Q/Q^\pi$. Since $\operatorname{Ext}(Q/Q^\pi, Z) = Z_n$, there is an extension $0 \to Z \to R \to P/P^\pi \to 0$ such that $R \cong Q$. R is projective implies that $R \cong P_p$ for some $p \in Z_n^*$.

The following proposition follows easily from Lemma 6.1 of [8].

PROPOSITION 2.9. *$Z\oplus P\oplus Z\pi\cong Z\oplus(Z\pi)^2$ iff $[P]\in \operatorname{im}\partial\subset \tilde{K}_0Z\pi$.*

PROPOSITION 2.10. *$Z\pi/(N)\oplus Z\pi\cong M\oplus Z\pi$ iff there exists a projective ideal P_M such that*

(a) $P_M/(P_M)^\pi\cong M$

(b) $[P_M]=0$ *in* $\tilde{K}_0Z\pi$.

Furthermore, let $Z\pi/(N)\oplus Z\pi\cong M'\oplus Z\pi$. Then $M\cong M'$ iff $P_{M'}\cong (P_M)_p$ for some $p\in Z_n^*$.

Proof. ($\Leftarrow$) By 2.7, $P_M\oplus Z\pi\cong(Z\pi)^2$ implies $Z\pi/(N)\oplus Z\pi\cong P_M/(P_M)^\pi\oplus Z\pi\cong M\oplus Z\pi$.

($\Rightarrow$) Consider the exact sequence

$$0\longrightarrow \underset{\substack{\| \\ ((N),\,0)}}{Z}\overset{i}{\longrightarrow}(Z\pi)^2\overset{j}{\longrightarrow}Z\pi/(N)\oplus Z\pi\longrightarrow 0\,.$$

Since α: $Z\pi/(N)\oplus Z\pi\cong M\oplus Z\pi$, we have

$$0\longrightarrow Z\overset{i}{\longrightarrow}Z\pi^2\overset{\alpha\circ j}{\longrightarrow}M\oplus Z\pi\longrightarrow 0$$

is exact. $Z\pi$ is a projective π-module implies that there exists a projective ideal P_M such that $\beta\colon P_M\oplus Z\pi\cong Z\pi\oplus Z\pi$ and

$$0\longrightarrow Z\longrightarrow P_M\oplus Z\pi\underset{\substack{\| \\ j'\oplus id}}{\overset{\alpha j\beta}{\longrightarrow}}M\oplus Z\pi\longrightarrow 0\,.$$

Thus $0\to Z\overset{i'}{\to}P_M\overset{j'}{\to}M\to 0$ is exact. M is torsion free and $M^\pi=0$ implies that $i'(Z)=P_M^\pi$. Thus $P_M/(P_M)^\pi\cong M$ and $[P_M]=0$ in $\tilde{K}_0Z\pi$.

For the second part, suppose that $P_{M'}\cong(P_M)_p$ for some $p\in Z_n^*$. Then $M'\cong P_{M'}/(P_{M'})^\pi\cong(P_M)_p/(P_M)_p^\pi\cong P_M/P_M^\pi\cong M$ by 2.3.

If $M\cong M'$, then $0\to Z\to P_{M'}\to M\to 0$ is exact. By 2.4, $\operatorname{Ext}(M,Z)\cong Z_n$ and the set of projective extensions is given by

$$\{0\longrightarrow P_M^\pi\longrightarrow(P_M)_p\longrightarrow(P_M)_p/(P_M)_p^\pi\cong P_M/P_M^\pi\cong M\longrightarrow 0\,|\,p\in Z_n^*\}\,.$$

Thus $P_{M'}\cong(P_M)_p$ for some $p\in Z_n^*$.

The following proposition has a proof which is similar to that of 2.10. For any projective ideal $P\subset Z\pi$, let $\varepsilon\colon P\to Z$ be the augmentation.

PROPOSITION 2.11. *$A(\pi)\oplus Z\pi\cong M\oplus Z\pi$ iff there exists a projective ideal P_M such that*

(a) $0\to\bar{M}\to P_M\overset{\varepsilon}{\to}Z\to 0$ *with* $\bar{M}\cong M$, *and*

(b) $[P_M] = 0$ *in* $\tilde{K}_0 Z\pi$.

Furthermore, let $A(\pi) \oplus Z\pi \cong M' \oplus Z\pi$. *Then* $M \cong M'$ *iff* $P_{M'} \cong (P_M)_p$ *for some* $p \in Z_n^*$.

We point out that the proof of 2.11 is not quite "dual" to that of 2.10, for it uses the relative injectivity of $Z\pi$ and the fact that, for any projective ideal P in $Z\pi$, $\mathrm{Ext}(Z, P/\ker \varepsilon) \cong Z_n$, etc.

Proof of Theorem 2.1. We prove only (a), as (b) is similar. Define a function $\nu: SW(\mathscr{P}'(\pi)) \to \mathscr{N}\mathscr{C}_{Z\oplus Z\pi}$ by $\nu(\{P\}) = \{Z \oplus P\}([P] \in \operatorname{im} \partial)$, where P is a projective ideal in $Z\pi$. Clearly ν is onto by 2.2(b). If $Z \oplus P \cong Z \oplus P'$, then (2.8) implies that $P' \cong P_p$ for some $p \in Z_n^*$.

3. Nontrivial $\mathscr{N}\mathscr{C}_N(\pi)$. In this section we show that both $\mathscr{N}\mathscr{C}_{Z\oplus Z\pi}(\pi)$ and $\mathscr{N}\mathscr{C}_{Z\oplus Z\pi}(\mathrm{Aut}\,\pi)$ are nontrivial for $\pi = GQ(32)$, the generalized quaternion group of order 32.

Definition. Let θ be an automorphism of π. Two π-modules M, M' are *θ-isomorphic* ($M \cong {}_\theta M'$) if there is a function $\beta: M \to M'$ which is bijective such that $\beta(x \cdot m) = \theta(x)\beta(m)$ for all $x \in \pi$, $m \in M$. β is called a *θ-isomorphism*. Let $\bar{*}_M$ denote the class of all modules stably isomorphic to M *and* θ-isomorphic to M for some $\theta \in \mathrm{Aut}\,\pi$. Clearly $*_M \subset \bar{*}_M$. Furthermore, let $\mathscr{N}\mathscr{C}_M(\mathrm{Aut}\,\pi)$ denote the set which is the union of $\bar{*}_M$ with the set of $\mathrm{Aut}\,\pi$-isomorphism classes of π-modules M' such that

(a) $M' \oplus (Z\pi)^2 \cong M \oplus (Z\pi)^2$

and

(b) M' is not θ-isomorphic to M for any $\theta \in \mathrm{Aut}\,\pi$.

Definition. A π-module M is *full* if for each $\theta \in \mathrm{Aut}\,\pi$, there is a θ-isomorphism $M \to M$.

For example, it is clear that $Z \oplus Z\pi$, $A(\pi)$, and $Z\pi/(N)$ are full π-modules.

Proposition 3.1. *If M is a full π-module, then* $*_M = \bar{*}_M$.

Proof. We must show that $M \cong M'$ if $M \cong {}_\theta M'$. Suppose β: $M \to M'$ is an θ-isomorphism. Let $\alpha: M \to M$ be a θ^{-1}-isomorphism. Then the composite $\beta \cdot \alpha: M \to M'$ is an *id*-isomorphism.

Corollary 3.2. *If M is a full π-module, then* $\mathscr{N}\mathscr{C}_M(\pi) \neq *_M$ *yields* $\mathscr{N}\mathscr{C}_M(\mathrm{Aut}\,\pi) \neq \bar{*}_M$.

Now let $G = GQ(32)$, the generalized quaternion group of order 32, and let P be the projective ideal in ZG defined in [10]. P has the following properties:

3.3 (a) $P \oplus ZG \cong (ZG)^2$

but

3.3 (b) $P \not\cong ZG$.

The proof of the following lemma was shown to me by R. Swan. It generalizes (3.3(b)).

LEMMA 3.4. *For any* $p \in Z_{32}^*$, $(p, N) \not\cong P$.

Proof. Suppose $P \cong (p, N)$ for some $p \in Z_{32}^*$. Then $P \oplus Z \cong ZG \oplus Z$. Let Λ be the order considered in [10] and apply $\Lambda \otimes_{ZG} -$ to the above obtaining

$$(3.5) \qquad (\Lambda \otimes_{ZG} P) \oplus (\Lambda \otimes_{ZG} Z) \cong \Lambda \oplus (\Lambda \otimes_{ZG} Z) .$$

The module $\mathscr{P} = \Lambda \otimes_{ZG} P \not\cong \Lambda$ [10, Lemma 1]. Now $\Lambda \otimes_{ZG} Z$ is a torsion module because $QG \cong Q\Lambda \times Q \times \cdots$, so $Q\Lambda \otimes_{QG} Q = 0$. Factoring out the torsion in (3.5) gives $\mathscr{P} \cong \Lambda$, which is contradiction.

COROLLARY 3.6. *For* $G = GQ(32)$ *and* $M = Z \oplus ZG$, $A(G)$, *or* $ZG/(N)$, $\mathscr{N}\mathscr{C}_M(G) \neq *_M$.

Proof. $P \not\cong (p, N)$ for any $p \in Z_{32}^*$ implies that $Z \oplus ZG \not\cong Z \oplus P$, by 2.8. Clearly $Z \oplus P \oplus ZG \cong Z \oplus (ZG)^2$ by 3.3(a). If $A' = \ker\{\varepsilon: P \to Z\}$, then by 2.11, $A(G) \oplus ZG \cong A' \oplus ZG$, but $A' \not\cong A(G)$. Letting $B = P/P^G$, 2.10 shows that $B \oplus ZG \cong ZG/(N) \oplus ZG$, but $B \not\cong ZG/(N)$, by 3.4.

4. **Roots in homotopy trees.** Let (π, m) be fixed, where π is a group and m an integer greater than or equal to two. Let $\chi_{\min} = \chi_{\min}(\pi, m) = \min\{(-1)^m \chi(X) \mid X \text{ is a } (\pi, m)\text{-complex}\}$. The *level* of a (π, m)-complex X is the number $(-1)^m \chi(X) - \chi_{\min}$. For π finite, it is known that roots occur only at levels 0 (minimal roots) or 1. In this section we give an example of a (π, m)-complex which is a root at level one. As pointed out in the introduction, these level one roots are rare (for π finite), occurring only when π is periodic and $m = k - 1$, where k is a period of π. Dunwoody's example is also at level one [3].

Question. Do roots occur at levels other than 0 or 1?

DEFINITION. The homotopy tree $HT(\pi, m)$ is a directed tree whose vertices $[X]$ consist of the homotopy classes of (π, m)-complexes X; a vertex $[X]$ is connected by an edge to vertex $[Y]$ iff Y has the homotopy type of the sum $X \vee S^m$ of X and the m-sphere S^m.

COROLLARY 3.7. *Let* $G = GQ(32)$. *Then each homotopy tree* $HT(G, 4i\text{-}1)(i > 0)$ *has nonminimal roots (at level one).*

Proof. Consider $\ker\{\partial\colon Z_{32}^* \to \tilde{K}_0 ZG\}$ (see § 2). Recent computations of S. Ullom [12, Prop. 3.5] show that

$$\ker \partial = \pm (Z_{32}^*)^2 .$$

Let $\mathscr{N}\mathscr{C} = \mathscr{N}\mathscr{C}_{Z\oplus ZG}(\operatorname{Aut} G)$. For each $\alpha \in \mathscr{N}\mathscr{C}$, choose a representative $Z \oplus P_\alpha \in \alpha$. It follows from Theorem 9.1 of [4] that the number of distinct homotopy classes of $(G; 4i\text{-}1)$-complexes at level one is given by order of the set

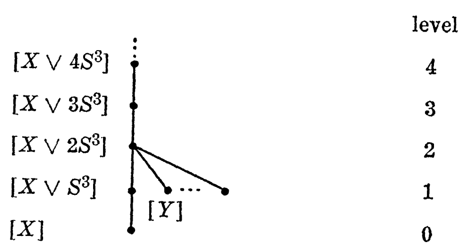
$$\dot{\bigcup_{\alpha \in \mathscr{N}\mathscr{C}}} \{\ker \partial / Q_{4i-1}(Z \oplus P_\alpha)\} .$$

For a definition of the subgroup $Q_{4i-1}(Z \oplus P_\alpha)$ of $\ker \partial$, see [4, page 272]. The number of distinct classes of *roots* is given by the order of the *nonempty* set

$$\dot{\bigcup_{\alpha \in \mathscr{N}\mathscr{C} - *}} \{\ker \partial / Q_{4i-1}(Z \oplus P_\alpha)\} .$$

We note that $\mathscr{N}\mathscr{C}_M(G) \neq *$ for $M = A(G)$ or $ZG/(N)$ implies that the homotopy trees $HT(G, 4i\text{-}2)$ or (respectively) $HT(G, 4i)$ have nontrivial minimal roots, with the possible exception of $HT(G, 2)$.

Finally, the computations of Ullom [12, 3.5] allow one to show that the homotopy tree $HT(G, 3)$ looks like:



where X is the unique $(G, 3)$-complex (up to homotopy type) having Euler characteristic zero and Y is the $(G, 3)$-complex at level 1 having $\pi_3(Y) \cong Z \oplus P$. It follows that $Q_3(Z \oplus P_\alpha) = \ker \partial$ for all $\alpha \in \mathscr{N}\mathscr{C}$ and hence the number of homotopy types of $(G, 3)$-complexes at level one is given by the order of the set

$$\mathcal{NC}_{Z\oplus ZG}(\mathrm{Aut}\, G) .$$

REFERENCES

1. H. Bass, *K-theory and stable algebra*, IHES, **22** (1965), 5-60.

2. W. Browning, *The homotopy classification of non-minimal 2-complexes with given finite fundamental group*, Preliminary report.

3. M. Dunwoody, *The homotopy type of a two-dimensional complex*, (preprint).

4. M. Dyer, *Homotopy classification of (π, m)-complexes*, J. of Pure and Applied Algebra, **7** (1976), 249-282.

5. ———, *Projective k-invariants*, Comment. Math. Helvetici, **51** (1976), 259-277.

6. K. Gruenberg, *Relation modules of finite groups*, CBMS Regional Conference Series in Math., no. 25, AMS publication, 1976.

7. H. Jacobinski, *Genera and decompositions of lattices over orders*, Acta Math., **121** (1968), 1-29.

8. R. Swan, *Periodic resolutions for finite groups*, Ann. Math., **72** (1960), 267-291.

9. ———, *Induced representations and projective modules*, Annals of Mathematics, **71** (1960), 552-578.

10. ———, *Projective modules over grovp rings and maximal orders*, Ann. Math., **76** (1962), 55-61.

11. R. Swan and G. Evans, *K-theory of finite groups and orders*, Lecture notes in mathematics, vol. 149, Springer-Verlag, 1970.

12. S. Ullom, *Nontrivial lower bounds for class groups of integral group rings*, Illinois J. Math., **20** (1976), 361-371.

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